1 Introduction

The static equation, $\vec{\nabla} \times \vec{E} = 0$ holds only for charge at rest. However, it will be found that $\vec{\nabla} \times \vec{E} = 0$ does not transform properly for charge in motion, so begin by studying the Lorentz force on charges in motion.

2 Law of induction

Consider a conducting rod which lies parallel to the x axis and moves with velocity, $V \hat{y}$. In the rest frame of the rod, the conduction charges are at rest, but in the moving frame, the charges move with the velocity of the rod. Now suppose a constant magnetic field lies in the z direction, as illustrated in Figure 1. There is then a Lorentz force given by:

$$\vec{F} + q[\vec{E} + \vec{V} \times \vec{B}] = qV B \hat{x}$$

Therefore, negative charges (electrons) are forced to move toward an end of the rod. In this example the negative charge moves in the negative x direction. This movement produces a charge distribution which creates an electric field. Charge movement continues until the electric force cancels the magnetic force, and when this happens the Lorentz force due to the electric field cancels that due to the magnetic field. However, the net charge distribution does not vanish. Now consider the motion in the reference frame moving with the rod. The

![Diagram of a conducting rod moving in a magnetic field](image-url)
velocity of the charge in this frame is zero, so there is no force due to the magnetic field. Still a force must still exist in this frame because a charge distribution has developed along the rod. Since a force due to a magnetic field requires the charge to have a velocity, any force in the rest frame of the rod must come from an electric field, but how is such a field created? The only way this can occur is for an electric field to be produced when the magnetic field is transformed from its own rest frame into the moving frame of the rod. Thus the created electric field has the value required to null the Lorentz force equation.

\[ \vec{E}' = -\vec{V}' \times \vec{B}' \]

To summarize, in the prime frame the velocity of the rod is zero, but there is an E field which moves negative charge in the -\( \hat{x} \) direction. In this frame \( \vec{V}' = -\vec{V} \). Thus one finds that the electric and magnetic fields are connected by transformations between moving coordinate frames.

Now suppose the conducting rod is bent into a current loop. This is illustrated in Figure 2, where the force on different elements of the loop are then determined. Given the geometry, the only forces which which matter come from elements 1 and 2, as shown in the figure. The work done in moving the charge around the loop is;

\[ \oint \vec{F} \cdot d\vec{s} = qV[B_1 - B_2]L \]

The integral around the loop is taken by the right hand rule, and the length of the elements 1 and 2 is L. Now divide by the charge, q.

\[ \oint \vec{E} \cdot d\vec{s} = V[B_1 - B_2]L \]
Obviously if \([B_1 - B_2] = 0\) no work is done and no charge moves. However, the right hand side is identified as the electromotive force around the loop, \(EMF\). This is the energy per unit charge which flows in the circuit. Since energy is put into the circuit, it must come from the kinetic energy of motion of the loop. There is a current flow around the loop which has velocity, \(\vec{V}\). Along side 1 for example, there is a force in the -\(\hat{y}\) direction given by \(\vec{V} \times \vec{B}\). This force then acts as a resistance to the initial motion of the loop. For example, if a force moves the loop at constant velocity, there is resistive force due to the current flow which balances the driving force.

The magnetic flux through the loop in figure 3 is:

\[
\phi = \int \vec{B} \cdot d\vec{\sigma} = \int dx \, dy \, B(x, y)
\]

Differentiate this flux by time, and look at figure 3. The loop area does not change, but it moves so that the lighter hatched area is replaced by the darker one. Thus the change in flux is;

\[
\Delta \phi = B_2L \, dy - B_1L \, dy = -[B_1 - B_2]L \, dy
\]

Then use \(\frac{dy}{dt} = V\);

\[
\frac{d\phi}{dt} = -LV[B_1 - B_2]
\]

Comparing this equation to the equation for the EMF above, one finds that;

\[
\oint \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{\sigma}
\]

