Energy, Momentum, and Symmetries - Lecture 9

1 Fields

The interaction of charges was described through the mathematics of fields. A field connects an interaction to a geometry in space-time. In the case of EM, a charge interacts with surrounding space such that it affects other charges at other points in space-time. While the abstraction of an interaction to include this intermediate step seems irrelevant with respect to Newtonian physics, it is the way relativity (*ie* the constancy of the velocity of light) can be satisfied. The Newtonian concept of action-at-a-distance (meaning instantaneous interaction between distant objects) is not consistent with the postulates of relativity.

Most think in terms of "force", and in particular, force acting instantaneously and directly between objects. You should now think in terms of fields, *i.e.* a modification to space-time geometry which affects a particle positioned at a geometric point in space-time. In fact, it is not the force fields, but the potentials which are fundamental to the dynamics of interactions.

2 Complex notation

Note that a description of the EM interaction involves the time dependent Maxwell equations. Time dependence can be removed by solving the equations in frequency rather that time space. Essentially this employs a Fourier transform of the equations. Remember in an earlier lecture, the Fourier transformation was introduced as follows.

$$\mathcal{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, F(t)\, e^{-i\omega t}$$

$$F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, \mathcal{F}(\omega)\, e^{i\omega t}$$

The assumption is that all related functions have the time dependence;

$$F(\vec{x}, t) \rightarrow F(\vec{x}) e^{i\omega t}$$

Solutions to Maxwell's equations are then valid for a particular frequency, ω. To obtain the time dependence, one can superimpose solutions of different frequencies using a weighted, inverse Fourier transform. However, note that this assumption results in the introduction of complex functions to describe measurable quantities. Obviously any functions of a physical quantity must be real, so this will require some further investigation. Thus as an example, look at Faraday's law.

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -i\omega \vec{B}(\vec{x})$$

In the above, the fields are complex since the time dependence $e^{i\omega t}$ was used instead of a real harmonic form such as $cos(\omega t)$. To get a real result, one could choose to take the real part of the function, *ie* ;

$$Re[Ee^{i\omega t}] = E\, cos(\omega t)$$

However, for most calculations of energy and power, only the time average of the fields is needed. Recall that frequencies in most cases of interest are large, and thus the instantaneous power is not as important as its time averaged value. Energy and power are obtained by squaring the fields or multiplying the field by a current. Therefore when multiplying the real component of two complex amplitudes one obtains;

$$Re[Ae^{i\omega t}] = 1/2[Ae^{-i\omega t} + A^* e^{i\omega t}]$$

$$Re[Be^{i\omega t}] = 1/2[Be^{-i\omega t} + B^* e^{i\omega t}]$$

$$\begin{aligned} &Re[A]Re[B] = \\ &\quad 1/4[ABe^{-i2\omega t} + A^*B + AB^* + A^*B^*e^{i2\omega t}] = \\ &\quad 1/2\, Re[A^*B + ABe^{i2\omega t}] \end{aligned}$$

In the above, A^* is the complex conjugate of A. Applying a time average, the complex harmonic term, $e^{i2\omega t}$, vanishes. Thus;

$$\langle Re[A]Re[B] \rangle = (1/2) A^*(\vec{x}) B(\vec{x})$$

Note A and B now depend only on spatial coordinates and a frequency (*ie* they are time independent). The mathematical convenience of using the complex exponential should be obvious.

3 Poynting vector

Recall that power input or loss in a mechanical system is found from $P = \vec{F} \cdot \vec{V}$ where $\vec{F}$ is the force and $\vec{V}$ the velocity. Use the current density, $\vec{J} = \rho \vec{V}$ with ρ the charge density, to write the power expended in moving a charge by an electric field, $\vec{E}$.

$$P = \int d\tau\, (\rho \vec{E}) \cdot \vec{V}) = \int d\tau\, \vec{E} \cdot \vec{J}$$

Insert from Ampere's Law the Maxwell equation;

$$\vec{J} = \vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t}$$

$$P = \int d\tau \, [\vec{E} \cdot (\vec{\nabla} \times \vec{H} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t})]$$

Then use;

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{H})$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

To obtain;

$$P = -\oint (\vec{E} \times \vec{H}) \cdot d\vec{a} - \int d\tau \, [\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}]$$

The energy density is;

$$\mathcal{W} = (1/2)(\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H})$$

so that;

$$-P = \int d\tau \, (\frac{\partial \mathcal{W}}{\partial t}) + \oint (\vec{E} \times \vec{H}) \cdot d\vec{a}$$

The above equation is interpreted as showing that the power expended by a moving charge, $P = \int d\tau \, (\vec{E} \cdot \vec{J})$, is obtained by decreasing the energy in a volume of the field after accounting for the outward flow of energy through a surface surrounding this volume. Then define the Poynting vector by;

$$\vec{S} = \vec{E} \times \vec{H}$$

This vector represents the energy flow through the surface of a volume. For a simple example, consider Figure 1 which represents the current flowing through a cylindrical resistor, R, of length L. The current, I, produces a magnetic field at the surface of the resistor with radius a. The magnetic field is obtained from Ampere's law, and points around the cylindrical surface in a direction given by the right hand rule.

$$\oint \vec{B} \cdot \vec{dl} = B(2\pi a) = \mu I$$

The $\vec{E}$ field driving the current acts in the axial direction and has a value of V/L where V is the potential drop over the distance, L. The power loss as obtained from the circuit

