**Lattice QCD**

- Best first principle-tool to extract predictions for the theory of strong interactions in the non-perturbative regime

- Uncertainties:
  - Statistical: finite sample, error $\sim \frac{1}{\sqrt{\text{sample size}}}$
  - Systematic: finite box size, unphysical quark masses

- Given enough computer power, uncertainties can be kept under control

- Results from different groups, adopting different discretizations, converge to consistent results

- Unprecedented level of accuracy in lattice data
Importance of continuum limit

- Lattice action: parametrization used to discretize the Lagrangian of QCD on a space-time grid

\[ N_t = \frac{1}{aT} \]

\[ N_s = \frac{L}{a} \]
Importance of continuum limit

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- Repeat the simulations on finer lattices (smaller \( a \) \( \leftrightarrow \) larger \( N_t \))
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- Lattice action: parametrization used to discretize the Lagrangian of QCD on a space-time grid

\[ N_t = \frac{1}{aT} \]

\[ N_s = \frac{L}{a} \]

- Repeat the simulations on finer lattices (smaller \( a \) ↔ larger \( N_t \))

![Graph showing data points and lines with equations:]

\[ f_1(N_t) = A + B \exp(-C/N_t^2) \]

\[ f_2(N_t) = A + B/N_t^2 + C/N_t^2 \log(N_t) \]

linear for \( N_t \leq 20 \)
Importance of continuum limit

- Lattice action: parametrization used to discretize the Lagrangian of QCD on a space-time grid

\[ N_t = \frac{1}{aT} \]

\[ N_s = \frac{L}{a} \]

- Repeat the simulations on finer lattices (smaller \( a \) \( \leftrightarrow \) larger \( N_t \))
Importance of continuum limit

- Observables are affected by discretization effects differently
Importance of continuum limit

- Observables are affected by discretization effects differently.

- In quantitative predictions, finite-$N_t$ results can lead to misleading information.

![Graph showing $\chi^2$ as a function of $1/N_t^2$](image)
Importance of continuum limit

- Observables are affected by discretization effects differently

- In quantitative predictions, finite-$N_t$ results can lead to misleading information

- **Message**: continuum extrapolated data always preferable
Low temperature phase: HRG model

Interacting hadronic matter in the ground state can be well approximated by a non-interacting resonance gas.

The pressure can be written as:

\[ \frac{p^{HRG}}{T^4} = \frac{1}{VT^3} \sum_{i \in \text{mesons}} \ln Z^M_{m_i} (T, V, \mu X^a) + \frac{1}{VT^3} \sum_{i \in \text{baryons}} \ln Z^B_{m_i} (T, V, \mu X^a) \]

where

\[ \ln Z^M_{m_i}/B = \mp \frac{Vd_i}{2\pi^2} \int_0^\infty dk k^2 \ln \left( 1 \mp z_i e^{-\varepsilon_i/T} \right) \]

with energies \( \varepsilon_i = \sqrt{k^2 + m_i^2} \), degeneracy factors \( d_i \) and fugacities

\[ z_i = \exp \left( \frac{\sum_{a} X^a_{i} \mu X^a}{T} \right) \]

\( X^a \): all possible conserved charges, including the baryon number \( B \), electric charge \( Q \), strangeness \( S \).

Up to which temperature do we expect agreement with the lattice data?
High temperature limit

- QCD thermodynamics approaches that of a non-interacting, massless quark-gluon gas:

\[
\left( \frac{P}{T^4} \right)_{\text{ideal}} = \frac{8\pi^2}{45} + \sum_{f=u,d,s} \left[ \frac{7\pi^2}{60} + \frac{1}{2} \left( \frac{\mu_f}{T} \right)^2 + \frac{1}{4\pi^2} \left( \frac{\mu_f}{T} \right)^4 \right]
\]

- We can switch on the interaction and systematically expand the observables in series of the coupling \( g \)

- Resummation of diagrams (HTL) or dimensional reduction are needed, to improve convergence

  Braaten, Pisarski (1990); Haque et al. (2014); Hietanen et al (2009)

- At what temperature does perturbation theory break down?
QCD Equation of state at $\mu_B=0$

- EoS available in the continuum limit, with realistic quark masses
- Agreement between stout and HISQ action for all quantities

