Review of Electrostatics

1 Gradient

Define the gradient operation on a field $F = F(x, y, z)$ by;

$$\vec{\nabla} F = \hat{x}\frac{\partial F}{\partial x} + \hat{y}\frac{\partial F}{\partial y} + \hat{z}\frac{\partial F}{\partial z}$$

This operation forms a vector as may be shown by its transformation properties under rotation and reflection. Write the following using the above definition of a partial derivative and the chain rule of differentiation.

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = \vec{\nabla} F \cdot \vec{ds}$$

Here $\vec{ds} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$. In Cartesian coordinates the gradient operator is defined by ;

$$grad = \vec{\nabla} = \hat{x}\,\frac{\partial}{\partial x} + \hat{y}\,\frac{\partial}{\partial y} + \hat{z}\,\frac{\partial}{\partial z}$$

Then write in polar coordinates using the length of the differential vector, $ds = \sqrt{dx^2 + dy^2 + dz^2}$;

$$\frac{\partial F}{\partial s} = |\vec{\nabla}\, F|\, cos(\theta)$$

The derivative, $\frac{\partial F}{\partial s}$, is a maximum only when $\vec{ds}$ is in the direction of $\vec{\nabla} F$, and has magnitude equal to $|\vec{\nabla} F|$.

2 Flux

The definition of flux comes from an analogy to flow through a surface. This is generalized to define the differential flux for any vector field $\vec{F}$ as;

$$d\, flux = \vec{F} \cdot d\vec{\sigma}$$

The dot product projects the direction of the field perpendicular to the surface. The total flux through an area is ;

$$flux = \int_{area} \vec{F} \cdot d\vec{\sigma}$$

3 Divergence

The divergence of a vector field, $\vec{F}$, is defined as the flux out of a volume per unit volume, and written, $Div\vec{F}$ or in Cartesian coordinates $\vec{\nabla} \cdot \vec{F}$. Develop this in a generalized, orthogonal set of coordinates, but note that applying the $\vec{\nabla}$ operator in non-Cartesian coordinates for the divergence is **NOT** correct. From the definition;

$$Div\vec{F} = \lim_{d\tau \to 0} \frac{\int \vec{F} \cdot d\vec{\sigma}}{\int d\tau}$$

In Cartesian coordinates;

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

4 The Circulation

Next consider the line integral of a vector field, F, between spatial points a and b. This is defined as;

$$I = \int_{a,\, path(L)}^{b} \vec{F} \cdot d\vec{l}$$

The circulation of the vector field, $\vec{F}$, is then;

$$\Gamma = \oint_L \vec{F} \cdot d\vec{l}$$

The differential operation of $Curl$ is the circulation per unit enclosed area. As with the divergence, do not directly use the $\vec{\nabla}$ operator on a vector field $\vec{F}$ unless in Cartesian coordinates.

$$Curl\vec{F} = \lim_{d\sigma \to 0} \frac{\oint \vec{F} \cdot d\vec{l}}{\int d\sigma}$$

In Cartesian Coordinates;

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_y \end{vmatrix}$$

5 Integral Theorems

Recall that the definition of the divergence operation is the flux out of a volume per unit volume, and direction of the area area vector is the outward normal to a closed surface. Thus the differential element of the flux through a small volume, $d\,\tau$, is;

$$d\,flux \;=\; (Div\vec{F})\,d\,\tau$$

Then sum all the volume elements contained within a finite volume to obtain the flux out of the volume. This is a statement of Gauss' theorem.

$$\oint\,\vec{F}\cdot d\vec{\sigma} \;=\; \int\,d\,\tau\,(Div\,\vec{F})$$

Here $d\,\sigma$ is an element of the surface area enclosing the volume, $d\,\tau$. Then the $Curl$ operation is defined as the circulation per unit area, so that for an infinitesimal area;

$$(Curl\vec{F})\cdot d\,\vec{\sigma} \;=\; (\vec{F}\cdot d\,\vec{l})_{closed\,path}$$

By combining small, infinitesimal paths around the perimeters of a set of infinitesimal areas one obtains;

$$\int\,(Curl\vec{F})\cdot d\,\vec{\sigma} \;=\; \oint\,\vec{F}\cdot d\,\vec{l}$$

This is stokes theorem. Finally, consider the line integral over a path from a to b which results in a scalar function.

$$\int_{path} d\,F \;=\; \int_{path} \vec{\nabla} F \cdot d\,\vec{s} \;=\; F(a) \;-\; F(b)$$

6 The Laplacian Operator

On a number of occasions we use the combination of the Div operating on the $Grad$ of a scalar (and later a vector) function. In Cartesian coordinates this takes the form ∇^2. Again be careful not to apply the Cartesian form of this operator in other coordinate systems, or to a vector function. We discuss later the differential equations that are produced by these various vector operators, and their physical implications.

