Systems of identical particles

Lecture notes 14 (based on CT, Section 14)
Two particles are called identical when they have the same intrinsic properties (mass, charge, spin).

For example, all electrons in the Universe are identical.

An electron and a positron are not identical because, even if they have the same spin and mass, their charge is opposite.

If a system contains two identical particles, when we exchange one with the other there is no change in its properties.
In classical mechanics, the presence of identical particles in a system poses no particular problems.

Each particle moves along a trajectory, which allows to distinguish it from the others and “follow” it through the evolution of the system.

At the time \( t = t_0 \), the state of the system is defined by specifying the position and velocity of each of the two particles \( \{ r_0, v_0 \}, \{ r'_0, v'_0 \} \).

We can choose

\[
\begin{align*}
 r_1(t_0) &= r_0 & r_2(t_0) &= r'_0 \\
 v_1(t_0) &= v_0 & v_2(t_0) &= v'_0
\end{align*}
\]

or

\[
\begin{align*}
 r_1(t_0) &= r'_0 & r_2(t_0) &= r_0 \\
 v_1(t_0) &= v'_0 & v_2(t_0) &= v_0
\end{align*}
\]
If we consider the evolution of the system, we can write the solution of the equations of motion corresponding to the first choice as:

\[ r_1(t) = r(t) \quad r_2(t) = r'(t) \]

The system is not changed if the particles exchange their roles: if we consider the second choice we have:

\[ r_1(t) = r'(t) \quad r_2(t) = r(t) \]

The two descriptions are perfectly equivalent
Statement of the problem

- The situation is different in Quantum Mechanics, where the particles don’t have definite trajectories.
- Even if at $t=t_0$ the wave packets associated with the two particles are completely separated in space, their subsequent evolution may mix them.
- We then lose track of the particles: when we detect one particle in a region of space in which both of them have a non-zero probability, we have no way of knowing whether the detected particle is number 1 or number 2.
We consider, as an example, the collision of two particles in the center of mass frame.

Before the collision, we have two well-separated wave packets which travel towards each other.

During the collision, they overlap.
Statement of the problem

- After the collision, the region in which the probability density of the two particles is non-zero looks like a spherical shell whose radius increases with time.

- We place a detector in a direction which forms an angle $\theta$ with the initial velocity of the wave packet 1.

- We suppose that the detector detects a particle.

- It is certain that the other particle moves away in the opposite direction (conservation of momentum).

- It is impossible to know whether the particle detected is 1 or 2.
There are two different paths that could have led to the final state found in the measurement:

Nothing enables us to determine which one was actually followed.

Fundamental difficulty: in order to calculate the probability of a given measurement, it is necessary to know the final state vectors associated with this result.
Consider a system of two identical spin ½ particles

Limit ourselves to study its spin degrees of freedom

Distinguish between the physical state and its mathematical description

It seems natural to suppose that, if we made a complete measurement of each of the two spins, we would know the physical state of the total system

We assume that \( S_z = +\hbar/2 \) for one particle and \( S_z = -\hbar/2 \) for the other one

\( S_1 \) and \( S_2 \) denote the two spins, \( \{ |\varepsilon_1,\varepsilon_2> \} \) is the orthonormal basis of common eigestates of \( S_{1z} \) and \( S_{2z} \)
Two different mathematical states can be associated with the same physical state:

\[
\begin{align*}
|\varepsilon_1 = +, \varepsilon_2 = - \rangle \\
|\varepsilon_1 = -, \varepsilon_2 = + \rangle
\end{align*}
\]

can both a priori describe the physical state.

These two kets span a two-dimensional subspace whose normalized vectors are of the form

\[
\alpha |+, - \rangle + \beta |-, + \rangle \text{ with } |\alpha|^2 + |\beta|^2 = 1
\]
Superposition principle: all the above mathematical kets can represent the same state (one spin pointing up, the other pointing down)

This is called exchange degeneracy

This creates fundamental difficulties

Let us determine the probability of finding the components of the two spins along x both equal to $+\hbar/2$
With this measurement result is associated a single ket of the state space:

$$\frac{1}{\sqrt{2}} [\epsilon_1 = + \rangle + \epsilon_1 = - \rangle] \otimes \frac{1}{\sqrt{2}} [\epsilon_2 = + \rangle + \epsilon_2 = - \rangle]$$

$$= \frac{1}{2} [\epsilon_1 = +, + \rangle + \epsilon_1 = -, + \rangle + \epsilon_1 = +, - \rangle + \epsilon_1 = -, - \rangle]$$

The probability for the vector $\alpha |+, -\rangle + \beta |-, +\rangle$

is $\left| \frac{1}{2} (\alpha + \beta) \right|^2$, which depends on $\alpha$ and $\beta$

We have to remove the exchange degeneracy, namely indicate unambiguously which of the above kets is to be used
Exchange degeneracy arises in the study of all systems containing an arbitrary number N of identical particles (with N>1)

Consider a three particle system. With each particle is associated a state space with observables acting on it

The state space of the three-particle system is the tensor product:

\[ \mathcal{E} = \mathcal{E}(1) \otimes \mathcal{E}(2) \otimes \mathcal{E}(3) \]

We consider an observable B(1) initially defined in \( \mathcal{E}(1) \)

We assume that it constitutes a C.S.C.O. in \( \mathcal{E}(1) \)
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We consider an observable B(1) initially defined in $\mathcal{E}(1)$.