WB: S. Borsanyi et al., 1309.5258
HotQCD: A. Bazavov et al., 1407.6387, PRD (2014)
The QCD path integral is computed by Monte Carlo algorithms which samples field configurations with a weight proportional to the exponential of the action

\[ Z(\mu_B, T) = \text{Tr} \left( e^{-\frac{H_{QCD} - \mu_B N_B}{T}} \right) = \int \mathcal{D}U e^{-S_G[U]} \det M[U, \mu_B] \]

- $\det M[\mu_B]$ complex $\rightarrow$ Monte Carlo simulations are not feasible

- We can rely on a few approximate methods, viable for small $\mu_B/T$:
  - Taylor expansion of physical quantities around $\mu=0$ (Bielefeld-Swansea collaboration 2002; R. Gavai, S. Gupta 2003)
  - Reweighting (complex phase moved from the measure to observables) (Barbour et al. 1998; Z. Fodor and S, Katz, 2002)
  - Simulations at imaginary chemical potentials (plus analytic continuation) (Alford, Kapustin, Wilczek, 1999; de Forcrand, Philipsen, 2002; D'Elia, Lombardo 2003)
Equation of state at $\mu_B > 0$

- Expand the pressure in powers of $\mu_B$
  \[
  \frac{p(\mu_B)}{T^4} = c_0 + c_2 \left( \frac{\mu_B}{T} \right)^2 + c_4 \left( \frac{\mu_B}{T} \right)^4 + c_6 \left( \frac{\mu_B}{T} \right)^6 + \mathcal{O}(\mu_B^8)
  \]

- Continuum extrapolated results for $c_2$, $c_4$, $c_6$ at the physical mass
Equation of state at $\mu_B>0$

- Calculate the EoS along the constant S/N trajectories
QCD phase diagram

Curvature $\kappa$ defined as:

$$\frac{T_c(\mu_B)}{T_c(\mu = 0)} = 1 - \kappa \left( \frac{\mu_B}{T_c(\mu_B)} \right)^2 + \lambda \left( \frac{\mu_B}{T_c(\mu_B)} \right)^4 \ldots$$

Recent results:

$$\kappa = 0.0149 \pm 0.0021$$
Curvature $\kappa$ defined as:

$$\frac{T_c(\mu_B)}{T_c(\mu = 0)} = 1 - \kappa \left( \frac{\mu_B}{T_c(\mu_B)} \right)^2 + \lambda \left( \frac{\mu_B}{T_c(\mu_B)} \right)^4 \ldots$$

Recent results:

$$\kappa = 0.020(4)$$

P. Cea et al., 1508.07599
QCD phase diagram

Curvature $\kappa$ defined as:

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Recent results:

$$\kappa = 0.020(4)$$  

P. Cea et al., 1508.07599

$$\kappa = 0.0135(20)$$  

C. Bonati et al., 1507.03571
QCD phase diagram

1. Kaczmarek et al., Nf=2+1, p4 staggered action, Taylor expansion, $\mu_s=0$, $N_t=8$
2. Falcone et al., Nf=2+1, p4 staggered action, analytic continuation, $\mu_s=\mu_u=\mu_d$, $N_t=4$
3. Bonati et al., Nf=2+1, stout staggered action, analytic continuation, $\mu_s=0$, continuum extrapolated
4. Bellwied et al. (WB), Nf=2+1, 4stout staggered action, analytic continuation, $<n_s>=0$, cont. extrap.
5. Cea et al., Nf=2+1, HISQ staggered action, analytic continuation, $\mu_s=\mu_u=\mu_d$, cont. extrapolated
Evolution of a Heavy Ion Collision

- **Chemical freeze-out**: inelastic reactions cease: the chemical composition of the system is fixed (particle yields and fluctuations)
- **Kinetic freeze-out**: elastic reactions cease: spectra and correlations are frozen (free streaming of hadrons)
- Hadrons reach the detector
Hadron yields

- \( E=mc^2 \): lots of particles are created
- **Particle counting** (average over many events)
- Take into account:
  - detector inefficiency
  - missing particles at low \( p_T \)
  - decays

**HRG model**: test hypothesis of hadron abundancies in equilibrium

\[
N_i = -T \frac{\partial \ln Z_i}{\partial \mu} = \frac{g_i V}{2\pi^2} \int_0^\infty \frac{p^2 dp}{\exp[(E_i - \mu_i)/T]} \pm 1
\]
The thermal fits