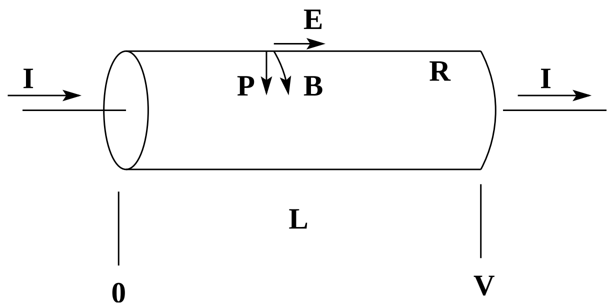

Figure 1: Evaluation geometry for the Poynting vector at the surface of a resistor

equations is VI. Now look at the Poynting vector;

$$\vec{P} = (1/\mu)\vec{E} \times \vec{B} = -(1/\mu)(V/L)(\frac{\mu I}{2\pi a})\,\hat{r} = -\frac{VI}{2\pi a L}\,\hat{r}$$

The Poynting vector P gives the power flowing into the resistor per unit surface area from the fields. Thus this analysis connects the functions of the circuit components to operations using fields.

In a further analysis, recall that a vanishing divergence equation can be related to a conservation principle. Define a 4-vector $(S^{\alpha} = \mathcal{W}, \vec{S})$, to form a relativistic Poynting 4-vector. If $\vec{J} \cdot \vec{E} = 0$, a relativistic scalar is obtained indicating that this result is independent of any coordinate transformation between inertial frames.

$$\frac{\partial S^{\alpha}}{\partial x^{\alpha}} = 0$$

Thus the divergence of the 4-vector, S^{α}, demonstrates conservation of energy.

4 Momentum flow

In a similar way to power flow conservation, use Maxwell's equations and the Lorentz force on an element of charge, dq,to develop the time change of momentum. The differential Lorentz force is written;

$$d\vec{F} = d\,q[\vec{E} + (\vec{v} \times \vec{B})]$$

Upon writing this for a volume charge density;

$$\frac{d\vec{P}_M}{dt} = \int [\rho\vec{E} + \vec{J} \times \vec{B}]\, d^3x$$

In the above expression, $\vec{P}_M$ is the mechanical momentum due to the moving charge. Substitute

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon; \quad \vec{\nabla} \times \vec{H} = \vec{J} + (\mu/c^2)\,\frac{\partial \vec{E}}{\partial t}$$

and

$$\vec{\nabla} \times \vec{F} \times \vec{G} = (\vec{G} \cdot \vec{\nabla})\vec{F} - (\vec{F} \cdot \vec{\nabla})\vec{B} + \vec{F})\vec{\nabla}\vec{G} - \vec{G} \cdot \vec{F}$$

Use the remaining Maxwell equations to write the equation;

$$\begin{aligned}\frac{d\vec{P}_M}{dt} + \frac{d}{dt}\int [\epsilon\,\vec{E} \times \vec{B}]\,d^3x = \\ \epsilon \int [\vec{E}(\vec{\nabla} \cdot \vec{E}) - \vec{E} \times (\vec{\nabla} \times \vec{E}) + \\ c^2\vec{B}(\vec{\nabla} \cdot \vec{B}) - c^2(\vec{B} \times (\vec{\nabla} \times \vec{B}))]\,d^3x\end{aligned}$$

Now note that $\vec{P}_{field}$ is identified by the momentum per unit volume in the field;

$$\vec{P}_{field} = (1/c^2)\int (\vec{E} \times \vec{H})\,d^3x$$

Described in words, the above equation expresses that the momentum per unit volume equals the energy flowing through the surface area divided by its velocity, c, and the volume in which contains the momentum density, $(ct)Area$, (Energy/c)/[$(ct)Area$]. The energy flow per unit time (power) through an area is given by the Poynting vector, $\vec{S} = \vec{E} \times \vec{H}$. Note that this power flows with a velocity, c, and the Energy/c equals the momentum of the electromagnetic wave. Divide the remaining term into its vector components. In the 1 direction;

$$[\vec{E}(\vec{\nabla} \cdot \vec{E}) - \vec{E} \times (\vec{\nabla} \times \vec{E})]_1 = \sum_i \frac{\partial}{\partial x^i}\,[E_1 E_i - (1/2)\vec{E} \cdot \vec{E}\delta_{i1}]$$

The right side of the above equation is the divergence of rank 2 tensors. Collecting all components for the $\vec{E}$ as well as the $\vec{B}$ field, the resulting tensor has the form;

$$\mathcal{T}_{\alpha\beta} = \epsilon[E_\alpha E_\beta + c^2 B_\alpha B_\beta - (1/2)(\vec{E} \cdot \vec{E} + c^2\vec{B} \cdot \vec{B})\delta_{\alpha\beta}]$$

This tensor, $\mathcal{T}$, is called the Maxwell stress tensor. In integral form, the above equation is;

$$\frac{d}{dt}[\vec{P}_M + \vec{P}_{Field}]_\alpha = \oint \mathcal{T}_{\alpha\beta}\,da_\beta$$