7 Coulomb's Law

The force between 2 static charges, q_1 and q_2, separated a distance r_{12} is described by the equation;

$$\vec{F} \;=\; \kappa \frac{q_1 q_2}{r_{12}^2}\, \hat{r}_{12}$$

If a charge is defined as a Coulomb which is defined by the measure of a current (*ie* charge per second), the force is measured in Newtons, then $\kappa = \frac{1}{4\pi\epsilon_0}$. This defines the rationalized MKS system of units. Here ϵ_0 is the permittivity of free space, and has the value 8.85×10^{-12} farad/m.

Charge is quantized in units of the electronic charge 1.6×10^{-19} Coulomb. Positive and negative charge have equal electronic units to the extent that this can be measured. Now return to the $1/r^2$ behavior of the force. Is the power of r really 2, or is this just approximate? Experimentally it can be tested, of course, but there is always be some uncertainty in a measured number. However one asserts the power is exactly 2 for other reasons. This law describes the interaction of charges through the exchange of photons, that is quanta of the electromagnetic field. These quanta in free space have zero mass, and for (and only for) for zero mass particles does the force decrease exactly as $1/r^2$.

8 Superposition

In addition to Coulomb's law, the law of superposition also applies. This law states that the total force on a charge is the vector sum of all the electrostatic forces acting on the charge.

$$\vec{F}_T = \kappa\, q_1 \sum_i \frac{q_i}{r_{1i}^2}\, \hat{r}_{1i}$$

The accumulation of forces could have been non-linear, and is not necessarily linear for fields in a medium. Superposition is a separate law and is applied with Coulomb's law in electrostatics.

9 The Electric Field

Now the electric field is discussed. Remember that a field has a mathematical definition but more importantly, it connects physics to space-time geometry. Assume that each point in space-time is effected by a charge so that if another charge is placed at that point, it experiences a force given by Coulomb's law. In this way the interaction is separated into two components. Thus one charge alters space-time at all geometric points, and the other interacts with the space time point at its geometric position, resulting in the force. Of course the interaction is mutual and is symmetric in the case of static charge. The description of interactions through fields is essential to allow a relativistic formulation of the electromagnetic interaction.

$$\vec{F} = q_1 \, [\kappa \sum_i \frac{q_i}{r_{1i}^2} \, \hat{r}_{1i}]$$

Here $\kappa = \frac{1}{4\pi\epsilon_0}$. Define the electric field as;

$$\vec{E} = [\kappa \sum_i \frac{q_i}{r_{1i}^2} \, \hat{r}_{1i}]$$

so that;

$$\vec{F} = q_1 \, \vec{E}$$

The electric field is a vector quantity that is a function of position. If the field is known at **ALL** points in space then the charge distribution which caused the field can be determined.

10 Gauss' Law

Gauss' law assumes both Coulomb's law and the law of superposition. Now find the flux which penetrates a surface enclosing a net charge composed of both positive and negative charges. For the moment, assume a positive charge density, ρ.

The elemental charge creating a quantity of elemental flux is $\rho\, d\tau$. The flux through the surface area, $d\vec{\sigma}$ is then;

$$d\,\text{flux} \;=\; \vec{E}\cdot d\vec{\sigma} \;=\; \kappa\,\frac{\rho\, d\tau}{r_2^2}\,\hat{r}_2\cdot d\sigma$$

The solid angle subtended by $d\sigma$ is $d\Omega$ so that;

$$r^2\, d\Omega \;=\; \hat{r}\cdot d\vec{\sigma}$$

Shifting the origin write;

$$d\text{flux} \;=\; \vec{E}\cdot d\vec{\sigma} \;=\; \kappa\, dQ\,\frac{(r - R\,cos(\theta))r^2 d\Omega}{[R^2 + r^2 - 2Rrcos(\theta)]^{3/2}}$$