- Fit is performed minimizing the $\chi^2$
- Fit to yields: parameters $T$, $\mu_B$, $V$
- Fit to ratios: the volume $V$ cancels out

Changing the collision energy, it is possible to draw the freeze-out line in the $T$, $\mu_B$ plane.
Fluctuations of conserved charges

- **Definition:**

\[
\chi_{lmn}^{BSQ} = \frac{\partial^{l+m+n} p/T^4}{\partial (\mu_B/T)^l \partial (\mu_S/T)^m \partial (\mu_Q/T)^n}.
\]

- **Relationship between chemical potentials:**

\[
\begin{align*}
\mu_u &= \frac{1}{3} \mu_B + \frac{2}{3} \mu_Q; \\
\mu_d &= \frac{1}{3} \mu_B - \frac{1}{3} \mu_Q; \\
\mu_s &= \frac{1}{3} \mu_B - \frac{1}{3} \mu_Q - \mu_S.
\end{align*}
\]

- They can be calculated on the lattice and compared to experiment.
Connection to experiment

- **Fluctuations** of conserved charges are the **cumulants** of their event-by-event distribution

  - Mean: \( M = \chi_1 \)
  - Variance: \( \sigma^2 = \chi_2 \)
  - Skewness: \( S = \frac{\chi_3}{\chi_2^{3/2}} \)
  - Kurtosis: \( \kappa = \frac{\chi_4}{\chi_2^2} \)
  - \( S\sigma = \frac{\chi_3}{\chi_2} \)
  - \( \kappa\sigma^2 = \frac{\chi_4}{\chi_2} \)
  - \( M/\sigma^2 = \frac{\chi_1}{\chi_2} \)
  - \( S\sigma^3/M = \frac{\chi_3}{\chi_1} \)

- **Lattice QCD results** are functions of temperature and chemical potential
  - By comparing lattice results and experimental measurement we can **extract** the freeze-out parameters from first principles

Things to keep in mind

- Effects due to volume variation because of finite centrality bin width
  - Experimentally corrected by centrality-bin-width correction method
    - V. Skokov et al., PRC (2013)
- Finite reconstruction efficiency
  - Experimentally corrected based on binomial distribution
    - A.Bzdak, V.Koch, PRC (2012)
- Spallation protons
  - Experimentally removed with proper cuts in $p_T$
- Canonical vs Gran Canonical ensemble
  - Experimental cuts in the kinematics and acceptance
- Proton multiplicity distributions vs baryon number fluctuations
  - Recipes for treating proton fluctuations
    - M. Asakawa and M. Kitazawa, PRC(2012), M. Nahrgang et al., 1402.1238
- Final-state interactions in the hadronic phase
  - Consistency between different charges = fundamental test
    - J. Steinheimer et al., PRL (2013)
“Baryometer and Thermometer”

Let us look at the Taylor expansion of $R_{31}^B$

$$R_{31}^B(T, \mu_B) = \frac{\chi_3^B(T, \mu_B)}{\chi_1^B(T, \mu_B)} = \frac{\chi_4^B(T, 0) + \chi_{31}^{BQ}(T, 0)q_1(T) + \chi_{31}^{BS}(T, 0)s_1(T)}{\chi_2^B(T, 0) + \chi_{11}^{BQ}(T, 0)q_1(T) + \chi_{11}^{BS}(T, 0)s_1(T)} + \mathcal{O}(\mu_B^2)$$

- To order $\mu_B^2$ it is independent of $\mu_B$: it can be used as a thermometer

- Let us look at the Taylor expansion of $R_{12}^B$

$$R_{12}^B(T, \mu_B) = \frac{\chi_1^B(T, \mu_B)}{\chi_2^B(T, \mu_B)} = \frac{\chi_2^B(T, 0) + \chi_{11}^{BQ}(T, 0)q_1(T) + \chi_{11}^{BS}(T, 0)s_1(T) \frac{\mu_B}{T}}{\chi_2^B(T, 0)} + \mathcal{O}(\mu_B^3)$$

- Once we extract $T$ from $R_{31}^B$, we can use $R_{12}^B$ to extract $\mu_B$
Freeze-out parameters from B fluctuations

- **Thermometer:** \( \frac{\chi^B_3(T, \mu_B)}{\chi^B_1(T, \mu_B)} = S_B^3 \sigma_B^2 / M_B \)

- **Baryometer:** \( \frac{\chi^B_1(T, \mu_B)}{\chi^B_2(T, \mu_B)} = \sigma_B^2 / M_B \)

- **Upper limit:** \( T_f \leq 151 \pm 4 \text{ MeV} \)

- **Consistency** between freeze-out chemical potential from electric charge and baryon number is found.