We assume that it constitutes a C.S.C.O. in $\mathcal{E}(1)$

Observables B(2) and B(3) constitute a C.S.C.O. in $\mathcal{E}(2)$ and $\mathcal{E}(3)$
Statement of the problem

- B(1), B(2), B(3) have the same spectrum

- We can construct an orthonormal basis of $\mathcal{E}$:
  \[
  \{ | 1 : b_i; 2 : b_j; 3 : b_k \rangle ; i, j, k = 1, 2, \ldots \} 
  \]

- The kets $| 1 : b_i; 2 : b_j; 3 : b_k \rangle$ are common eigenvectors of the extensions of B(1), B(2), B(3) in $\mathcal{E}$, with eigenvalues $b_i, b_j, b_k$

- The three particles are identical: we cannot measure B(1), B(2), B(3) because the numbering has no significance

- We can measure the physical quantity B for each of the three particles
Suppose such measurement has resulted in three different eigenvalues $b_n$, $b_p$, $b_q$.

Exchange degeneracy appears: the state of the system after the measurement can be represented by any of the following kets:

$| 1: b_n; 2: b_p; 3: b_q \rangle, \quad | 1: b_q; 2: b_n; 3: b_p \rangle, \quad | 1: b_p; 2: b_q; 3: b_n \rangle,$

$| 1: b_n; 2: b_q; 3: b_p \rangle, \quad | 1: b_p; 2: b_n; 3: b_q \rangle, \quad | 1: b_q; 2: b_p; 3: b_n \rangle$

A complete measurement on each of the particles does not permit the determination of a unique ket of the state space of the system.

The indeterminacy disappears if the three eigenvalues are identical.
We introduce some operators which permute the various particles of the system.

Consider a system of two particles with the same spin $s$.

They do not need to be identical: the numbers (1) and (2) indicate their nature (e.g. (1) is a proton, (2) an electron).

We choose a basis $\{|u_i>\}$ in the state space $\mathcal{E}(1)$ of particle (1).

The two particles have the same spin: $\mathcal{E}(2)$ is isomorphic to $\mathcal{E}(1)$ and it can be spanned by the same basis.
Permutation operators

- We construct in \( \mathcal{E} \) the basis:

\[
\{ | 1 : u_i; 2 : u_j \rangle \}
\]

- The order is not important in a tensor product: we have:

\[
| 2 : u_j; 1 : u_i \rangle \equiv | 1 : u_i; 2 : u_j \rangle
\]

- However, of course we have:

\[
| 1 : u_j; 2 : u_i \rangle \neq | 1 : u_i; 2 : u_j \rangle \quad \text{if} \quad i \neq j
\]

- The permutation operator \( P_{21} \) is defined as the linear operator whose action on the basis vector is given by:

\[
P_{21} | 1 : u_i; 2 : u_j \rangle = | 2 : u_i; 1 : u_j \rangle = | 1 : u_j; 2 : u_i \rangle
\]
If we choose the basis formed by the common eigenstates of the position $\mathbf{R}$ and spin component $S_z$ we have:

$$P_{21} | 1 : \mathbf{r}, \varepsilon ; 2 : \mathbf{r}', \varepsilon' \rangle = | 1 : \mathbf{r}', \varepsilon' ; 2 : \mathbf{r}, \varepsilon \rangle$$

Any ket $| \psi \rangle$ of the state space can be represented by a set of $(2s+1)^2$ functions of six variables:

$$| \psi \rangle = \sum_{\varepsilon, \varepsilon'} \int d^3r \ d^3r' \ \psi_{\varepsilon, \varepsilon'}(\mathbf{r}, \mathbf{r}') | 1 : \mathbf{r}, \varepsilon ; 2 : \mathbf{r}', \varepsilon' \rangle$$

with

$$\psi_{\varepsilon, \varepsilon'}(\mathbf{r}, \mathbf{r}') = \langle 1 : \mathbf{r}, \varepsilon ; 2 : \mathbf{r}', \varepsilon' | \psi \rangle$$
Permutation operators

We then have:

\[ P_{21} | \psi \rangle = \sum_{\varepsilon, \varepsilon'} \int d^3r \, d^3r' \, \psi_{\varepsilon, \varepsilon'}(r, r') | 1 : r', \varepsilon' ; 2 : r, \varepsilon \rangle \]

By changing the name of the dummy variables:

\[ \varepsilon \leftrightarrow \varepsilon' \]
\[ r \leftrightarrow r' \]

we can write:

\[ P_{21} | \psi \rangle = \sum_{\varepsilon, \varepsilon'} \int d^3r \, d^3r' \, \psi_{\varepsilon', \varepsilon}(r', r) | 1 : r, \varepsilon ; 2 : r', \varepsilon' \rangle \]