The integral over $d\,a_\beta$ is over the area surrounding the volume containing the field momentum. Now identify $\mathcal{T}_{\alpha\alpha}$ as the pressure $(force/area)$ on the surface α defined by the outward normal, $\vec{\alpha}$. The integral has units of energy density (Force/area). The tensor element $\mathcal{T}_{\alpha\beta}$

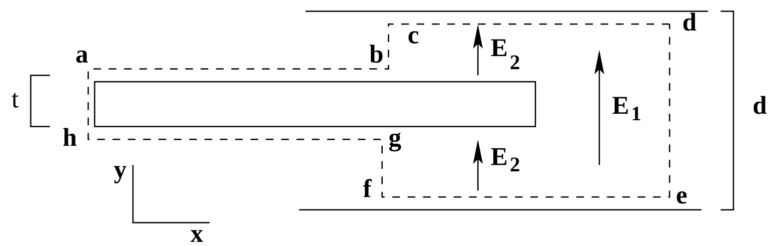

Figure 2: The surfaces on which to evaluate the Maxwell stress tensor.

when $\alpha \neq \beta$, is the momentum density times the velocity, or a shearing force on the area element α. In differential form the above equation is;

$$\vec{\nabla} \cdot \mathcal{T} \,+\, (\rho \vec{E} + \vec{J} \times \vec{B}) \,=\, -\frac{\partial \vec{G}}{\partial t}$$

In this equation, $\vec{G} \,=\, (1/c^2)\vec{E} \times \vec{H}$ is the momentum density. The equation represents momentum conservation. In summary, the momentum density flowing through a surface plus the mechanical momentum density of the charge currents equals the time change of the momentum density in the enclosed volume contained within the surface.

5 Examples

Look at the z component of the force applied by the fields in Cartesian coordinates.

$$\mathcal{T}_{zz} \,=\, (\epsilon/2)[E_z^2 - E_y^2 - E_x^2] \,+\, (\epsilon c^2/2)[B_z^2 - B_y^2 - B_x^2]$$

$$\mathcal{T} zy \,=\, \epsilon(E_z E_y \,+\, c^2\, B_z B_y)$$

$$\mathcal{T} zx \,=\, \epsilon(E_z E_x \,+\, c^2\, B_z B_x)$$

5.1 Example 1

As an example, calculate the force on a conducting plate inserted in a capacitor. Simple analysis shows that the field of the capacitor induces charge on the conducting plate, and this causes a force on the plate pulling it into the capacitor. Consider Figure 2. Ignore dimensions out of the plane of the drawing and assume that the fields within the capacitor are uniform and perpendicular to the surfaces of the plates, *ie* the field of an infintite parallel plate capacitor.

The dashed line indicates a surface (include dimensions in and out of the drawing) over which the stress tensor is evaluated. There is only an E field to consider. Integration over ha is essentially zero as there is little field at this point. Integration over ab and gh cancel as does integration over cd and ef. Then integration over bc and fg are equal and only the field components E_y are non-zero.

$$E_1 = V/d$$

$$E_2 = V/(d-t)$$

Let the width of the conducting plate be w. Use the outward normal as the direction of the area vectors. The field tensor is then;

$$\mathcal{T} = \epsilon \begin{pmatrix} -(1/2)E_y^2 & 0 & 0 \\ 0 & (1/2)E_y^2 & 0 \\ 0 & 0 & -(1/2)E_y^2 \end{pmatrix}$$

The force on the plate is;

$$F_x = \int \mathcal{T}_{xx} dA_x$$

$$F_x = (\epsilon/2)\frac{V^2}{(d-t)^2}\, w(d-t) \; - \; (\epsilon/2)\frac{V^2}{d^2}\, wd$$

$$F_x = \frac{\epsilon V^2 wt}{2d(d-t)}$$

In this simple case the same result can be obtained from the field energy. To make this comparison, suppose the plate extends into the capacitor a distance x. Use the energy density in the fields to obtain;

$$W = (1/2)[\epsilon E^2 \,+\, (1/\mu)B^2] \times Volume$$

$$W = (1/2)\epsilon[\frac{V^2}{d^2}\, d(L-x)w \,+\, \frac{V^2}{(d-t)^2}\,(d-t)xw]$$

$$F_x = -\frac{dW}{dx} \,=\, (1/2)\epsilon[-\frac{V^2}{d}\, w \,+\, \frac{V^2}{(d-t)}\, w]$$

Which results in the same answer.

5.2 Example 2

Now consider another, more complicated example. Find the radiation pressure on a spherical, metalic particle of radius, a. The particle is assumed to be perfectly conducting. First develop an expression for the radiation pressure using an approximation. The Poynting vector divided by c^2 is the momentum density in the EM fields ($Momentum - c/(area - time - c^2)$. The force is the time change of the momentum density, $\vec{G}$ integrted over volume, so (assume a plane EM wave in the z direction);

$$\frac{dP_z}{dt} = F_z = 2c \int \vec{G}_z dA_z$$

In the above dA_z is the surface area of the spherical particle struck by the wave. The factor of 2 comes from the change of momentum between the incoming and outgoing wave. The integral extends over the lower hemisphere. Then obtain the time average. For a plane wave in MKS units $B_0 = E_0/c$.

$$F_z = 2/c \int r^2 \, d\,cos(\theta) \, d\phi \, cos(\theta) \, ((1/2)Re[\vec{E} \times \vec{H}^*]_z)$$