WB: S. Borsanyi et al., PRL (2014)
STAR collaboration, PRL (2014)
Freeze-out parameters from B fluctuations

- Thermometer: \( \frac{\chi^B_3(T, \mu_B)}{\chi^B_1(T, \mu_B)} = S_B \sigma_B^3/M_B \)
- Baryometer: \( \frac{\chi^B_1(T, \mu_B)}{\chi^B_2(T, \mu_B)} = \sigma_B^2/M_B \)

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Freeze-out parameters from B fluctuations

- Thermometer: $\frac{\chi_3^B(T, \mu_B)}{\chi_1^B(T, \mu_B)} = S_B \sigma_B^3/M_B$
- Baryometer: $\frac{\chi_1^B(T, \mu_B)}{\chi_2^B(T, \mu_B)} = \sigma_B^2/M_B$

- Upper limit: $T_f \leq 151\pm4$ MeV
- Consistency between freeze-out chemical potential from electric charge and baryon number is found.

*WB: S. Borsanyi et al., PRL (2014)  
STAR collaboration, PRL (2014)*
Curvature of the freeze-out line

- Parametrization of the freeze-out line:
  \[ T_f(\mu_B) = T_{f,0} \left( 1 - \kappa_2^f \mu_B^2 - \kappa_4^f \mu_B^4 \right) \]

- Taylor expansion of the “ratio of ratios” \( R_{12}^{QB} = \left[ \frac{M_Q}{\sigma_Q^2} \right] / \left[ \frac{M_B}{\sigma_B^2} \right] \)
  \[ R_{12}^{QB} = R_{12}^{QB,0} + \left( R_{12}^{QB,2} - \kappa_2^f T_{f,0} \frac{dR_{12}^{QB,0}}{dT} \right) \hat{\mu}_B^2 \]
Curvature of the freeze-out line

- Parametrization of the freeze-out line:
  \[ T_f(\mu_B) = T_{f,0} \left( 1 - \kappa_2^f \tilde{\mu}_B^2 - \kappa_4^f \tilde{\mu}_B^4 \right) \]

- Taylor expansion of the “ratio of ratios” \( R_{12}^{QB} = \frac{[M_Q/\sigma_Q^2]}{[M_B/\sigma_B^2]} \)

\[ R_{12}^{QB} = \left( \kappa_2^f < 0.011 \right) \left( \tilde{\mu}_B^2 \right) \]

\[ T_{f,0} = (147 \pm 2) \text{ MeV} \]

\[ \kappa_P \sigma_P^2 < S_P \sigma_P \]

A. Bazavov et al., 1509.05786
STAR0.8: PRL (2013)

STAR2.0: X. Luo, PoS CPOD 2014
PHENIX: 1506.07834
Freeze-out line from first principles

- Use T- and $\mu_B$-dependence of $R_{12}^Q$ and $R_{12}^B$ for a combined fit:

$$R_{12}^Q(T, \mu_B) = \frac{\chi_1^Q(T, \mu_B)}{\chi_2^Q(T, \mu_B)} = \frac{\chi_{11}^Q(T, 0) + \chi_2^Q(T, 0)q_1(T) + \chi_{11}^Q(T, 0)s_1(T)}{\chi_2^Q(T, 0)} \frac{\mu_B}{T} + \mathcal{O}(\mu_B^3).$$

$$R_{12}^B(T, \mu_B) = \frac{\chi_1^B(T, \mu_B)}{\chi_2^B(T, \mu_B)} = \frac{\chi_{11}^B(T, 0) + \chi_{11}^B(T, 0)q_1(T) + \chi_{11}^B(T, 0)s_1(T)}{\chi_2^B(T, 0)} \frac{\mu_B}{T} + \mathcal{O}(\mu_B^3).$$

WB: S. Borsanyi et al., in preparation
What about strangeness freeze-out?