The functions \( \psi_{\varepsilon, \varepsilon'}(r, r') = \langle 1 : r, \varepsilon ; 2 : r', \varepsilon' | P_{21} | \psi \rangle \)

which represent the ket \( |\psi'\rangle = P_{21} |\psi\rangle \) can be obtained from the functions which represent the ket \( |\psi\rangle \) by inverting \((r, \varepsilon)\) with \((r', \varepsilon')\):

\[ \psi_{\varepsilon', \varepsilon}(r, r') = \psi_{\varepsilon, \varepsilon'}(r', r) \]
Permutation operators

- We see from the definition that \((P_{21})^2 = 1\)

- The operator \(P_{21}\) is its own inverse

- \(P_{21}\) is Hermitian: \(P^\dagger_{21} = P_{21}\)

- Its matrix elements in the \(\{ 1 : u_i; 2 : u_j \}\) basis are:
  
  \[
  \langle 1 : u_{i'} ; 2 : u_{j'} | P_{21} | 1 : u_i ; 2 : u_j \rangle = \langle 1 : u_{i'} ; 2 : u_{j'} | 1 : u_j ; 2 : u_i \rangle = \delta_{i',j} \delta_{j',i}
  \]

- Those of \(P^\dagger_{21}\) are:
  
  \[
  \langle 1 : u_{i'} ; 2 : u_{j'} | P^\dagger_{21} | 1 : u_i ; 2 : u_j \rangle = (\langle 1 : u_i ; 2 : u_j | P_{21} | 1 : u_{i'} ; 2 : u_{j'} \rangle)^*
  = (\langle 1 : u_i ; 2 : u_j | 1 : u_{j'} ; 2 : u_{i'} \rangle)^* = \delta_{i,j'} \delta_{j,i'}
  \]

- \(P_{21}\) is also unitary: \(P^\dagger_{21} P_{21} = P_{21} P^\dagger_{21} = 1\)
Permutation operators

- $P_{21}$ is hermitian, therefore its eigenvalues are real.
- Since $(P_{21})^2 = 1$, its eigenvalues are $+1$ and $-1$.
- The eigenvectors of $P_{21}$ corresponding to the eigenvalue $+1$ are symmetric, those corresponding to $-1$ are antisymmetric.

\[ P_{21} \left| \psi_S \right\rangle = \left| \psi_S \right\rangle \quad \Rightarrow \quad \left| \psi_S \right\rangle \text{ symmetric} \]
\[ P_{21} \left| \psi_A \right\rangle = -\left| \psi_A \right\rangle \quad \Rightarrow \quad \left| \psi_A \right\rangle \text{ antisymmetric} \]

- Consider the two operators:

\[ S = \frac{1}{2}(1 + P_{21}) \quad A = \frac{1}{2}(1 - P_{21}) \]
- They are projectors: $S^2 = S$, $A^2 = A$ and hermitian:

\[ S^\dagger = S \quad A^\dagger = A \]
Permutation operators

- S and A are projectors on orthogonal subspaces, since:
  \[ SA = AS = 0 \]

- These subspaces are supplementary: \( S + A = 1 \)

- If \( |\psi> \) is an arbitrary ket of the state space, \( S|\psi> \) is a symmetric ket and \( A|\psi> \) an antisymmetric ket:
  \[
P_{21} S |\psi> = S |\psi> \quad \quad \quad \quad \quad \quad \quad\quad\quad \quad \quad\quad\quad\quad\quad\quad P_{21} A |\psi> = - A |\psi>\]

- S and A are called symmetrizer and antisymmetrizer

- Notice that
  \[
  SP_{21} |\psi> = S |\psi> \quad \quad \quad \quad \quad \quad \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad AP_{21} |\psi> = - A |\psi>
  \]
Permutation operators

- Let us now see how observables are transformed by permutation.
- Consider an observable \( B(1) \) initially defined in \( \mathcal{E}(1) \) and then extended into \( \mathcal{E} \).
- It is possible to construct the basis in \( \mathcal{E}(1) \) from eigenvectors of \( B(1) \).
- What is the action of the operator \( P_{21}B(1)P_{21}^\dagger \) on an arbitrary basis ket of \( \mathcal{E} \)?

\[
P_{21}B(1)P_{21}^\dagger \begin{bmatrix} 1 : u_i \; ; \; 2 : u_j \end{bmatrix} = P_{21}B(1) \begin{bmatrix} 1 : u_j \; ; \; 2 : u_i \end{bmatrix} = b_j P_{21} \begin{bmatrix} 1 : u_j \; ; \; 2 : u_i \end{bmatrix} = b_j \begin{bmatrix} 1 : u_i \; ; \; 2 : u_j \end{bmatrix}
\]
We would obtain the same result applying $B(2)$ on the basis ket:

$$P_{21} B(1) P_{21}^\dagger = B(2)$$

Similarly we get:

$$P_{21} B(2) P_{21}^\dagger = B(1)$$

For observables which involve both indices simultaneously we have:

$$P_{21} [B(1) + C(2)] P_{21}^\dagger = B(2) + C(1)$$

$$P_{21} B(1) C(2) P_{21}^\dagger = P_{21} B(1) P_{21}^\dagger P_{21} C(2) P_{21}^\dagger = B(2) C(1)$$

