Dielectric wave guides, resonance, and cavities

1 Dielectric wave guides

Instead of a cavity constructed of conducting walls, a guide can be constructed of dielectric material. In analogy to a conducting wave guide, the dielectric can be made to conduct light along the guide, or trap light in a cavity if the guide is closed by placing mirrors on its ends (optical cavity). An optical cavity is essentially the geometry of a laser. Since the geometry is cylindrical, the mathematics is not developed in detail here, although this is not so difficult. Remember this requires use of the vector Laplacian. However, the basic equations are already developed. In the last lecture the wave equation obtained by separating Maxwell's equations in a source free region of space were obtained. These were then written in terms of the transverse and longitudinal components of the fields to match a longitudinal geometry with the wave moving along the guide in the $\hat{z}$ direction. To summarize these mathematics for cylindrical coordinates the fields have the form;

$$\begin{bmatrix} \vec{E}(\vec{r}) \\ \vec{B}(\vec{r}) \end{bmatrix} = \begin{bmatrix} \vec{E}(\rho, \phi) \\ \vec{B}(\rho, \phi) \end{bmatrix} e^{i(kz-\omega t)}$$

As previously, the fields are written in terms of longitudinal and transverse components. When decoupled;

Inside the dielectric with ϵ, μ ;

$$[\nabla_T^2 + \gamma^2] \begin{bmatrix} E_z \\ B_z \end{bmatrix} = 0$$

$$\gamma^2 = \mu\epsilon\omega^2 - k^2$$

Outside the dielectric with ϵ_0, μ_0 ;

$$[\nabla_T^2 - \beta^2] \begin{bmatrix} E_z \\ B_z \end{bmatrix} = 0$$

$$\beta^2 = k^2 - \mu_0\epsilon_0\omega^2$$

The equations are written so that the solutions represent a wave propagating inside the dielectric and an exponentially decaying wave outside the dielectric. The inside and outside solutions are matched at the dielectric boundary requiring that the tangential components of E and H are continious, as developed in the lectures on geometric optics. In a fiber-optic cable only a few modes are usually excited. A simple picture of the reflection of an optical wave at the dielectric boundary is shown in Figure 1. Upon total internal reflection there is a phase change which mixes the z components of the fields, and this somewhat complicates the solutions.

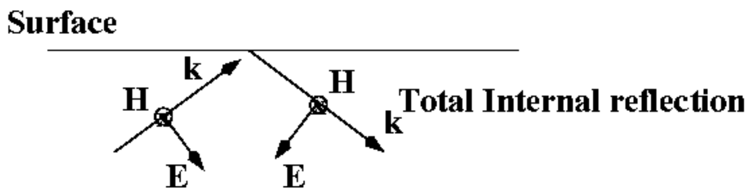

Figure 1: Reflection of an optical wave within a fiber optic element

2 Resonance

You are familiar with mechanical resonance. If a driving force is a harmonic of the natural modes (a periodic oscillation represented in Fourier harmonics) of a system, one or more of these modes will absorb energy from the driving force, amplifying the amplitude of the motion. Here consider a simple LRC circuit which exhibits resonance, Figure 2. The equation representing the current (charge) flow in the circuit is given by;

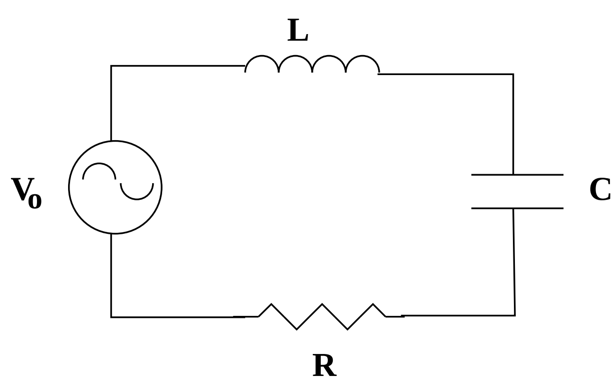

Figure 2: A simple LRC circuit exhibiting resonance.