- Yield fits seem to hint at a higher temperature for strange particles.
Initial analysis of LHC data

- Fluctuation data not yet available
- Assuming Skellam distribution, can use yields: $\tilde{\chi}_N = \frac{1}{\sqrt{T^3}} \left( \langle N_q \rangle + \langle N_{-q} \rangle \right)$

- Slightly higher temperature than at RHIC: $(150 < T_f < 163)$ MeV
- Looking forward to fluctuation measurements at the LHC

P. Braun-Munzinger et al., PLB (2015)
Missing strange states?

- Quark Model predicts not-yet-detected (multi-)strange hadrons
- QM-HRG improves the agreement with lattice results for some observables but it worsens it for some other ones
- The effect is only relevant at finite $\mu_B$
- Feed-down from resonance decays neglected

A. Bazavov et al., PRL (2014)

R. Bellwied et al., in preparation
Flavor-dependent freeze-out?

Lattice data hint at possible flavor-dependence in transition temperature

Possibility of strange bound-states above $T_c$?

See talk by R. Bellwied on Thursday
Onset of deconfinement for charm quarks:

- Partial meson and baryon pressures described by HRG at $T_C$ and dominate the charm pressure then drop gradually. **Charm quark only dominant dof at $T>200$ MeV**

A. Bazavov et al., PLB (2014)

S. Mukherjee, P. Petreczky, S. Sharma 1509.08887
Fluctuations at high temperatures

HTL: N. Haque et al., JHEP (2014); DR: S. Mogliacci et al., JHEP (2013)
Conclusions

- Unprecedented precision in lattice QCD data allows a direct comparison to experiment for the first time

- QCD thermodynamics at $\mu_B=0$ can be simulated with high accuracy

- Extensions to finite density are under control up to $O(\mu_B^6)$

- Challenges for the near future
  - Sign problem
  - Real-time dynamics
Transport properties

- Matter in the region \((1-2)T_c\) is highly non-perturbative
- Significant modifications of its transport properties
- Common problem:
  - Transport properties can be explored through the analysis of certain correlation functions:

\[
G_H(\tau, \vec{p}, T) = \int_0^\infty \frac{d\omega}{2\pi} \rho_H(\omega, \vec{p}, T) \frac{\cosh(\omega(\tau - 1/2T))}{\sinh(\omega/2T)} = \int d^3x \ e^{i\vec{p}\cdot\vec{x}} \langle J^\alpha(0, 0) J^{\beta\dagger}(\tau, \vec{x}) \rangle
\]

- **Challenge**: integrate over discrete set of lattice points in \(\tau\) direction
- Use inversion methods like Maximum Entropy Method or modeling the spectral function at low frequencies
Quarkonia properties

- Three main approaches:
  - *Potential models* with heavy quark potential calculated on the lattice
    - Solve Schrödinger’s equation for the bound state two-point function
  - Extract *spectral functions* from Euclidean temporal correlators
  - Study *spatial correlation functions* of quarkonia and their in-medium screening properties
Inter-quark potential

- Static quark-antiquark free-energy

Borsanyi et al. JHEP(2015)

- Continuum extrapolated result with $N_f=2+1$ flavors at the physical mass
Inter-quark potential

- Quark-antiquark potential in $N_f=2+1$ QCD

- Real part of the complex potential lies close to the color singlet free energy

- Central potential: combination of pseudoscalar and vector potentials:

$$V_C(r) = \frac{1}{4}V_{PS} + \frac{3}{4}V_V$$

Burnier et al. (2014)
Allton et al. (2015)
Charmonium spectral functions in quenched approximation and preliminary studies with dynamical quarks yield consistent results: all charmonium states are dissociated for $T \gtrsim 1.5T_c$.

Bottomonium ($N_f=2+1$, $m_\pi=400$ MeV), MEM:

- S-wave ground state survives up to $1.9 T_c$, P-wave ground state melts just above $T_c$.