Generalization:

$$P_{21} \mathcal{O}(1, 2) P_{21}^\dagger = \mathcal{O}(2, 1)$$
An observable is said to be symmetric if:

\[ \mathcal{O}_s(2, 1) = \mathcal{O}_s(1, 2) \]

All symmetric observables satisfy:

\[ P_{21} \mathcal{O}_s(1, 2) = \mathcal{O}_s(1, 2)P_{21} \]

namely \[ [\mathcal{O}_s(1, 2), P_{21}] = 0 \]

Symmetric observables commute with the permutation operator.
In the state space of a system composed of $N$ particles, $N!$ permutation operators can be defined (one of them is the identity).

If $N>2$, their properties are more complex than those of $P_{21}$.

To get an idea, we briefly consider $N=3$.

We have three particles which have the same spin.

We build a basis by taking a tensor product:

$$\{ | 1 : u_i ; 2 : u_j ; 3 : u_k \rangle \}$$

There are six permutator operators:

$$P_{123}, P_{312}, P_{231}, P_{132}, P_{213}, P_{321}$$
Permutation operators

- Their action on the basis vectors is:

\[ P_{npq} \left| 1 : u_i; 2 : u_j; 3 : u_k \right> = \left| n : u_i; p : u_j; q : u_k \right> \]

- For example:

\[ P_{231} \left| 1 : u_i; 2 : u_j; 3 : u_k \right> = \left| 2 : u_i; 3 : u_j; 1 : u_k \right> \]
\[ = \left| 1 : u_k; 2 : u_i; 3 : u_j \right> \]

- \( P_{123} \) is the identity operator
The set of permutation operators constitutes a group:

- $P_{123}$ is the identity operator
- The product of two permutation operators is also a permutation operator:

$$P_{312}P_{132} = P_{321}$$

Proof:

$$P_{312}P_{132} | 1 : u_i; 2 : u_j; 3 : u_k \rangle = P_{312} | 1 : u_i; 3 : u_j; 2 : u_k \rangle = P_{312} | 1 : u_i; 2 : u_k; 3 : u_j \rangle = | 3 : u_i; 1 : u_k; 2 : u_j \rangle = | 1 : u_k; 2 : u_j; 3 : u_i \rangle$$

- Each permutation operator has an inverse, which is also a permutation operator:

$$P_{123}^{-1} = P_{123}; \quad P_{312}^{-1} = P_{231}; \quad P_{231}^{-1} = P_{312}$$

$$P_{132}^{-1} = P_{132}; \quad P_{213}^{-1} = P_{213}; \quad P_{321}^{-1} = P_{321}$$
Permutation operators

- A transposition is a permutation which exchanges the roles of two of the particles.

- The last three of these operators are transposition operators:

  \[ P_{123}, P_{312}, P_{231}, P_{132}, P_{213}, P_{321} \]

- They are Hermitian and each of them is the same as its inverse: they are also unitary.

- Any permutation operator can be broken down into a product of transposition operators:

  \[ P_{312} = P_{132}P_{213} = P_{321}P_{132} = P_{213}P_{321} = P_{132}P_{213}(P_{132})^2 = \ldots \]
Permutation operators

- Permutation operators are unitary, because they are the products of transposition operators, all of which are unitary.
- They are not necessarily Hermitian.
- The adjoint of a permutation operator has the same parity as that of the operator itself.
Permutation operators

- The permutation operators do not commute for $N > 2$
- Therefore, it is not possible to build a basis of common eigenvectors of these operators
- $P_\alpha$ is a permutation operator associated to a system of $N$ particles with the same spin. $\alpha$: arbitrary permutation of the first $N$ integers
- A ket $|\psi_S\rangle$ such that $P_\alpha |\psi_S\rangle = |\psi_S\rangle$ for any permutation $P_\alpha$ is said to be completely symmetric
- A completely antisymmetric state satisfies: $P_\alpha |\psi_A\rangle = \varepsilon_\alpha |\psi_A\rangle$
  \[
  \varepsilon_\alpha = +1 \quad \text{if } P_\alpha \text{ is an even permutation} \\
  \varepsilon_\alpha = -1 \quad \text{if } P_\alpha \text{ is an odd permutation}
  \]
Permutation operators

- The set of completely symmetric kets constitutes a vector subspace $\mathcal{E}_S$ of the state space; the set of completely antisymmetric kets a subspace $\mathcal{E}_A$.