For the plane wave, $H_0 = B_0/\mu = E_0/c\mu = \epsilon \, c \, E_0$. Substitution and integration yields;

$$F_z = \pi a^2 \epsilon E_0^2$$

Now to precisely work this problem, one must use the stress tensor. This means the fields surrounding the spherical particle must be obtained. Therefore the solution involves the scattering of a plane from a conducting sphere. First assume that these fields are known, *i.e.* both the incident and the scattered wave at the spherical surface of the particle are known. The fields should be expressed in spherical coordinates, and the force on the particle is obtained from;

$$F_z = (1/2)[\int da_Z \, |\mathcal{T}_{zz}| + \int dA_x \, |\mathcal{T}_{zx}| + \int dA_y \, |\mathcal{T}_{zy}|]$$

The factor of (1/2) comes from the time average of the harmonic wave fields. The time-averaged, stress-tensor components $|\mathcal{T}_{\alpha\beta}|$ are used. Put theses in spherical coordinates and average over the azimuthal angle. Note that the fields must be the total field (*i.e* incident plus scattered field) at the surface $d\vec{\sigma}$. To obtain these field components, consider Figure 2. The direction cosines of the axes in spherical coordinates are;

$$l = sin(\theta) \, cos(\phi)$$

$$m = sin(\theta) \, sin(\phi)$$

$$n = cos(\theta)$$

and $l \, d\sigma = d\sigma_x$, etc. Then;

$$E_x = E_r \, sin(\theta) \, cos(\phi) + E_\theta \, cos(\theta) \, cos(\phi) - E_\phi \, sin(\phi)$$

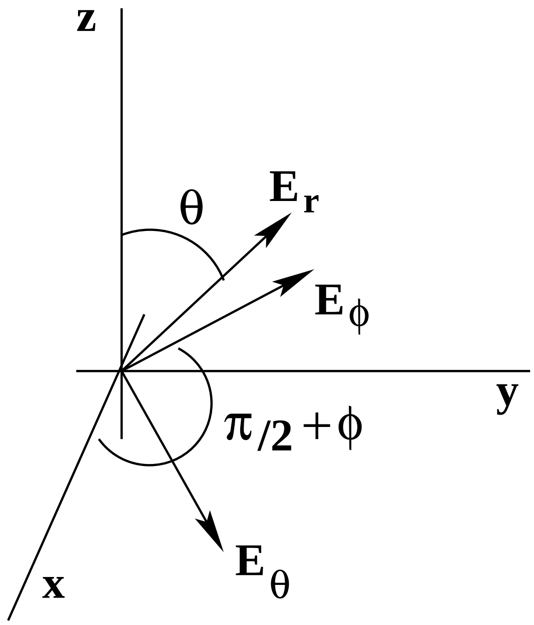

Figure 3: The surfaces on which to evaluate the Maxwell stress tensor.

$$E_y \;=\; E_r\, sin(\theta)\, sin(\phi) \;+\; E_\theta\, cos(\theta)\, sin(\phi) \;+\; E_\phi\, cos(\phi)$$

$$E_z \;=\; E_r\, cos(\theta) \;-\; E_\theta\, sin(\theta)$$

There is substantial algebra in order to collect terms when substituting the fields into the stress tensor, and averaging over the azimuthal angle. Here one only needs $\mathcal{T}_{zz}$ to continue the development using the approximations previously applied. Thus, let $ct = cos(\theta)$ and $st = sin(\theta)$

$$\mathcal{T}_{zz} \;=\; (\epsilon/2)[E_r^2(ct^2 - st^2) + E_\theta^2(st^2 - ct^2) - E_\phi^2 - 4E_r E_\theta\, ct\, st]$$

Ignore the scattered wave which is assumed small, and choose a plane wave incident along the z axis. For this case $\mathcal{T}_{zy} = \mathcal{T}_{zx} = 0$.

$$\mathcal{T}_{zz} \;=\; (\epsilon/2)E_x^2 \;+\; (1/2\mu)B_y^2$$

Integration proceeds easily to obtain in approximation the previous result.

5.3 Example 3

As another example, find the force on the Northern hemisphere exerted by the Southern hemisphere of a uniformly charged sphere with density $\rho = \frac{Q}{4\pi R^3/3}$. Here Q is the total charge and R is the radius. Use the stress tensor, $Tij = \epsilon(E_i E_j - 1/2\,\delta_{ij} E^2)$ where the B field component of the tensor is set to zero. Since spherical coordinates match the geometry, the field is written as;

$$\vec{E}(r) = E_x \hat{x} + E_y \hat{y} + E_z \hat{z} = \kappa(Q/R^2)\hat{r}$$

$$\vec{E}(r) = \kappa(Q/R^2)[sin(\theta)\,cos(\phi)\,\hat{x} + sin(\theta)\,sin(\phi)\,\hat{y} + cos(\theta)\,\hat{z}]$$

The net force must be in the $\hat{z}$ direction and is divided into 2 parts, 1) the hemispherical cap and 2) the disk at the equator. For these forces, one obtains;