Keep $r = r_0$ (a constant) and integrate over angles.

$$d\text{flux} \;=$$
$$(dQ/2\epsilon)\,\int\, d\,cos(\theta)\,\frac{1 - (R/r_0)cos(\theta)}{[1 + (R/r_0)^2 - 2(R/r_0)cos(\theta)]^{3/2}}$$

This integrates to dQ/ϵ_0 which should not be surprising. One can always draw a sphere centered on the point containing the charge dq. The total flux through this sphere is given by Coulomb's law and equals dQ/ϵ_0. However, the flux out of this sphere is the same as the flux out of any surface which encloses the sphere. Finally, the total flux by superposition of all the elements of enclosed charge is obtained.

$$\text{Flux} = \oint \vec{E} \cdot d\vec{\sigma} = Q_{Total}/\epsilon_0$$

This is Gauss' law. Use the divergence theorem (Gauss' theorem) to write;

$$\int div\vec{E}\, d\tau = Q/\epsilon_0 = \int \rho/\epsilon_0\, d\tau$$

In differential form, because the volume is arbitrary;

$$div\vec{E} = \rho/\epsilon_0$$

Now a combination of both positive and negative charge could lie within the volume. However, by superposition the charge Q_{enc} is the net enclosed charge within the volume.

11 Line Integrals and the Electric Potential

Using mechanics the energy difference, W, generated by moving a charge, q, between positions a and b along a path in a static electric field is obtained from the expression of the electrostatic force, $\vec{F} = q\vec{E}$, where $\vec{E}$ is the electric field.

$$W = \int_{a,\,Path}^{b} d\vec{l} \cdot \vec{F}$$

$$W = -\int_{Path} qE\,dr = \frac{q\,Q}{4\pi\epsilon} \int_{b}^{a} (1/r^2)\,dr = \frac{q\,Q}{4\pi\epsilon}[1/r]_a^b$$

Thus the energy is independent of the path, and depends only on the end points. By superposition, add contributions for each charge of a charge distribution. This shows that the energy obtained by moving a charge in any electric field is independent of the path. The energy obtained from an integral over a closed path vanishes.

$$\oint \vec{E} \cdot d\vec{l} = 0 = \int (\vec{\nabla} \times \vec{E}) \cdot d\vec{\sigma}$$

The above result is obtained from Stokes theorem where the integral on the right hand side is over the area enclosed by the path integral on the left hand side. Since the path and the area are arbitrary, the integrand must vanish;

$$\vec{\nabla} \times \vec{E} = 0$$

This result is a consequence of Coulomb's law, superposition, and the fact that the charge is static.

The energy is a **numerical value** for a specific charge distribution, as opposed to the electric potential which is a field function. From the above equations;

$$dW = q\,dV = -q\,\vec{E}\cdot d\vec{s}$$

This results in the definition of the electric potential, V, which is a scalar **FIELD** obtained from the vector electric field, $\vec{E}$. If a unit charge is placed at a geometric position in an electric potential, then the result is the potential energy of this dual charge distribution. The potential **energy** is always measured with respect to another point in space, (*ie* in this case, the potential at a is measured relative to that at b which are the integration limits). From the definition of the gradient, the electric field may be obtained from the electric potential.

$$\vec{E} = -\vec{\nabla}V$$

Again the above results are for a static charge distribution.

12 Including the Electric Potential in Gauss' Law

Gauss' law in differential form is $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon$. Insert the above expression for the electric field in terms of the electric potential. This results in;

$$\vec{\nabla} \cdot \vec{\nabla} V = -\rho/\epsilon_0$$

The above is a partial differential equation (Poisson's equation), which can be solved to obtain the electric potential, and from the potential, the electric field. In the case where $\rho = 0$ the equation is called Laplace's equation and the operator, ∇^2, is the Laplacian. These equations are 2^{nd} order, linear, partial differential equations.

