Exam II, 04/17/2018

Physics 6316: Quantum Mechanics 2, Spring 2018

Solve the following problems, for a maximum of 60 points:

Problem 1 (20 points)

The unperturbed Hamiltonian of a two-state system is represented by

\[ H_0 = \begin{pmatrix} E_0^1 & 0 \\ 0 & E_0^2 \end{pmatrix}. \]

There is, in addition, a time-dependent perturbation

\[ V(t) = \begin{pmatrix} 0 & \lambda \cos(\omega t) \\ \lambda \cos(\omega t) & 0 \end{pmatrix} \]

with \( \lambda \) real. At \( t = 0 \) the system is known to be in the first eigenstate of \( H_0 \), represented by

\[ |i\rangle = |\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]

Using time-dependent perturbation theory, and assuming that \( E_0^1 - E_0^2 \) is not close to \( \pm \hbar \omega \), derive an expression for the probability that the system is found in the second state, represented by

\[ |f\rangle = |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

at the time \( t > 0 \).

Solution

The matrix element we need is:

\[ \langle f|V|i\rangle = \lambda \cos(\omega t) \]

so that the desired probability is:

\[ P_{fi} = \frac{1}{\hbar^2} \left| \int_0^t dt' e^{i \omega t - i \omega t'} \lambda \cos(\omega t') \right|^2 \]
We can write the integral as:

\[
\int_0^t dt' e^{iωf_1 t'} \lambda ω \cos(ω t') = \frac{\lambda}{2} \int_0^t dt' \left[ e^{i(ωf_1 + ω)t'} + e^{i(ωf_1 - ω)t'} \right] = \frac{\lambda}{2} \left[ \frac{e^{i(ωf_1 + ω)t} - 1}{i(ωf_1 + ω)} + \frac{e^{i(ωf_1 - ω)t} - 1}{i(ωf_1 - ω)} \right]
\]

from which we get

\[P_{fi} = \frac{λ^2}{4ℏ^2} \left| \frac{e^{i(ωf_1 + ω)t} - 1}{(ωf_1 + ω)} + \frac{e^{i(ωf_1 - ω)t} - 1}{(ωf_1 - ω)} \right|^2 = \frac{λ^2}{ℏ^2} \left[ \frac{\sin^2 \left( \frac{ωf_1 + ω}{2} t \right)}{(ωf_1 + ω)^2} + \frac{\sin^2 \left( \frac{ωf_1 - ω}{2} t \right)}{(ωf_1 - ω)^2} + \frac{\cos ω t (\cos ω t - \cos ωf_1 t)}{(ωf_1 - ω^2)} \right]\]

Problem 2 (20 points)

Two identical non-relativistic fermions of mass \(m\) and spin \(1/2\) are in a 1-dimensional square well of length \(L\) with \(V\) infinitely high outside the well. The fermions are subjected to a repulsive potential \(V(x_1, x_2)\) which may be treated as a perturbation.

1. List the three lowest energy states, their degeneracy and corresponding eigenfunctions, in terms of the individual particles and spin states

2. Calculate, to 1st order in the perturbation, the perturbative correction to the energy of the second and third lowest states. Leave your result in the form of an integral

(Tip: The single-particle energy levels are \(E_n = \frac{π^2 ℏ^2}{2mL^2} n^2, n = 1, 2, 3, \ldots\). Express the corresponding eigenfunctions as \(ψ_n(x)\), no need to write them explicitly).

Solution.

1. The spin-part of the wave-function can be singlet

\[|0, 0⟩ = \frac{1}{\sqrt{2}} |+⟩ |−⟩ - |−⟩ |+⟩ \]

or triplet:

\[|1, 1⟩ = |+⟩ + |⟩

\[|1, 0⟩ = \frac{1}{\sqrt{2}} (|+⟩ |−⟩ - |−⟩ |+⟩) \]

\[|1, −1⟩ = |−⟩ − |⟩

which we can generically indicate as \(|1, M⟩\). The energy of the two-particle system is:

\[E_{n,m} = \frac{π^2 ℏ^2}{2mL^2} (n^2 + m^2).\]