H. Ding et al., PRD (2012)
G. Aarts et al., PRD (2007)
WB: S. Borsanyi et al., JHEP (2014)

G. Aarts et al. JHEP (2014)
Quarkonia spectral functions

- Charmonium spectral functions in quenched approximation and preliminary studies with dynamical quarks yield consistent results: all charmonium states are dissociated for $T \gtrsim 1.5 T_c$

- Bottomonium ($N_f=2+1$, $m_{\pi}=160$ MeV), Bayesian method:
  - $\Upsilon(1S)$ signal survives at $T=249$ MeV
  - $\chi_b(1P)$ signal survives at $T=249$ MeV

- S-wave ground state and P-wave ground state survive up to $T \sim 250$ MeV

H. Ding et al., PRD (2012)
G. Aarts et al., PRD (2007)
WB: S. Borsanyi et al., JHEP (2014)

S. Kim et al. PRD (2015)

Talk by A. Rothkopf on Tuesday 23/26
Electric conductivity and charge diffusion

- Definitions:

\[ \sigma = \frac{C_{em}}{6} \lim_{\omega \to 0} \lim_{p \to 0} \omega \sum_{i=1}^{3} \rho_{ii}(\omega, p, T) \]

\[ D_Q = \frac{\sigma}{\chi_2^Q} \]

- Electric conductivity measures the response of the medium to small perturbations induced by an electromagnetic field.
Viscosity

- Shear viscosity in the pure gauge sector of QCD

- Challenge: very low signal-to-noise ratio for the Euclidean energy-momentum correlator

![Graph showing shear viscosity over temperature]

- Meyer’07
- Meyer’09
- Haas’13
- Nakamura & Sakai ‘05
- S. Borsanyi et al. ‘14
Freeze-out parameters from Q fluctuations

- Studies in HRG model: the different momentum cuts between STAR and PHENIX are responsible for more than 30% of their difference.
- Using continuum extrapolated lattice data, lower values for $T_f$ are found.

A. Bazavov et al. (2014)  
WB: Borsanyi et al. PRL (2013)  
F. Karsch et al., 1508.02614
Effects of kinematic cuts

- Rapidity dependence of moments needs to be studied for $1<\Delta\eta<2$
- Difference in kinematic cuts between STAR and PHENIX leads to a 5% difference in $T_f$

V. Koch, 0810.2520

Talk by J. Thaeder on Monday

Talk by F. Karsch on Monday
Strangeness fluctuations

Lattice data hint at possible flavor-dependence in transition temperature

Possibility of strange bound-states above $T_c$?

Additional strange hadrons

Discrepancy between lattice and HRG for $\mu_S/\mu_B$ can be understood by introducing higher mass states predicted by the Quark Model.

Discrepancy between QM predictions and lattice data for $\chi_4^S/\chi_2^S$ needs to be understood.

Their effect on freeze-out conditions needs to be investigated taking into account their decay feed-down into stable states.

A. Bazavov et al., PRL (2014)

Poster by P. Alba

P. M. Lo et al., 1507.06398
Columbia plot

- Pure gauge theory: $T_c=294(2)$ MeV  
  Francis et al., 1503.05652

- $N_f=2$ QCD at $m_\pi>m_\pi^{\text{phys}}$:
  - O(a) improved Wilson, $N_t=16$  
    - $m_\pi=295$ MeV $T_c=211(5)$ MeV  
    - $m_\pi=220$ MeV $T_c=193(7)$ MeV  
  Brandt et al., 1310.8326

- Twisted-mass QCD
  - $m_\pi=333$ MeV $T_c=180(12)$ MeV  
  Burger et al., 1412.6748

- $N_f=2+1$ O(a) improved Wilson
  - Continuum results  
  Borsanyi et al., 1504.03676

- HISQ action, $N_t=6$, no sign of 1\textsuperscript{st} order phase transition at $m_\pi=80$ MeV  
  HotQCD, 1312.0119, 1302.5740
Equation of state at $\mu_B > 0$

- Expand the pressure in powers of $\mu_B$ (or $\mu_L = 3/2(\mu_u + \mu_d)$)

\[
\frac{p(T, \{\mu_i\})}{T^4} = \frac{p(T, \{0\})}{T^4} + \frac{1}{2} \sum_{i,j} \frac{\mu_i \mu_j}{T^2} \chi_{ij}^2
\]

with

\[
\chi_{ij}^2 \equiv \frac{T}{V T^2} \left. \frac{\partial^2 \log Z}{\partial \mu_i \partial \mu_j} \right|_{\mu_i = \mu_j = 0}
\]

- Continuum extrapolated results at the physical mass

---

S. Borsanyi et al., JHEP (2012)
Alternative methods for thermodynamics

- **Gradient flow**: EoS in the quenched approximation
- **Twisted mass Wilson fermions**: EoS available so far for heavier-than-physical quark masses and $N_f=2$