For part 1

$$(T \cdot d\vec{a})_z|_{r=R} = T_{zx} da_x + T_{zy} da_y + T_{zz} da_z$$

Then $d\vec{a} = R^2\,sin(\theta)\,d\theta\,d\phi\,\hat{r}$

$$da_x = (R^2\,sin(\theta)\,d\theta\,d\phi)(sin(\theta)\,cos(\phi)\,\hat{x}$$

$$da_y = (R^2\,sin(\theta)\,d\theta\,d\phi)(sin(\theta)\,sin(\phi)\,\hat{y}$$

$$da_x = (R^2\,sin(\theta)\,d\theta\,d\phi)(cos(\theta)\,\hat{z}$$

Substitute these into the contraction of the stress tensor with the area vector.

$$(T \cdot d\vec{a})_z|_{r=R} = \epsilon\,(\frac{Q}{4\pi\epsilon R})^2[sin^3(\theta)\,cos(\theta)cos^2(\phi)\,d\theta\,d\phi +$$

$$sin^3(\theta)\,cos(\theta)sin^2(\phi)\,d\theta\,d\phi +$$

$$(1/2)\,(cos^2(\theta) - sin^2(\theta))\,sin(\theta)\,cos(\theta)\,d\theta\,d\phi]$$

Collect terms and integrate over the top hemisphere. The result is;

$$F1_z = \kappa\frac{Q^2}{8R^2}$$

For part 2

Use the outward normal for the outward disk. This $d\vec{a} = -da_z\,\hat{z}$. The field on this surface is;

$$\vec{E}_{disk} = -\kappa(Q/R^3)\vec{r}|_{\theta=\pi/2}$$

$$\vec{E}_{disk} = -\kappa(Q/R^3)r[cos(\phi)\hat{x} + sin(\phi)\hat{y}]$$

The force evaluated by the integral of $T_{zz}da_x$ over the disk with is;

$$T_{zz} = -1/2[\epsilon(E_z^2 - E_y^2 - E_x^2)]$$

$$F2_z = -\kappa\frac{Q^2}{16R^2}$$

The total force is the difference of the two components.

5.4 Example 4

Previously an expression for the magnetic field in the interior of a long solenoid using Ampere's law was developed. If there are N turns of wire per unit length each carrying a current, I, the magnetic field inside the solenoid is constant, directed along the solenoidal axis, and equal to $\vec{B} = \mu_0 NI\,\hat{z}$. The stress tensor is diagonal and has the form;

$$\mathcal{T}_{zz} = \frac{1}{2\mu_0}\,B_z^2$$

$$\mathcal{T}_{zy} = -\frac{1}{2\mu_0}\,B_z^2$$

$$\mathcal{T}_{zx} = -\frac{1}{2\mu_0}\,B_z^2$$

There is an outward, radial pressure on the windings of the solenoid. To find the outward force on the upper cylindrical surface with length, L, evaluate the equation,

$$F_y = \int \mathcal{T}_{yy} d\,A_y$$

Then $d\,\vec{A}_y = R\,d\phi\,dz\,\hat{r}\cdot\vec{y} = R\,d\phi\,dz\,sin(\phi)$

$$F_y = R\,L\int_0^\pi d\phi\,\mathcal{T}_{yy}\,sin(\phi) = \frac{2RLB_z^2}{2\mu_0} = \mu_0 RL(NI)^2$$

This problem can also be worked using the Lorentz force $\vec{F} = q\vec{V}\times\vec{B}$, see figure 4. Use $qV = NI\,dl$ and for a length L of the solenoid, the force on the upper cylindrical surface is;

$$F_y = NI\,RL\,(B^2/2)\int_0^\pi sin(\phi)\,d\phi = \mu_0 RL(NI)^2$$

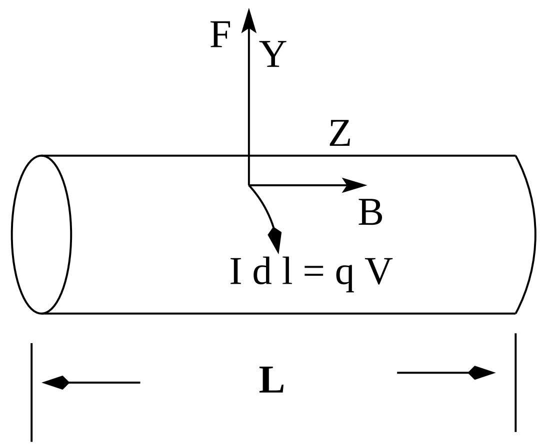

Figure 4: A figure used to find the outward pressure on the windings of a long solenoid

The factor of 1/2 in the expression for the force can be obtained either by recognizing that 1/2 of the current causes a B field which interacts with the other half of the current, or that the B force decreases from the interior value to zero across the solenoid windings. The resulting answer is then the same when calculated by the stress tensor.

Such forces can be substantial in a high field high current superconducting solenoid and cause the windings to move. Coil movement can quench the superconductor. If that occurs, the high current in the coil must pass through the a resistive component in the coil heating the coil and raising additional components of the coil above the critical temperature. This can result in a run-away condition which converts the energy stored in the magnetic field of the magnet into heat which can destroy the magnet.