- Consider the two operators:

$$S = \frac{1}{N!} \sum_{\alpha} P_\alpha$$

$$A = \frac{1}{N!} \sum_{\alpha} \varepsilon_\alpha P_\alpha$$

where the summations are performed over the $N!$ permutations of the first $N$ integers and

$$\varepsilon_\alpha = +1 \quad \text{if } P_\alpha \text{ is an even permutation}$$

$$\varepsilon_\alpha = -1 \quad \text{if } P_\alpha \text{ is an odd permutation}$$
Permutation operators

- We now show that $S$ and $A$ are projectors onto $E_S$ and $E_A$
- They are called symmetrizer and antisymmetrizer
- They are Hermitian operators:
  
  $$S^\dagger = S \quad A^\dagger = A$$

- The adjoint of $P_\alpha$ is another permutation operator of the same parity which coincides with $P_\alpha^{-1}$

- Therefore, taking the adjoints of the right hand sides of the definitions of $S$ and $A$ amounts to changing the order of the terms in the summations
If $P_{\alpha_0}$ is an arbitrary permutation operator, we have:

$$P_{\alpha_0}S = SP_{\alpha_0} = S$$

$$P_{\alpha_0}A = AP_{\alpha_0} = \varepsilon_{\alpha_0}A$$

This is due to the fact that $P_{\alpha_0}P_{\alpha} = P_{\beta}$ is also a permutation operator such that $\varepsilon_{\beta} = \varepsilon_{\alpha_0}\varepsilon_{\alpha}$

If we fix $P_{\alpha_0}$ and choose successively for $P_{\alpha}$ all the permutations of the group, we see that the $P_{\beta}$ are each identical to one and only one of these permutations. Therefore:

$$P_{\alpha_0}S = \frac{1}{N!} \sum_{\alpha} P_{\alpha_0}P_{\alpha} = \frac{1}{N!} \sum_{\beta} P_{\beta} = S$$

$$P_{\alpha_0}A = \frac{1}{N!} \sum_{\alpha} \varepsilon_{\alpha} P_{\alpha_0}P_{\alpha} = \frac{1}{N!} \varepsilon_{\alpha_0} \sum_{\beta} \varepsilon_{\beta}P_{\beta} = \varepsilon_{\alpha_0}A$$
Similarly, we could prove analogous relations in which we multiply $S$ and $A$ by $P_{\alpha_0}$ on the right.

From the above we see that:

\[ S^2 = S \]

\[ A^2 = A \]

This is because:

\[ S^2 = \frac{1}{N!} \sum_{\alpha} P_{\alpha}S = \frac{1}{N!} \sum_{\alpha} S = S \]

\[ A^2 = \frac{1}{N!} \sum_{\alpha} \epsilon_{\alpha} P_{\alpha}A = \frac{1}{N!} \sum_{\alpha} \epsilon_{\alpha}^2 A = A \]

Furthermore:

\[ AS = \frac{1}{N!} \sum_{\alpha} \epsilon_{\alpha} P_{\alpha}S = \frac{1}{N!} S \sum_{\alpha} \epsilon_{\alpha} = 0 \]

\[ AS = SA = 0 \]
Therefore, S and A are projectors

They project onto $S$ and $A$ since

$$P_{x_0} S \left| \psi \right> = S \left| \psi \right>$$
$$P_{x_0} A \left| \psi \right> = \varepsilon_{x_0} A \left| \psi \right>$$

The following relations hold:

$$S P_\alpha \left| \psi \right> = S \left| \psi \right> \quad A P_\alpha \left| \psi \right> = \varepsilon_\alpha A \left| \psi \right>$$

For N>2, the symmetrizer and antisymmetrizer are not projecting onto supplementary subspaces:

$$S + A = \frac{1}{3} (P_{123} + P_{231} + P_{312}) \neq 1$$
When a system includes several identical particles, only certain kets of its state space can describe its physical state. Physical kets are, depending on the nature of the identical particles, either completely symmetric or completely antisymmetric with respect to permutation of these particles. Those particles for which the physical kets are symmetric are called \textit{bosons}; those for which they are antisymmetric are called \textit{fermions}. 
The symmetrization postulate

- Therefore, the symmetrization postulate limits the state space for a system of identical particles.

- This space is no longer the tensor product of the individual state spaces of the particles constituting the system: it is only a subspace, $\mathcal{E}_S$ or $\mathcal{E}_A$, depending on whether the particles are bosons or fermions.

- All particles existing in nature are divided into two categories: particles of half-integral spin are fermions, particles of integral spin are bosons.

- This also applies to composite particles: nuclei whose mass number $A$ is even are bosons, those whose $A$ is odd are fermions.
The symmetrization postulate

This new postulate removes the exchange degeneracy and the corresponding difficulties.

Let $|u\rangle$ be a state which can mathematically describe a well-defined physical state of a system containing $N$ identical particles.

For any permutation operator $P_\alpha$, $P_\alpha |u\rangle$ can describe this physical system as well as $|u\rangle$.

This is true for any ket belonging to the same subspace $\mathcal{E}_u$ spanned by $|u\rangle$ and all its permutations.

Depending on the choice of $|u\rangle$, the dimension of $\mathcal{E}_u$ can vary between 1 and $N!$. If it is $>1$ we have an exchange degeneracy.
The symmetrization postulate

- The new postulate restricts the class of kets able to describe a physical state: they must belong to $\mathcal{E}_S$ for bosons and $\mathcal{E}_A$ for fermions.

- The difficulties related to the exchange degeneracy are eliminated if we can show that $\mathcal{E}_u$ contains a single ket of $\mathcal{E}_S$ or a single ket of $\mathcal{E}_A$.