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + q/C = V_0 e^{i\omega t}$$

There is a harmonic driving term, $V_0 e^{i\omega t}$. The natural modes of the above equation are obtained by looking for solutions with $V_0 = 0$. Choose a solution of the form, $q = Q_0 e^{i\omega t}$, so that the following algebraic equation results;

$$-\omega^2 L \,+\, i\omega R + 1/C \,=\, 0$$

This results in the natural mode;

$$\omega \,=\, i(R/2L) \,\pm \sqrt{1/LC \,-\, (R/2L)^2}$$

When R is small, the value of ω is approximately;

$$\omega \,=\, \sqrt{1/LC} \,+\, i(R/2LC)$$

This equation demonstrates oscillation with frequency, $\omega_0 \,=\, \sqrt{1/LC}$, and the amplitude is damped by a time constant $2LC/R$.

Now consider the inhomogeneous equation with a harmonic driving term. Again a solution of the form, $q \,=\, Q_0 e^{i\omega t}$, results in an algebraic equation;

$$-\omega^2 LQ_0 \,+\, i\omega RQ_0 \,+\, Q_0/C \,=\, V_0$$

The solution is;

$$Q_0 \,=\, \frac{V_0 C}{[(1-\omega^2 LC) \,+\, i\omega RC]}$$

The above equation shows that the amplitude is maximized when the driving frequency $\omega = \omega_0 \,=\, \sqrt{1/LC}$. Note the harmonic solution found above is the steady state solution (*ie* equilibrium solution). The energy in the circuit is obtained from the maximum energy stored in the capacitor during an oscillation cycle;

$$W \,=\, \frac{|q|^2}{2C} \,=\, \frac{V_0^2 c^2}{2C[(1-\omega^2 LC)^2 + \omega^2 R^2 c^2]}$$

The maximum energy as a function of ω is ;

$$\frac{dW}{d\omega} \,=\, 0$$

This results in;

$$\omega_0^2 \,=\, 1/LC \,-\, R^2/2L^2$$

For a given driving frequency, it is clear that the energy in the circuit will increase till it reaches an equilibrum described by the above equations. When this occurs, the energy into the circuit per cycle is lost as heat in the resistance. The equibrium power stored in the circuit as a function of the driving frequency, is shown in Figure 3.

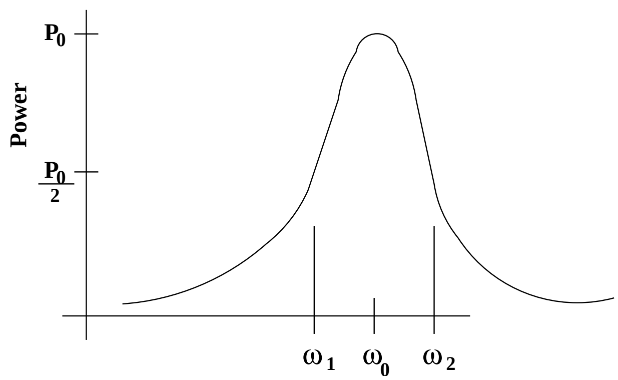

Figure 3: The stored power in the circuit as a function of the driving frequency

Define the quality factor of the circuit as;

$$Q = \frac{\omega_0 \,\text{Stored Energy}}{\text{Power Loss/Cycle}}$$

The stored energy is W and the power loss is $\frac{dW}{dt}$. Therefore

$$\frac{dW}{dt} = -\frac{\omega_0}{Q}\, W$$

$$W = W_0 e^{-\omega_0 t/Q}$$

Quality factors of a superconducting cavity can be greater than 10^7

3 Resonant cavities

Now change the variable in the last section from the charge in a circuit to the electric field in a cavity. Close the ends of a wave guide by conducting walls when $z = -d/2,\, d/2$. For a perfect conductor, this traps the fields inside the cavity. In the case of the TE mode of a wave guide, the fields are;

$$\vec{B}_T = \frac{ik\vec{\nabla}_T B_z}{\gamma}$$

$$\vec{E}_T = -\frac{i(\omega/c)\,\hat{z}\times\vec{\nabla}_T B_z}{\gamma}$$

$$B_z = B_0\, cos(\tfrac{m\pi}{a}\, x)\, cos(\tfrac{n\pi}{b}\, y)\, e^{i[kz\pm\omega t]}$$

$$\gamma^2 = \mu\epsilon\omega^2 - k^2$$

Now B_z must vanishe at the walls at $z = -d/2$, $d/2$. The solution is a superposition of waves traveling in the $\pm$ z directions so that it must have the form;

$$B_z \;=\; B_0\, cos(\tfrac{m\pi}{a}\, x)\, cos(\tfrac{n\pi}{b}\, y)\, cos(\tfrac{l\pi}{d}\, z)$$