6 Momentum Conservation in Current Flows

Consider a long coaxial cable of length, L, with inner radius a and outer radius b. There is a charge per unit length of λ on the conductors and they carry a current I in opposite directions. The fields between the conductors are;

$$\vec{E} = \frac{1}{2\pi\epsilon} \frac{\lambda}{\rho} \hat{\rho}; \;\; \vec{B} = \frac{\mu}{2\pi} \frac{I}{\rho} \hat{\phi}$$

The power flow down the cable is;

$$\text{Power} = \int \vec{S} \cdot d\vec{A} = (1/\mu) \int (\vec{E} \times \vec{B}) \cdot \hat{z}\, dA_z$$

$$\text{Power} = (1/\mu) \frac{\lambda\mu I}{4\pi^2\epsilon} \int d\phi \int \frac{\rho}{\rho^2} d\rho =$$

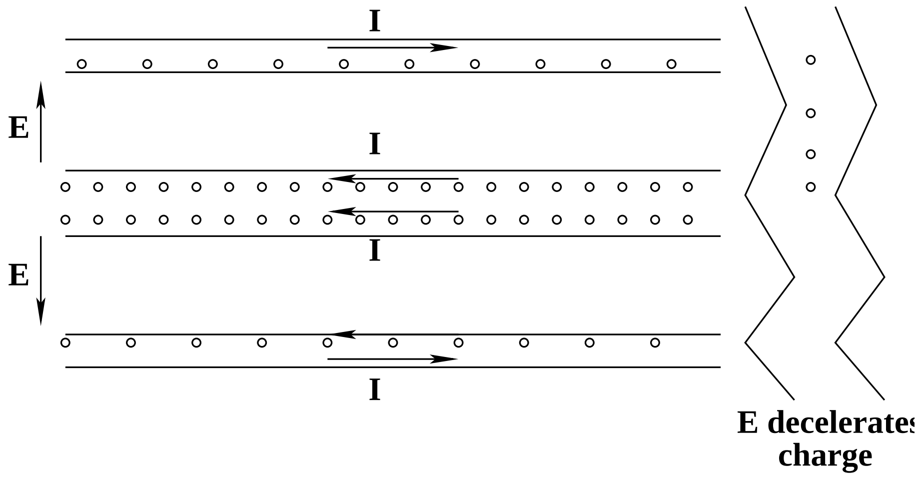

Figure 5: The movement of charges in the power flow in the coaxial cable to show the hidden momentum.

$$I[\frac{\lambda}{2\pi\epsilon} ln(b/a)] = IV$$

The momentum in the field is ;

$$\vec{P}_{field} = (1/c^2) \int \vec{S} \, d^3x = \frac{\mu\lambda I L}{2\pi} ln(a/b) \, \hat{z}$$

Nothing appears to move, at least the CM of the system is constant, so what does the field momentum represent? Look at the current loop in Figure 5. Charges on the left are accelerated toward the top and charges on the right decelerated toward the bottom. The current $I = \lambda U$ is the same in all the segments, but the number of charges, N_t, at the top move faster than number of charges, N_b, at the bottom. Thus since the current top and bottom are the same there must be less charge at the top.

$$I = \frac{QN_t}{L} U_t = \frac{QN_b}{L} U_b$$

Classically the momentum of each charge carrier is mU for U the velocity. Therefore;

$$P_{classical} = mN_tU_t - mN_bU_b$$

Substitute for the number of charges;

$$P_{classical} = \frac{mIL}{Q} - \frac{mIL}{Q} = 0$$

However relativistically the momentum is $m\gamma U$. Substitute for NU as above to obtain;

$$P_{rel} = \gamma_t m N_t U_t - \gamma_b m N_b U_b = \frac{mIL}{Q}[\gamma_t - \gamma_b]$$

Note there is work done to move the charge. The field momentum provides this work, and the momentum is canceled by the resistance to the charge flow.

7 Symmetry

A symmetry is "observed" when a description of a physical system remains unchanged when the coordinates used to describe the system transformed into the coordinates of another inertial system. In general, such changes in representation can be either continuous or discrete. An example of a continuous change is a rotation about some axis since a rotation can take any angle value. A discrete change is represented by a reflection of coordinates, for example a change of x to -x. Symmetries are important because the mathematical representation of a physical system must express the physical symmetries of an interaction. For example, recall that the homogeneous symmetry of spacetime allowed the formulation of the Lorentz transformations.

All symmetries lead to invariance (conservation) of some quantity. Thus invariance under time translation yields conservation of energy, while invariance under spatial translation yields conservation of momentum, and invariance under rotations yields conservation of angular momentum. Inversely, corresponding to each conserved quantity there is a symmetry. Symmetries however, are not always related to geometry.

7.1 Time Reversal

Time reversal is an example of a discrete symmetry, $t \rightarrow -t$. This is occurs in the mathematics describing a physical system through the square of the time parameter, or in conjunction with other parameters which change sign under time reversal, T. Thus consider Ampere's law;

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{B} = \mu_o I \int \frac{d\vec{l} \times \vec{r}}{r^3}$$

When $t \rightarrow -t$ both t and $\vec{B}$ change sign since $I = \frac{dq}{dt}$. Macroscopically, T is not a good symmetry. However, when quantum mechanics is applied to a microscopic system;

$$
\begin{aligned}
T^\dagger \mathrm{H} T &= \mathrm{H}; \\
\mathrm{H}\psi &= i\hbar \frac{\partial}{\partial t}\psi; \\
T\mathrm{H}\psi &= [T\psi]; \\
\mathrm{H}[T\psi] &= -i\hbar\,[T\psi].
\end{aligned}
$$

Thus Ψ and $[T^\dagger\psi]$ are not equivalent, and $T^\dagger$ requires $t \rightarrow -t$ and $i \rightarrow -i$. Thus one constructs observables in Quantum Mechanics by bilinear forms, (*i.e.* by products of operators and wave functions) so that microscopic time reversibility holds.