- To show this, we shall use the relations $S = SP_x$ or $A = \varepsilon_x AP_x$.

- We obtain: $S \left| u \right> = SP_x \left| u \right> \quad A \left| u \right> = \varepsilon_x AP_x \left| u \right>$

- The symmetrization postulate indicates the ket of $\mathcal{E}_u$ to be associated with the physical state: $S|u>$ for bosons and $A|u>$ for fermions. This is the physical ket.
The symmetrization postulate

- Rules to construct the unique physical ket of a system of N identical particles:
  - Number them arbitrarily, and construct the ket |u> corresponding to the physical state considered and to the numbers given to the particles
  - Apply S or A to |u>, depending on whether the particles are bosons or fermions
  - Normalize the ket so obtained
The symmetrization postulate

- We consider a system of two identical particles

- One of them is in the individual state characterized by the normalized ket $|\varphi\rangle$, the other one is in the individual state characterized by the normalized ket $|\chi\rangle$

- We first consider the case in which these two kets are distinct
  - We label with the number 1 the particle in the state $|\varphi\rangle$ and with the number 2 the particle in the state $|\chi\rangle$
  - We symmetrize $|u\rangle$ if the particles are bosons, antisymmetrize $|u\rangle$ if they are fermions

$$S|u\rangle = \frac{1}{2} [ |1: \varphi; 2: \chi\rangle + |1: \chi; 2: \varphi\rangle ]$$

$$A|u\rangle = \frac{1}{2} [ |1: \varphi; 2: \chi\rangle - |1: \chi; 2: \varphi\rangle ]$$
The symmetrization postulate

- We have to normalize the states: if we assume $|\varphi\rangle$ and $|\chi\rangle$ to be orthogonal, the normalized physical ket can be written as:

$$
|\varphi;\chi\rangle = \frac{1}{\sqrt{2}} \left[ |1:\varphi;2:\chi\rangle + \epsilon |1:\chi;2:\varphi\rangle \right]
$$

with $\epsilon=+1$ for bosons and $\epsilon=-1$ for fermions

- We assume that the two individual states are identical: $|\varphi\rangle = |\chi\rangle$

- We then get: $|u\rangle = |1:\varphi;2:\varphi\rangle$

- The state is already symmetric: If the particles are bosons, $|u\rangle$ is their physical state. If they are fermions we have:

$$
A |u\rangle = \frac{1}{2} \left[ |1:\varphi;2:\varphi\rangle - |1:\varphi;2:\varphi\rangle \right] = 0
$$
The symmetrization postulate

- This is the **Pauli exclusion principle**: two identical fermions cannot be in the same individual state

- We now want to generalize all this to the case of N particles

- We start with N=3

- Consider a physical system defined by specifying the three individual normalized states: \( | \varphi \rangle, | \chi \rangle \text{ and } | \omega \rangle \)

- \( |u\rangle \) can be chosen as:

\[
| u \rangle = | 1 : \varphi ; 2 : \chi ; 3 : \omega \rangle
\]
The symmetrization postulate

- If the particles are bosons we have:

\[ S |u\rangle = \frac{1}{3!} \sum_{\alpha} P_{\alpha} |u\rangle \]

\[ = \frac{1}{6} [ |1:\phi;2:\chi;3:\omega\rangle + |1:\omega;2:\phi;3:\chi\rangle + |1:\chi;2:\omega;3:\phi\rangle \\
+ |1:\phi;2:\omega;3:\chi\rangle + |1:\chi;2:\phi;3:\omega\rangle + |1:\omega;2:\chi;3:\phi\rangle ] \]

- We then just need to normalize the ket

- We assume that the three individual kets are orthogonal

- The six kets which compose \( S|u\rangle \) are also orthogonal

- To normalize the state, we replace \( 1/6 \) by \( 1/\sqrt{6} \).
The symmetrization postulate

- If \( | \varphi \rangle \text{ and } | \chi \rangle \) coincide, but remain orthogonal to \( | \omega \rangle \), only three distinct kets now appear:

- The normalized physical ket can be written as:

\[
| \varphi ; \varphi ; \omega \rangle = \frac{1}{\sqrt{3}} \left[ | 1 : \varphi ; 2 : \varphi ; 3 : \omega \rangle + | 1 : \varphi ; 2 : \omega ; 3 : \varphi \rangle + | 1 : \omega ; 2 : \varphi ; 3 : \varphi \rangle \right]
\]

- Finally, if the three individual kets are the same, we get:

\[
| u \rangle = | 1 : \varphi ; 2 : \varphi ; 3 : \varphi \rangle
\]

which is already symmetric and normalized.
In the case of fermions, the application of $A$ to $|u>$ gives:

$$A | u > = \frac{1}{3!} \sum_{\alpha} \varepsilon_{\alpha} P_{\alpha} | 1 : \varphi ; 2 : \chi ; 3 : \omega >$$

The signs of the various terms follow the determinant rule:

$$A | u > = \frac{1}{3!} \begin{vmatrix}
| 1 : \varphi > & | 1 : \chi > & | 1 : \omega > \\
| 2 : \varphi > & | 2 : \chi > & | 2 : \omega > \\
| 3 : \varphi > & | 3 : \chi > & | 3 : \omega > \\
\end{vmatrix}$$

$A | u >$ is zero if two of the individual states coincide.