13 Total energy of a charge distribution

Assume a set of positive charges which are placed at various positions in space. Energy is required to assemble this distribution, and this can be calculated by moving each charge from far away to its position. Note here that the energy is assumed to vanish at infinity, and energy is always measured relative to an arbitrary position set to zero energy. As each charge is assembled, the next charge experiences a potential due to the other charges previously moved into position. The energy is given by the value of the charge multiplied by the electric potential created by the assembled charge. Write for this energy;

$$W = \kappa\,[q_1 q_2/r_{12} + q_1 q_3/r_{13} + \cdots$$

$$q_2 q_3/r_{23} + q_2 q_4/r_{24} + \cdots]$$

This is;

$$W = \kappa \sum_{i>j} q_i q_j/r_{ij} = (\kappa/2) \sum_{i\neq j} q_i q_j/r_{ij}$$

Note the sum can be rearranged to be written in the form;

$$W = (\kappa/2)\,[q_1 \sum_{1\neq j} q_j/r_{1j} + q_2 \sum_{2\neq j} q_j/r_{2j} + \cdots]$$

Here the electric potential is just;

$$V_i = \kappa \sum_{i\neq j} q_i/r_{ij}$$

Which can be used above to obtain;

$$W = (1/2) \sum_i q_i V_i$$

Now change the discrete charge distribution to a continuous one so that the sum is changed to an integral.

$$W = (1/2) \int d\tau \, \rho \, V$$

Then apply Poisson's equation $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$

$$W = (\epsilon_0/2) \int d\tau \, \rho \, \vec{\nabla} \cdot \vec{E}$$

Here;

$$\vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

Substitute into the above equation and integrate by parts or use the identity $\vec{\nabla} \cdot (V\vec{E}) = (\vec{\nabla}V) \cdot \vec{E} + V(\vec{\nabla} \cdot \vec{E})$. If the surface terms in the integration by parts vanish (*ie* taken at large distances) we obtain;

$$W = \int d\tau \, (\epsilon_0/2)(\vec{E} \cdot \vec{E})$$

From the above, assign an energy per unit volume of $(\epsilon_0/2)\vec{E}^2$ to the electric field. This is the energy which was needed to assemble the charge distribution creating the field.

14 Boundary conditions

The electric field must be perpendicular to a conducting surface, and must vanish within an enclosed conducting volume. This means that the potential has a constant value on the surface. There is no field within the conductor so no flux penetrates the Gaussian surface. Outside the conductor, the E field is perpendicular to the surface at the interface. Shrink the dimensions of a Gaussian cylinder so that the outer end cap approaches the surface. The field out of this surface is perpendicular to the area vector and equal to the perpendicular value of E at that point. The flux is then $E_{\perp}\, Area \,=\, Q/\epsilon_0$;

$$\vec{E} \,=\, (\sigma/\epsilon_0)\,\hat{n}$$

where σ is the surface charge density and $\hat{n}$ is the outward normal. Then the surface charge on a conductor is given by;

$$\sigma \,=\, -\epsilon_0 \vec{\nabla} V \cdot \hat{n}$$

$$\Gamma = E_{\parallel\, above} dl - E_{\parallel\, below} dl$$

For static charge $\Gamma = 0$ so that $E_{\parallel\, above} = E_{\parallel\, below}$, and for a conductor both equal zero.

15 Uniqueness

Look for solutions to a second order partial differential equation. These solution can be determined in several ways, and will usually be represented in the form of a series of special functions obtained from the solutions to a set of eigenvalue equations. Thus the solution may take different forms, and an important question arises. How do we know that the solution we find is unique? There are mathematical proofs that unique solutions to various second order differential equations are unique if they satisfy the differential equation and also have a specified value on a set of boundaries in the geometric space in which the equation applies. These conditions are specified in table 1. Finding a proper solution is called a boundary value problem.

The Dirichlet boundary conditions require specification of the value of the solution on the boundary. The Neumann boundary conditions require the specification of the

Table 1: Boundary conditions required for unique solutions to various 2^{nd} order partial differential equations

	Poisson's Eqn $\nabla^2 V = \rho/\epsilon$	Wave Eqn $\nabla^2 V = (1/c^2)\frac{\partial^2 V}{\partial t^2}$	Diffusion Eqn $\nabla^2 V = (1/a)\frac{\partial V}{\partial t}$
Dirichlet			
Open Surface	not enough	not enough	unique
Closed Surface	unique	too much	too much
Neumann			
Open Surface	not enough	not enough	unique
Closed Surface	unique	too much	too much
Cauchy			
Open Surface	unstable	unique	too much
Closed Surface	too much	too much	too much

derivative of the solution on the boundary. The Cauchy boundary conditions require the specification of both the value of the solution and its normal derivative on the boundary. For electrostatics, we are interested in the solution to Laplace's (or Poisson's) equation which requires specification of the value of the solution (**or**) its derivative on a closed surface.