7.2 Parity

Another example of a discrete symmetry is spatial inversion which leads to the conservation of parity, P. The normal modes of a string have either even or odd symmetry. This also occurs for stationary states in Quantum Mechanics. The transformation is called parity exchange. In the QM description of the harmonic oscillator there are 2 distinct types of wave function solutions, characterized by the selection of the starting integer in a series representation. This selection produces a series in odd or even powers of the coordiante so that the wave function is either odd or even upon reflections about the origin, $x = 0$. Since the potential energy function depends on the square of the position, x^2, the energy eigenvalue is always positive and independent of whether the eigenfunctions are odd or even under reflection. In 1-D, parity is a symmetry operation, $x \rightarrow -x$. The strong interaction is invariant under the symmetry of parity.

$$\vec{r} \rightarrow -\vec{r}$$

Parity is a mirror reflection plus a rotation of 180°, and transforms a right-handed coordinate system into a left-handed one. Our Macroscopic world is clearly "handed", but "handedness" in fundamental interactions is more involved. As previously discussed, vectors (tensors of rank 1), as illustrated in the definition above, change sign under Parity. Scalars (tensors of rank 0) do not. One can then construct, using tensor algebra, new tensors which reduce the tensor rank and/or change the symmetry of the tensor. Thus a dual of a symmetric tensor of rank 2 is a pseudovector (cross product of two vectors), and a scalar product of a pseudovector and a vector creates a pseudoscalar.

7.3 Gauge invariance

Symmetries can not only be continuous or discrete, but they can be global or local. A global symmetry requires a universal change at all spacetime points, while a local symmetry is valid near one spacetime point, but is not extended to nearby points. Local symmetries form the basis of all theories involving gauge transformations.

A gauge symmetry of a system requires that the mathematical representation is invariant under a continuous, local transformation. Historically, gauge symmetry and gauge transformations were explored in classical electrodynamics. Thus the static electric field is determined from the scalar potential, $\vec{E} = -\vec{\nabla}V$. However, if the potential is changed by a scalar, C, such that $V \to V + C$, the field is not changed. Note that the force and energy, which are observables, depend only on the field, so that Maxwell's equations are invariant under such a change. In addition, in electrodynamics;

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$$

Insert a function $\lambda(\vec{x}, t)$ in the above equations such that;

$$\vec{A} \to \vec{A} + \vec{\nabla}\lambda$$

$$V = \to V - \frac{\partial \lambda}{\partial t}$$

This change in the potentials does not change Maxwell's equations. Thus gauge symmetry represents a mathematical mapping which leaves Maxwell's equations invariant. Previously Maxwell's equations in covariant form are;

$$\frac{\partial F^{\alpha\beta}}{\partial x^{\alpha}} = \mu_0 \, J^{\alpha}$$

In this equation, $F^{\alpha\beta}$ is the field tensor, and J^{α} is the charge/current density 4-vector. The Proca equation, which is a form of Maxwell's equations with a non-zero photon mass, μ_{γ} was discussed. This was written;

$$\frac{\partial F^{\alpha\beta}}{\partial x^{\beta}} + \mu_{\gamma}^2 = \mu_0 J^{\alpha}$$

Take the divergence of the above equation;

$$\frac{\partial^2 F^{\alpha\beta}}{\partial x^{\alpha} \, \partial x^{\beta}} + \mu_{\gamma}^2 = \mu_0 \frac{\partial J^{\alpha}}{\partial x^{\alpha}}$$

The first term on the left vanishes as $F^{\alpha\beta}$ is antisymmetric in (α, β). Only when $\mu_{\gamma}^2 = 0$ does one find charge conservation, $\frac{\partial J^\alpha}{\partial x^\alpha} = 0$. Note that if this does not occur, the mass term depends on the potential A^α so the equation is not gauge invariant.

The Lorentz condition expresses this symmetry as;

$$\vec{\nabla} \cdot \vec{A} + (1/c)^2 \frac{\partial V}{\partial t} = 0$$

In covariant form this is;

$$\partial_\alpha A^\alpha = 0$$

Satisfying the Lorentz condition results in applying the Lorentz gauge, and the result is a relativistic invariant. However, one could also choose a gauge in which $\vec{\nabla} \cdot \vec{A} = 0$. This is the Coulomb gauge, and in this gauge;

$$\nabla^2 V = -\rho/\epsilon$$

with solution;

$$V = \frac{1}{4\pi\epsilon} \int d\tau' \frac{\rho(\vec{x}', t)}{|\vec{x}' - \vec{x}|}$$

This gauge is not Lorentz invariant. The scalar potential is the instantaneous Coulomb potential, which is obviously not Lorentz invatient. The vector potential is much more complicated, as it is the solution of the pde obtained after substituting, $\vec{\nabla} \cdot \vec{A} = 0$

$$\nabla^2 \vec{A} - \epsilon\mu \frac{\partial^2 \vec{A}}{\partial t^2} = \mu \vec{J} + \epsilon\mu \vec{\nabla}(\frac{\partial \phi}{\partial t})$$

A gauge transformation does not change the value of the fields which determine the Lorentz force and energy conservation. The fields are the observables, so a gauge transformation is a symmetry of the electromagnetic interaction, which can be associated with the vanishing mass of the photon.