If the three states were orthogonal, the six kets which compose $A | u >$ are orthogonal: to normalize it, we replace $1/3!$ by $\frac{1}{\sqrt{3!}}$. 

The symmetrization postulate
The symmetrization postulate

- All of this can be generalized to the case of $N$ identical particles.
- For $N$ identical bosons, it is always possible to construct the physical state $S|u>$ from arbitrary individual states.
- For fermions, the physical ket $A|u>$ can be written in the form of an $N\times N$ Slater determinant.
- This excludes the case in which two individual states coincide.
The symmetrization postulate

- Consider a system of \( N \) identical particles
- Starting with a basis \( \{|u_i>\} \) in the state space of a single particle, we can construct the basis
  \[
  \{|1 : u_i; 2 : u_j; \ldots; N : u_p\rangle\}
  \]
in the tensor product space \( \mathcal{E} \)
- Since the physical state space is either \( \mathcal{E}_S \) or \( \mathcal{E}_A \), we need to determine a basis in it
- By applying \( S \) or \( A \) to the kets of the above basis, we obtain a set of vectors spanning \( \mathcal{E}_S \) or \( \mathcal{E}_A \)
The symmetrization postulate

- Consider an arbitrary ket $| \varphi \rangle$ of $\mathcal{E}_S$

- Since $| \varphi \rangle$ belongs also to $\mathcal{E}$, we can write it as:

$$| \varphi \rangle = \sum_{i,j,\ldots,p} a_{i,j,\ldots,p} |1 : u_i; 2 : u_j; \ldots N : u_p \rangle$$

- Since $| \varphi \rangle$ belongs to $\mathcal{E}_S$, we have that $S | \varphi \rangle = | \varphi \rangle$

- Applying $S$ to both sides we get that $| \varphi \rangle$ can be written as a linear combination of the kets $S | 1 : u_i; 2 : u_j; \ldots N : u_p \rangle$

- The various kets $S | 1 : u_i; 2 : u_j; \ldots N : u_p \rangle$ are not independent: if we permute the role of the various particles in one of the kets, application of $S$ on this state leads to the same ket of $\mathcal{E}_S$
We introduce the concept of occupation number: for the ket $\ket{1 : u_i; 2 : u_j; \ldots; N : u_p}$, the occupation number $n_k$ of the individual state $\ket{u_k}$ is equal to the number of times that the state $\ket{u_k}$ appears in the sequence $\{\ket{u_i}, \ket{u_j}; \ldots; \ket{u_p}\}$, namely the number of particles in the state $\ket{u_k}$.

Of course we have: $\sum_k n_k = N$

Two different kets $\ket{1 : u_i; 2 : u_j; \ldots; N : u_p}$ for which the occupation numbers are equal can be obtained from each other through a permutation.

After the action of the symmetrizer, they give the same physical state, denoted by $\ket{n_1, n_2, \ldots, n_k, \ldots}$.
The symmetrization postulate

\[ |n_1, n_2, ..., n_k, ... \rangle = c \ S \left| \begin{array}{cc} 1 : u_1 ; 2 : u_2 ; ... n_1 : u_1 ; n_1 + 1 : u_2 ; ... ; n_1 + n_2 : u_2 ; ... \end{array} \right\] \]

\[ n_1 \text{ particles} \quad \text{in the state } |u_1 \rangle \quad \text{and} \quad n_2 \text{ particles} \quad \text{in the state } |u_2 \rangle \]

- For fermions, S would be replaced by A

- Properties:
  - The scalar product of two kets \[ |n_1, n_2, ..., n_k, ... \rangle \quad \text{and} \quad |n'_1, n'_2, ..., n'_k, ... \rangle \] is different from zero only if all the occupation numbers are equal
  - If the particles are bosons, the occupation numbers \( n_k \) are arbitrary and the kets form an orthonormal basis
The symmetrization postulate

- If the particles are fermions, a basis in $\mathcal{F}_A$ is obtained by choosing the set of kets $|n_1,n_2,\ldots,n_k,\ldots>$ in which all the occupation numbers are either 1 or 0.
- If one of the occupation numbers is larger than one, the vector is zero.
- The vector is never zero if all the occupation numbers are equal to one or zero.
The symmetrization postulate

- We still have to show that the general postulates of Quantum Mechanics can be applied in light of the symmetrization postulates and that no contradiction arises.

- We will see that a measurement process can be described with kets belonging only to $\mathcal{E}_S$ or $\mathcal{E}_A$.

- We will see that the time evolution of the system does not take the ket out of its subspace.

- All the quantum mechanical formalism can be applied inside $\mathcal{E}_S$ or $\mathcal{E}_A$. 
The symmetrization postulate

- Consider a measurement performed on a system of identical particles.

- The ket $|\psi(t)\rangle$ describing the quantum state of the system before the measurement must belong to $\mathcal{H}_S$ or $\mathcal{H}_A$.