Then use stokes theorem to convert the integral on the left hand side to an integral over the enclosed area. Since this area is arbitrary, the integrands are equal.
\n\n\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \n\]

The use of the partial derivative here is not so clear, and will be addressed later. The above equation is Faraday’s Law, which can also be written as:

\[ \text{EMF} = -\frac{\partial \phi}{\partial t} \]

3 Examples

3.1 Sliding Rod in a Magnetic Field

A conducting rod of length, \( L \), and mass, \( M \), slides under gravity without friction, down conducting rails which have an angle, \( \theta \), with respect to the horizontal, see the figure. The rails are connected through a resistance, \( R \), and placed in a constant, vertical magnetic field, \( B \). Find the terminal velocity (velocity which remains constant in time) if the rod is released from rest in the Earth’s gravitational field.

![Figure 4: The geometry for problem 2.](image)

Solution

The magnetic flux through a loop of length, \( s \), is;

\[ \phi = BLs \cos(\theta) \]

\[ \text{EMF} = -\frac{d\phi}{dt} = -BL \frac{ds}{dt} \cos(\theta) = -BLV \cos(\theta) \]

In the above, \( V \), is the downward velocity. The downward force on the rod is, \( F_d = Mg \sin(\theta) \). This force introduces a power \( F_dV \) into the motion of the rod. By energy conservation, the power lost in the resistor, \( I^2R \), balances this power input when equilibrium is
reached. Thus the current flow is;

$$I = (BLV \cos(\theta)/R)$$

$$\left(\frac{BLV \cos(\theta)}{R}\right)^2 R = MgV \sin(\theta)$$

$$V = \frac{MgR \sin(\theta)}{L^2B^2 \cos^2(\theta)}$$

### 3.2 Rotating Loop in a Static Magnetic Field

Suppose a loop is rotating in a static magnetic field as illustrated in Figure 5. The flux through the loop is;

$$\phi = \int \vec{B} \cdot d\vec{\sigma} = B \cos(\theta) \int d\sigma = B(WL) \cos(\theta)$$

By Faraday’s Law;

$$EMF = -\frac{d\phi}{dt} = B(WL)\omega \sin(\theta)$$

Here $\omega = \frac{d\theta}{dt}$. The EMF and thus the current, has a harmonic form as expected (*ie an alternating current is produced*).
4 Homopolar Generator

On the other hand, a homopolar generator generates a direct current. A type of homopolar generator was devised by Faraday (Faraday Disk) and is shown in Figure 6. A conducting disk is rotated between the poles of a magnetic field, and an EMF is generated between the radial edge of the disk and its axis. A constant rotational speed generates a constant EMF. This is most easily seen by using the Lorentz force acting on the rotating charges in the disk.

\[ \vec{E} = \vec{V} \times \vec{B} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ 0 & r\omega & 0 \\ 0 & 0 & B \end{vmatrix} \]

\[ \vec{E} = r\omega B \hat{r} \]

The EMF generated between the axis and the rim of the disk is;

\[ EMF = \int_0^a \, dr \, r \omega B = \omega B a^2 / 2 \]

This assumes that the field is constant over the entire disk. The potential difference is low but the current flow can be large. However, the efficiency of the Faraday Disk is low as the current is not completely radial due to eddy currents as shown in Figure 6. A more efficient homopolar generator uses a uniform, static magnetic field through the entire disk surface. Modern designs can create DC currents up to mega-amps.
5 MHD Generator

A magnetodynamic (MHD) generator has no moving, mechanical components. Gas at high temperature produces a plasma (a gas of ions and electrons), and this plasma is then allowed to expand at high velocity through a magnetic field separating the ions and electrons, creating an EMF between the conducting plates. These generators are highly efficient. However, they produce direct currents and have other practical problems, Figure 7.