16 The Sturm-Liouville Problem

The Sturm-Liouville solutions are eigenfunctions and the value of λ_n are the corresponding eigenvalues. The general Sturm-Liouville equation is;

$$\frac{d}{dz}[p(z)\,\frac{dF}{dz}] \,+\, [q(z) \,+\, \lambda\, r(z)]F \,=\, 0$$

These functions form an orthogonal set of functions in the space defined within the problem boundaries, $a \leq z \leq b$. Thus any function, $F(z)$, can be represented by a linear sum of eigenfunctions;

$$F(z) \,=\, \sum_{n=0}^{\infty} A_n\, \eta_n(z)$$

so that;

$$\lim_{m \to \infty} \int_a^b dz\, [F(z) \,-\, \sum_{n=0}^{m} A_n\, \eta_n(z)]^2 r(z) \,=\, 0$$

This represents convergence in the mean. In addition;

$\int dz\, r(z)\, \eta_n(z)\, \eta_m(z) \,=\, 0$ when $m \neq n$ and $(\lambda_n - \lambda_m) \neq 0$

So that the eigenfunctions are orthogonal using a weighting factor $r(z)$ which comes from the ode.

17 Potential and Multipole Expansions

First look at the potential of a charge distribution ρ. It is given by;

$$V \;=\; \kappa \int d\tau' \,\frac{\rho(\vec{r'})}{|\vec{r} - \vec{r'}|}$$

Now suppose $r > r'$ and look at the term;

$$\frac{1}{|\vec{r} - \vec{r'}|} \;=\; (1/r)\frac{1}{\sqrt{1 + (r'/r)^2 - 2\,(r'/r)\,cos(\theta)}}$$

Make a power expansion of the fraction;

$$\frac{1}{\sqrt{1 + (r'/r)^2 - 2\,(r'/r)\,cos(\theta)}} \;=\; [1 + (r'/r)\,cos(\theta) + (r'/r)^2[1 + (2/3)cos^2(\theta) + \cdots]$$

$$\frac{1}{\sqrt{1 + (r'/r)^2 - 2\,(r'/r)\,cos(\theta)}} = 1 + \sum_{n=1}^{\infty} \frac{2n-1)!!}{2n!!}\,[2(r'/r)\,cos(\theta) - (r'/r)^2]$$

Here introduce the Legendre polynomials;

$$P_n(x) \;=\; \sum_{k=0}^{n/2} (-1)^k)\,\frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!}\,x^{n-2k}$$

For even n, P_n has even powers of x and for odd n, odd powers of x. Some examples are;

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = 3/2(x^2 - 1/2)$$

$$\frac{1}{|\vec{r} - \vec{r'}|} = [\sum_{n=0}^{\infty} (r'/r)^n P_n(cos(\theta))]$$

The Legendre polynomials are an orthogonal set of functions;

$$\int_{-1}^{1} dx P_n(x) P_m(x) = \frac{2}{2n+1}$$

18 Spherical Harmonics

In the case that there is no axial symmetry, one must include eigenfunctions of ϕ in the solution. This introduces the associated Legendre polynomials as well. Combine these angular functions into an orthonormal set called the spherical harmonics.

$$Y_l^M(\theta,\phi) \;=\; \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(cos(\theta))\,e^{im\phi}$$

Note that;

$$Y_l^{-m} \;=\; (-1)^m\,Y_l^{m\,*}$$

The functions are orthogonal;

$$\int\, d\Omega\, Y_{l'}^{m'\,*}\,Y_l^m \;=\; \delta_{l,l'}\,\delta_{m,m'}$$

$$Y_0^0 \;=\; \frac{1}{4\pi}$$

$$Y_1^1 \;=\; -\frac{3}{8\pi}\,sin(\theta)\,e^{i\phi}$$

$$Y_0^1 \;=\; -\frac{3}{4\pi}\,cos(\theta)$$

$$Y_2^2 \;=\; -\frac{15}{32\pi}\,sin^2(\theta)\,e^{i2\phi}$$

$$Y_1^2 \;=\; -\frac{15}{8\pi}\,sin(\theta)\,cos(\theta)\,e^{i\phi}$$

$$Y_0^2 \;=\; -\frac{5}{4\pi}\,((3/2)cos^2(\theta) - 1/2)$$

An arbitrary function can be expanded in spherical harmonics, for example;

$$f(\theta, \phi) \;=\; \sum_{l,m} A_{lm} Y_l^M(\theta, \phi)$$

19 The Addition Theorem

The addition theorem allows representations in abritrary spherical coordinate frames.