The total wave-function must be anti-symmetric. The ground state is non-degenerate:

\[ψ_0(x_1, x_2) = ψ_1(x_1)ψ_1(x_2)|0, 0⟩\]
and it has energy $E_{1,1} = \frac{\pi^2\hbar^2}{mL^2}$. The first excited state corresponds to $n = 1$, $m = 2$, is 4-fold degenerate:

$$
\psi_1(x_1, x_2) = \begin{cases} 
\frac{1}{\sqrt{2}} \big[ \psi_1(x_1)\psi_2(x_2) - \psi_1(x_2)\psi_2(x_1) \big] |1, M\rangle \\
\frac{1}{\sqrt{2}} \big[ \psi_1(x_1)\psi_2(x_2) + \psi_1(x_2)\psi_2(x_1) \big] |0, 0\rangle
\end{cases}
$$

and it has energy $E_{1,2} = \frac{5\pi^2\hbar^2}{2mL^2}$. The second excited state is non-degenerate:

$$
\psi_2(x_1, x_2) = \psi_2(x_1)\psi_2(x_2)|0, 0\rangle
$$

and it has energy $E_{2,2} = \frac{8\pi^2\hbar^2}{2mL^2}$.

2. The perturbation is spin-independent. The corrections are:

$$
\Delta E_A^1 = \int dx_1 \int dx_2 |\psi_1^A(x_1, x_2)|^2 V(x_1, x_2)
$$

$$
\Delta E_S^1 = \int dx_1 \int dx_2 |\psi_S^1(x_1, x_2)|^2 V(x_1, x_2)
$$

$$
\Delta E_2 = \int dx_1 \int dx_2 |\psi_2(x_1, x_2)|^2 V(x_1, x_2).
$$

**Problem 3 (20 points)**

Consider two particles with spin 1/2 in a harmonic oscillator. They also interact with a spin-spin coupling $V = 2A\vec{S}_1 \cdot \vec{S}_2$. Determine the energy levels of the system in the case in which the particles are different and in the case in which they are equal.

**Solution.**

The Hamiltonian is:

$$
H = H_0 + V
$$

where $H_0$ is the harmonic oscillator Hamiltonian for the two particles and $V = 2A\vec{S}_1 \cdot \vec{S}_2$. Since $H_0$ and $V$ act on different spaces, we can write the eigenvalue equation for $H$ as:

$$
H|\psi\rangle = E|\psi\rangle, \quad |\psi\rangle = |n_1 n_2\rangle |\chi_S\rangle
$$

with $|n_1 n_2\rangle$ eigenstate of $H_0$ with eigenvalue $E_{n_1 n_2} = (n_1 + n_2 + 1)\hbar\omega$ and $|\chi_S\rangle$ eigenstate of $V$ with eigenvalue $E_S$. The total energy will be $E = E_{n_1 n_2} + E_S$.

The eigenstates of $V$ are the singlet state, $|0, 0\rangle$, with eigenvalue $E_{sing} = -\frac{3}{2}Ah^2$ and the triplet states $|1, M\rangle$ with eigenvalue $E_{trip} = \frac{1}{2}Ah^2$.

If the particles are different, we will have for the singlet ground state

$$
E_0 = E_{00} - \frac{3}{2}Ah^2 = \hbar\omega - \frac{3}{2}Ah^2.
$$

and for the triplet

$$
E_0 = E_{00} + \frac{1}{2}Ah^2 = \hbar\omega + \frac{1}{2}Ah^2.
$$
If the particles are identical, for the singlet the ground state is

\[ E_0 = E_{00} - \frac{3}{2} \hbar^2 = \hbar \omega - \frac{3}{2} A \hbar^2. \]

but for the triplet we must have \( n_1 \neq n_2 \):

\[ E_0 = E_{01} + \frac{1}{2} \hbar^2 = 2 \hbar \omega + \frac{1}{2} A \hbar^2. \]

Extra problem for a 10 point-bonus:

**Problem 4 (10 points)**

Consider a system of 3 identical particles with permutation operator \( P_{npq} \). Prove that

- \( P_{312}P_{132} = P_{321} \)
- \( P_{231}^{-1} = P_{312} \)