6 Energy Conservation

As discussed previously, the current induced by Faraday’s Law in a moving rod, Figure 8, interacts with the magnetic field causing a force which acts as a resistance to the motion of the rod. The EMF due to the motion is;

\[ \text{EMF} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{\sigma} = BLV. \]

The current which flows in the circuit expends power in the resistor equal to \( I (\text{EMF}) = IBLV \). The current flowing in the circuit is \( I = (\text{EMF})/R \) from which the force \( F \) on the wire due to the current flow is;

\[ F = ILB \]

The power input required to keep the velocity constant (or decelerate the rod) is;

\[ P = FV = ILBV \]

This shows conservation of energy.
7 Lenz’s law

The total system energy is conserved if we take into account the energy loss by dissipation. To conserve energy when an EMF is created, a current flow is induced which tends to keep the flux through a current loop constant. This induced current creates a force which resists the motion of the loop (if the loop moves), and a magnetic field opposite to the field which initially produced the flux. This allows energy to be removed from the mechanical motion and/or reduces the energy in the field. This energy goes into the movement of charge (EMF) in the loop.

8 Conductivity

As voltage is increased between conductors embedded in a medium, the flow of charge between the conductors increases in linear proportion to the voltage. This is Ohm’s law, which is usually expressed as;

\[ \vec{J} = \sigma \vec{E} \]

In the above equation, \( \vec{J} \) is the current density, \( \vec{E} \) is the electric field in the medium, and \( \sigma \) is the conductivity of the medium. The resistivity of the medium is the inverse of the conductivity, \( \rho = 1/\sigma \). Conductivity is measured in Siemens (one Siemen equals one Ampere produced by a potential difference of one volt). The Ohm is the unit of resistance. Table 1 gives some resistivities and indicates the sensitivity of the resistivity to temperature, \( \alpha \) where

\[ \sigma_{T'} = \frac{\sigma_T}{1 + \alpha(T - T')} \]

Generally as the temperature decreases the resistivity also decreases. Although, with the exception of superconductors, resistance remains non-zero even at 0 K. However, superconductors in addition to having zero resistance at some finite temperature, also expell magnetic fields from their interior, the Meisner effect.
Table 1: Resistivities and temperature coefficients for some metals

<table>
<thead>
<tr>
<th>Metal</th>
<th>Resistivity $\Omega \cdot m$ 20° C</th>
<th>Temperature Coefficient K$^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Silver</td>
<td>$1.59 \times 10^{-8}$</td>
<td>0.0038</td>
</tr>
<tr>
<td>Copper</td>
<td>$1.72 \times 10^{-8}$</td>
<td>0.0039</td>
</tr>
<tr>
<td>Gold</td>
<td>$2.44 \times 10^{-8}$</td>
<td>0.0034</td>
</tr>
<tr>
<td>Aluminum</td>
<td>$2.82 \times 10^{-8}$</td>
<td>0.0039</td>
</tr>
<tr>
<td>Tungsten</td>
<td>$5.60 \times 10^{-8}$</td>
<td>0.0045</td>
</tr>
<tr>
<td>Zinc</td>
<td>$5.90 \times 10^{-8}$</td>
<td>0.0037</td>
</tr>
<tr>
<td>Nickel</td>
<td>$6.99 \times 10^{-8}$</td>
<td>0.0059</td>
</tr>
</tbody>
</table>

Figure 9: The connection between resistance and resistivity

The resistance of a filament along an electric field line is illustrated in Figure 9. The cross sectional area of the filament is $A$, and its length is $L$, as shown in the figure. The differential resistance, $dR$, in terms of the resistivity, $\rho$, is then:

$$dR = \rho \frac{dL}{A}$$

All metals are reasonably good conductors because electrons in the outer atomic shells are loosely bound and can easily travel in the potential wells of a metallic crystal. This is illustrated in Figure 10. The temperature dependence of the resistivity is given by the Bloch-Grüneisen equation.

$$\rho(T) = \rho(0) + A(T_\text{R})^n \int_0^{\Theta_\text{R}/T} dx \frac{x^n}{(e^x - 1)(1 - e^{-x})}$$