$$P_l(\cos(\gamma)) \;=\; \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_l^{m\,*}(\theta', \phi')\, Y_l^m(\theta, \phi)$$

The following expansion results;

$$\frac{1}{|\vec{r} \,-\, \vec{r'}|} \;=\; 4\pi \sum_{l,m} \frac{1}{2l+1} \,(r_<^l / r_>^{l+1})\, Y_l^{m\,*}(\theta', \phi')\, Y_l^m(\theta, \phi)$$

20 Induced and Permanent Electric Moments

Approximate the macroscopic potential due to an arbitrary charge distribution. From previous lectures this can be written as;

$$V \;=\; \kappa \int d\tau' \,\frac{\rho(\vec{r'})}{|\vec{r} - \vec{r'}|}$$

Now make the assumption that $r > r'$, and expand right hand side of the equation in Legendre polynomials. The first 2 terms have the form;

$$V \;=\; \frac{\kappa}{r}\,[\int\, d\tau'\,\rho(\vec{r'}) \;+\; (1/r)\,\hat{r}'\cdot\vec{p} + \;\cdots]$$

Here $\vec{p}$ is the dipole moment of the charge distribution, and the terms in the above series are called the multipole moments of the charge distribution. The full expansion in terms of the Spherical harmonics is;

$$V \;=\; \sum_l \sum_{m=-l}^{l} \frac{1}{\epsilon_0(2l+1)}\,\frac{g_l^m}{r^{l+1}}\,Y_l^m(\theta,\phi)$$

The multipole q_l^m is;

$$q_l^m \;=\; \int d\tau'\,Y_l^{*\,m}(\theta',\phi')\,r'^{\,l}\,\rho(\hat{r}')$$

The first few terms are;

$$q_0^0 \;=\; Q/\sqrt{4\pi}$$

$$q_1^0 \;=\; \sqrt{3/4\pi}\,p_z$$

$$q_1^1 = -\sqrt{3/8\pi}\,(p_x - ip_y)$$

Using the above, the dipole moment $\vec{p}$ as ;

$$\vec{p} = \int d\tau'\, \vec{r}'\rho(\vec{r}')$$

21 Gauss' Law and the Electric Displacement

Suppose we apply Gauss' Law inside a dielectric material. The Gaussian surface cuts through atoms/molecules such that some induced charge is always included within the surface. Gauss' Law still holds, but it must be modified to include all charge enclosed within the surface. Therefore for a system of free, Q_{free}, and induced, Q_{Ind}, charge, Gauss' Law is;

$$\oint \vec{E}\cdot d\vec{A} = (Q_{free}+Q_{Ind})/\epsilon_0 = (1/\epsilon_0) \int d\tau\,(\rho_{free}+\rho_{Ind})$$

Introduce a new vector called the electric displacement;

$$\vec{D} = \epsilon_0\vec{E} + \vec{P}$$

The vector, $\vec{P}$, above is the dipole moment per unit volume of the polarization. Then form a new Gauss' Law dependent only on the free charge enclosed within a volume;

$$\int \vec{D} \cdot d\vec{A} = \int d\tau \, \rho_{free} = Q_{free}$$

In differential form this is;

$$\vec{\nabla} \cdot \vec{D} = \rho_{free}$$

Note that the field is still static so $\oint \vec{E} \cdot d\vec{l} = 0$ has not changed. Then for a class A dielectric the polarization, $\vec{P}$, is in the direction of, and proportional to, the applied field. Thus write;

$$\vec{D} = \epsilon_0 \epsilon_r \vec{E} = \epsilon \vec{E}$$

The Polarization is then;

$$\vec{P} = (\epsilon - \epsilon_0)\vec{E}$$

22 Magnetic Force

The magnetic force on a charge moving with a velocity, $\vec{v}$, is obtained from the Lorentz force equation;

$$\vec{F} = q[\vec{E} + \vec{v} \times \vec{B}]$$

In the above, both the electric field and a magnetic field are included. The form of the electric field is given by previous expressions, and the magnetic field can be obtained from the Biot-Savart Law;

$$\vec{B} = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l} \times \hat{r}}{r^2}$$