8 Angular momentum

There is also angular momentum in the EM field. This is obtained from;

$$\vec{L}_{field} = (1/c^2) \int d^3x \, [\vec{x} \times (\vec{E} \times \vec{H})]$$

In terms of a 4-dimensional tensor formulation;

$$\mathcal{M}^{\alpha\beta\gamma} = H^{\alpha\beta} x^{\gamma} - H^{\alpha\gamma} x^{\beta}$$

Conservation of angular momentum requires that a divergence equation demonstrate the conservation of angular momentum;

$$\partial_{\alpha} M^{\alpha\beta\gamma} = 0$$

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This requires that $H^{\alpha\beta}$ is symmetric, and thus the symmetric form for the stress tensor is required for a 4-dimensional representation.

8.1 Example

Now investigate an example of angular momentum in a static electromagnetic field. The geometry of the example is shown in Figure 6. There is a spherical permanent magnet inside a spherical shell. The magnet has a non-conducting interior with the exception of a thin layer on its surface which allows it to serve as the inner conductor of a spherical capacitor. The outer shell is non-magnetic, but conducting, and acts as the outer surface of the spherical capacitor. A charge $\pm Q$ is placed on the capacitor surfaces by applying a potential between the spherical surfaces. There is an electric field between these surfaces given by;

$$\vec{E} = \kappa \frac{Q}{r^2} \hat{r}$$

Both spheres are free to rotate about the z axis. The inner magnetic sphere has uniform magnetization M in its interior pointed in the $\hat{z}$ direction. A dipole magnetic field is produced having the value;

$$\vec{B} = \frac{Ma^3}{3r^3} [2 \, cos(\theta) \, \hat{r} + sin(\theta) \, \hat{\theta}]$$

In the above equation a is the radius of the magnetic sphere. Then find the Poynting vector.

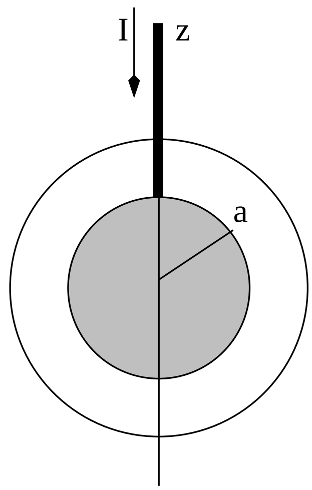

Figure 6: An example showing power flow around a magnetic sphere, representing angular momentum in the field required to cancel the rotation of the spheres

$$\vec{S} = (1/\mu_0)\vec{E} \times \vec{B}$$

$$\vec{S} = \frac{QMa^3 sin(\theta)}{12\pi\epsilon_0\mu_0 r^5}\,\hat{\phi}$$

Thus in the space between the shell and the magnetic sphere, energy flows in a circular motion around the $\hat{z}$ axis. To understand why, consider the initial state of the system with the capacitor uncharged and the spheres at rest. There is no Poynting vector because $\vec{E} = 0$. To charge the capacitor, a current flows down a conductor along the z axis to the surface of the magnetic sphere. These currents interact with the magnetic field to produce an equal but opposite angular momentum in the spheres. The difference in this angular momentum between the spheres is equal to that in the fields.

9 Bohm-Aharanov effect

Classically the force on a charge is independent of the gauge. However, in QM the phase of the wave function depends on the gauge properties. In global symmetry the phase must be the same at all points and at all times, and connects the spatial values of the vector potential. Consider the plane wave function of a non-interacting particle wave.

$$\psi_0 = A\, e^{i(\vec{p}\cdot\vec{x})/\hbar}$$

In the presence of a static vector potential $\vec{B} = \vec{\nabla} \times \vec{A} = 0$. But the kinetic energy term in the Schrodinger equation has the form;

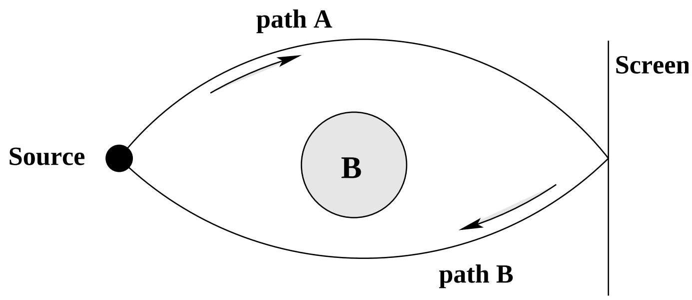

Figure 7: A path integral around a Magnetic field

$$-(\hbar^2/2m)[\vec{\nabla} + \frac{iqA}{\hbar c}]^2 \psi = E\psi$$

The solution is $\psi = e^{i\phi}\psi_0$

Now choose a path around the magnetic field as shown in the figure. Here $\phi = (q/\hbar c) \int_{path} \vec{A} \cdot \vec{dl}$ As the phase depends on the path length it will not necessarily return to the same value. This can be related to a geometric effect of moving a vector around a surface in 3-D. Thus the potentials have an effect which is not contained in the fields.