- To apply the postulates of Quantum Mechanics concerning measurements, we need to take the scalar product of $|\psi(t)\rangle$ with the ket $|u\rangle$ corresponding to the physical state of the system after the measurement.

- The probability amplitude $\langle u | \psi(t) \rangle$ can be expressed in terms of two vectors, both belonging either to $\mathcal{H}_S$ or $\mathcal{H}_A$. 
The symmetrization postulate

- If the measurement is complete, the physical ket $|u>\,$ is unique.
- If the measurement is incomplete, several orthogonal physical states are obtained, and the corresponding probabilities must be summed.
The symmetrization postulate

- Sometimes it is possible to specify the measurement performed on the system by giving the explicit expression of the corresponding observable in terms of \( R_1, P_1, S_1, R_2, P_2, S_2 \), etc.

- Some observables that can be measured in a three-particle system
  - Position of the center of mass \( R_G \), total momentum \( P \) and total angular momentum \( L \)
    
    \[
    R_G = \frac{1}{3} (R_1 + R_2 + R_3)
    \]
    
    \[
    P = P_1 + P_2 + P_3
    \]
    
    \[
    L = L_1 + L_2 + L_3
    \]

- Electrostatic repulsion energy
    
    \[
    W = \frac{q^2}{4\pi\varepsilon_0} \left( \frac{1}{|R_1 - R_2|} + \frac{1}{|R_2 - R_3|} + \frac{1}{|R_3 - R_1|} \right)
    \]

- Total spin \( S = S_1 + S_2 + S_3 \)
The symmetrization postulate

- The observables involve various particles symmetrically
- This is because the particles are identical: they must play a symmetric role in any measurable observable
- Therefore, any observable $G$ must be invariant under all permutations of the $N$ identical particles

$$[G, P_\alpha] = 0 \quad \text{for all} \quad P_\alpha$$

- E.g.: the observable $\mathbf{R}_1 - \mathbf{R}_2$ is not invariant under the effect of permutation and indeed it is not a physical observable: a measurement of it, assumes that particle (1) can be distinguished from particle (2).
The symmetrization postulate

- Therefore $\mathcal{E}_S$ and $\mathcal{E}_A$ are both invariant under the action of a physical observable $G$

- We show that, if $|\psi\rangle$ belongs to $\mathcal{E}_A$, also $G|\psi\rangle$ belongs to $\mathcal{E}_A$

- The fact that $|\psi\rangle$ belongs to $\mathcal{E}_A$ means that: $P_\alpha |\psi\rangle = \varepsilon_\alpha |\psi\rangle$

- Now we calculate $P_\alpha G |\psi\rangle = GP_\alpha |\psi\rangle = \varepsilon_\alpha G |\psi\rangle$

- Since $P_\alpha$ is arbitrary, this means that $G |\psi\rangle$ is completely antisymmetric and therefore it belongs to $\mathcal{E}_A$

- Therefore, all operations usually performed on an observable can be performed on $G$ within one of the subspaces $\mathcal{E}_S$ and $\mathcal{E}_A$
The symmetrization postulate

- However, not all the eigenvalues of $G$ which exist in the total space are necessarily found in the subspaces $\mathcal{E}_S$ and $\mathcal{E}_A$.
- No new eigenvalues are added to the spectrum.
- The Hamiltonian of a system of identical particles must be a physical observable.
- For example, the Hamiltonian which describes the motion of the two electrons of the helium atom about the nucleus can be written as:

$$H(1, 2) = \frac{P_1^2}{2m_e} + \frac{P_2^2}{2m_e} - \frac{2e^2}{R_1} - \frac{2e^2}{R_2} + \frac{e^2}{|R_1 - R_2|}$$
The symmetrization postulate

- The kinetic energy is the same because the two masses are equal
- The attraction of the nucleus is the same because the two charges are equal
- The last term is the mutual interaction of the electrons
- It is symmetrical since neither of the electrons is in a privileged position
- Therefore, \([H, P_{\alpha}] = 0\)
The symmetrization postulate

- If the ket $|\psi(t_0)\rangle$ describing the state of the system at $t=t_0$ is a physical ket, the same must be true of $|\psi(t)\rangle$ obtained from $|\psi(t_0)\rangle$ by solving Schroedinger’s equation.

- According to this equation we have:

$$|\psi(t + dt)\rangle = \left(1 + \frac{dt}{i\hbar} H\right)|\psi(t)\rangle$$

- Applying $P_\alpha$ we get:

$$P_\alpha |\psi(t + dt)\rangle = \left(1 + \frac{dt}{i\hbar} H\right)P_\alpha |\psi(t)\rangle$$

- If $|\psi(t)\rangle$ is eigenstate of $P_\alpha$, $|\psi(t+dt)\rangle$ is also an eigenvector of $P_\alpha$ with the same eigenvalue.
The symmetrization postulate

- Since $|\psi(t_0)>$ is either completely symmetric or antisymmetric, this property is conserved over time.

- Therefore, Schroedinger’s equation does not remove $|\psi(t)>$ from $\mathcal{E}_S$ or $\mathcal{E}_A$. 