In the above expression, the multiplying constant $\frac{\mu_0}{4\pi}$ provides a force in Newtons for current, I, in amperes. The term, μ_0, is the permeability of a material for free space, and the integral is along the current filament $d\vec{l}$. More correctly, $\vec{B}$ is called the magnetic induction while the magnetic field is defined by $\vec{H}$, to be introduced later. Units of $\vec{B}$ are Tesla or Weber/m^2. The Gaussian unit of magnetic induction is the Gauss. For conversion, 1 Webber/$m^2 = 10^4$ Gauss. In what follows, we interchange the terms "magnetic field" and "magnetic induction" unless the meaning is not clear.

If the current flows through a filament of finite size, introduce a current density, $\vec{J}$ such that;

$$dI = \vec{J} \cdot d\vec{A}$$

where dA is the infinitesmal area perpendicular to the current flow. The Biot-Savart law is then written;

$$\vec{B} = \frac{\mu_0}{4\pi} \int d\tau' \frac{\vec{J} \times (\vec{r'} - \vec{r})}{|\vec{r} - \vec{r}|^3}$$

23 Ampere's Law and the Vector Potential

In the case of electrostatics, the various results were developed by applying the differential opperators for the divergence, curl, and gradient. Now apply these to the magnetic field, $\vec{B}$. First define a magnetic flux by $\phi_M = \int \vec{B} \cdot d\vec{A}$. Then apply the divergence theorem (Gauss's Law).

$$\int \vec{B} \cdot d\vec{A} = \int d\tau \, (\vec{\nabla} \cdot \vec{B})$$

Substitute the expression for $\vec{B}$ using the Biot-Savart law;

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \hat{r}}{r^2}$$

This gives $\vec{\nabla} \cdot \vec{B} = 0$ so that $\vec{B} = curl \vec{A}$

$$\vec{A} = \frac{\mu_0}{4\pi} \int d\tau \, \vec{J}/r$$

Note the similarity to the expression for the electric potential;

$$V = \kappa \int d\tau \, \frac{\rho}{r}$$

$\vec{A}$ is called the vector potential. Now consider $\vec{\nabla} \times \vec{B}$.

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

Remember from an earlier discussion one must be careful with ∇^2 operations on a vector. In this case one can ignore the complications of this operation. Now look at each term in the above expression. This results in

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

This is Ampere's Law in differential form. In integal form it is;

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{A} = \mu_0 I$$

24 Equation of Continuity

The current density $\vec{J}$ is the charge density multiplied by the velocity. Obviously the velocity gives the direction and for a differential distance in the direction of the current flow, $v = dx/dt$. The charge density is $\rho = dQ/(\text{area}\, dx)$. Then

$$\rho\, v = dQ/(\text{area} - dt) = I/\text{area}$$

Here, the area is perpendicular to the current flow in the direction dx. From the definition of the current density, $\vec{J} \cdot d\,\vec{\text{area}} = dI$, and as a result $\vec{J} = \rho\,\vec{v}$. Now consider $\int \vec{\nabla} \cdot \vec{J}\, d\tau = \int \vec{J} \cdot d\,\vec{\text{area}}$. This last equation represents the current flux out of the volume. That is the flow of charge out of the volume per unit time. This must equal the change per unit time of the enclosed charge within the volume.

$$\int d\tau \, (\vec{\nabla} \cdot \vec{J}) = -\int d\tau \, \frac{\partial \rho}{\partial t}$$

This is the equation of continuity which represents conservation of charge. Similar equations appear for other conserved quantities, such as energy or mass.