In the above, $\Theta_\text{R}$ is the Debye temperature and $n$ is an integer which depends on the electron interactions in the material (i.e. electron scattering by phonons, electron scattering from atomic electrons, and/or electron-electron Coulomb interaction). The resistivity can be represented as a complex number for an alternating current. The real component is connected to an Ohmic resistance and the imaginary component connected to reactance (energy placed in temporary storage in the electric and magnetic fields but can be recovered in the second
Currents flow along the field lines. Therefore in a 3-dimensional object the field and equipotential lines are needed to obtain the resistance. As previously, this requires the solution of Laplace’s equation, $\nabla^2 V = 0$, subject to boundary conditions in order to obtain the potential function, $V$. In the electrostatic case the value of the potential, $V_s$, or its normal derivative at the conducting surfaces is required. The derivative normal to the surface gives the charge on the surface of a conductor. To obtain the resistance, consider the current flow. Specification of a constant potential in a conductor, and specification of the current is equivalent to specification of the charge. This is because the value of $E$ at a conducting surface is equal to the normal derivative of the potential. In electrostatics:

$$Q = \varepsilon \int da \frac{\partial V}{\partial n}$$

In the case of currents, use the conductivity $\sigma = J/E$, where $J$ is the current density and $E$ is the electric field.

$$I = \int da (\sigma \frac{\partial V}{\partial n})$$

There is a connection between the capacitance of a set of conductors and the resistance between them. Recall that the capacitance is $C = Q/V$. If the capacitance is known and $\sigma$ is constant, the resistance is obtained by multiplying the inverse of the capacitance by $\sigma/\varepsilon = 1/\rho\varepsilon$. Thus;

$$R = \frac{V}{I} = \frac{Q}{I_C} = \frac{\rho\varepsilon}{C} = \frac{\rho\varepsilon V}{Q}$$
9 Examples

9.1 The Resistance Between Concentric Spheres

A resistive medium fills the space between 2 concentric spheres as illustrated in Figure 11. The potential between the spheres is obtained in spherical coordinates;

\[ V = \kappa \frac{Q}{r} \]

\[ \vec{E} = \kappa \frac{Q}{r^2} \hat{r} \]

The potential difference is then;

\[ V_a - V_b = \kappa Q \left[ \frac{1}{r_a} - \frac{1}{r_b} \right] \]

The capacitance is;

\[ C = \frac{Q}{V} = \frac{4\pi \varepsilon r_a r_b}{r_a - r_b} \]

The resistance between the spheres is;

\[ R = \frac{\rho \varepsilon}{C} = \frac{\rho}{4\pi} \left[ \frac{1}{r_a} - \frac{1}{r_b} \right] \]

This could have been obtained from the resistivity using the equation for the resistance of a filament as in the above section, \( dR = \rho dL/A \). Here \( A \) is the surface area of the conductor perpendicular to \( \vec{E} \) at the surface.

\[ dR = \frac{\rho dr}{4\pi r^2} \]
Integration over $r$ between $a$ and $b$ gives the result obtained above.

### 9.2 Resistivity in a Cylindrical System

A voltage, $V_0$, is placed between the conductors of a long coaxial cable. The space between the conductors is filled with a material of dielectric constant, $\epsilon$, and resistivity, $\rho$. What current per unit length flows through the material between the conductors in length, $L$, of the cable?

![Figure 12: The geometry for problem 1.](image)

The surfaces of constant potential are cylinders, concentric with the inner conductor. The current follows the electric field lines in the radial direction. Thus for the resistance of a length of cable, $L$;

$$dR = \frac{\rho}{2\pi r L} dr$$

Integrate;

$$R = \int_a^b \frac{\rho}{2\pi L} \frac{dr}{r} = \frac{\rho}{2\pi L} \ln(b/a)$$

$$I = \frac{V}{R} = \frac{2\pi LV_0}{\rho \ln(b/a)}$$