25 Magnetic pressure and energy

Consider two parallel current sheets separated by a distance, d, with uniform, constant currents flowing in opposite directions. The magnetic field is obtained using Ampere's law. Because of symmetry the magnetic field must be directed parallel to the sheet, and can only depend on the perpendicular distance from the sheet to the field point. Evaluation of the integral form of Ampere's law gives;

$$\oint \vec{B} \cdot d\vec{l} = 2BL = \mu_0 I$$

The factor of 2 comes from adding the fields above and below the sheet, and L is the distance parallel to the sheet over the path along $\vec{B}$. I is the current which flows through this Amperian loop. Thus for one sheet;

$$B = \mu(I/L)/2 = (\mu/2)\,\mathcal{I}$$

In the above, $\mathcal{I}$ is the current per unit width on the sheet. The direction of the magnetic field is given by the right-hand-rule. Note that the field is independent of the distance from the sheet. Thus the fields when superimposed from the two sheets, add between the sheets and cancel outside the sheets. Finally we also see that the force generated by the magnetic field on one sheet interacting with the current on the other is repulsive. Visualize this situation by thinking of the magnetic field as creating a pressure between the plates tending expand the distance between them.

Use the Lorentz force to calculate this force. In the equation for the Lorentz force, substitute $I\,dL$ for qV. The field and current direction are perpendicular, so $F_2 = I_2 L B_1 = I_2 x_2 (\mu/2)\mathcal{I}_1$. Now the total magnetic field between the plates comes from the superposition of both fields which add to $B_T = 2(\mu/2)\mathcal{I}_{1,2}$. Then $I_{1,2} = BL/\mu$ which we substitute for I_1 in the force equation yielding;

$$\frac{F}{Lx} = \frac{1}{2\mu_0} B^2$$

The above is the force per unit surface area (pressure) of one current sheet on the other. This pressure attempts to push the plates apart. Now suppose work is done against this pressure, by compressing the plates a distance, d. The movment of the plates removes a volume of the magnetic field, Lxd, under the Amperian loop. and puts an energy into the system given by $W = Fd$. Remove the geometry in the equation by dividing by the volume to obtain the energy per unit volume which we assign to the magnetic field. Compare this energy density, $(1/2\mu_0)B^2$, to the energy density of the electric field, $(\epsilon_0/2)E^2$.

26 Macroscopic Equations

Previously it was assumed that $\vec{J}$ was known or could be determined. In the presence of matter this is not true or irrevelent on the atomic scale, because all atoms have currents due to the movement of atomic charge. Of interested here is a macroscopic average of these currents over a sufficiently large volume so one can treat fields in matter as a continuous distribution of mass and current. The solution to this problem is similar to the way the equations for the electric field in materials were developed.

The current density is divided into 2 components, 1) a conduction current density, $\vec{J_c}$ and 2) an atomic current density, $\vec{J_a}$. In analogy to the electric field case, $\vec{J_a}$ is a bound current density.

$$\vec{A} = \frac{\mu_0}{4\pi} \int d\tau \, \frac{\vec{J}c(r')}{|\vec{r} - \vec{r'}|} + \frac{\mu_0}{4\pi} \int d\tau \, \frac{\vec{J}a(r')}{|\vec{r} - \vec{r'}|}$$

Expand the second term by a multipole expansion.

Then identify a new current density;

$$\vec{J}_a^* = \vec{\nabla} \times \vec{M}$$

In the above M is the magnetic moment per unit volume or the magentization, so that

$$\vec{J}^* = \vec{J_c} + J_a^*$$

Ampere's Law then becomes;

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}^* = \mu_0[\vec{J_c} + \vec{\nabla} \times \vec{M}]$$

$$\vec{\nabla} \times (\vec{B} - \mu_0 \vec{M}) = \vec{J}_c$$

Define a new quantity, H;

$$\mu_0 \vec{H} = \vec{B} - \mu_0 \vec{M}$$

The variable, H, is usually called the magnetic field and B the magnetic induction. For comparison, the relationship between the electric displacement, electric field, and the polarization is;

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

In the case of the magnetic field the relationship between the magnetic field, magnetic induction, and magnetization is;

$$\vec{H} = (1/\mu_0)\vec{B} - \vec{M}$$

27 Static Maxwell Equations

$$\vec{\nabla} \cdot \vec{D} = \rho$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = 0$$

$$\vec{\nabla} \times \vec{H} = \vec{J}$$

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

$$\vec{H} = (1/\mu_0)\vec{B} - \vec{M}$$