The perpendicular component of $\vec{B}$ vanishes at the walls $z = -d/2$, $d/2$ which must occur at the surface of a perfect conductor. There are 3 eigenvalues given by $l\pi/d$, $m\pi/a$, $n\pi/b$. This means $k^2 \;=\; [\frac{l\pi}{d}]^2 + [\frac{m\pi}{a}]^2 + [\frac{n\pi}{b}]^2 \;=\; \mu\epsilon\omega^2$. The cavity has a resonant frequency given by the 3-D mode defined by, l, m, n. There is energy stored in the cavity, and if the walls are not perfectly conducting, there will be energy loss in the walls. Resonant behavior, as described by the equations in the last section, will occur. The energy is obtained from the energy density which is proportional to $|E|^2$. Using the definition of the quality factor and the equations which followed, the E field should take the form;

$$E(t) \;=\; E_0\, e^{-\omega_0 t/2Q}\, e^{-i(\omega_0 + \Delta\omega)t}$$

Here the frequency is allowed to be slightly off resonance. Break this equation into Fourier components using a Fourier transformation;

$$\tilde{\mathrm{E}}(\omega) \;=\; (1/\sqrt{2\pi})\, \int\, E_0 e^{-\omega t/2Q}\, e^{i(\omega-\omega_0)t}\, dt$$

$$|\tilde{\mathrm{E}}(\omega)|^2 \;\propto\; \frac{1}{(\omega-\omega_0)^2 + (\omega_0/2Q)^2}$$

When $\omega - \omega_0 - \delta\omega \;=\; \omega_0/2Q$ occurs, the amplitude is approximately 1/2 the resonant value at ω_0. Thus the full width at half maximum of the power curve equals ω_0/Q. This illustrates the quality factor can either be defined as;

$$Q \;=\; \frac{\omega_0\,\text{Stored Energy}}{\text{Power Loss/Cycle}}$$

or as ;

$$Q \;=\; \frac{\omega_0}{\omega_2 - \omega_1}$$

where $\omega_{1,2}$ are the half power points. The later equation is used most often in circuit analysis and the former for resonant cavities. Resonant cavities are useful for storing EM energy, filtering or amplifying a specific frequency.

4 The Diffusion Equation

Previously the solution for waves traveling in a conducting medium was found without a dissipative term. However a finite conductivity can be enterted in Maxwell's equations using

Ohm's law,
$vecJ = \sigma\vec{E}$. This results in the equation for the electric field;

$$\nabla^2\vec{E} - \mu\epsilon\sigma\frac{\partial\vec{E}}{\partial t} - \mu\epsilon\frac{\partial^2\vec{E}}{\partial t} = 0$$

Choose a solution of harmonic form (*ie* and apply a Fourier transformation on the electric field vector).

$$\vec{E} = E_0\, e^{i[kz-\omega t]}\,\hat{x}$$

Substitution into the equation and solving for the square of the wave vector gives the dispersion relation.

$$-k^2 + i\mu\sigma\omega + \mu\epsilon\omega^2 = 0$$

The solution shows that the wave vector is complex and is given by the following equation.

$$k^2 = \mu\epsilon\omega^2[1 + i\sigma/(\epsilon\omega)]$$

$$\vec{E} = \vec{E}_0\, e^{i[\alpha z-\omega t]}\, e^{-\beta z}$$

where $k = \alpha + i\beta$. Then identify;

$$\alpha = \omega\sqrt{\mu\epsilon}\left[\frac{\sqrt{1+(\sigma/(\epsilon\omega))^2} + 1}{2}\right]^{1/2}$$

$$\beta = \omega\sqrt{\mu\epsilon}\left[\frac{\sqrt{1+(\sigma/(\epsilon\omega))^2} - 1}{2}\right]^{1/2}$$

The absorption coefficient (approximately $1/\beta$ for sea water is shown in Figure 4. From the figure, penetration at any depth requires very low frequencies. Thus communication with submarines or oil exploration uses high power low fequency EM waves. This impacts the resulting resolution.

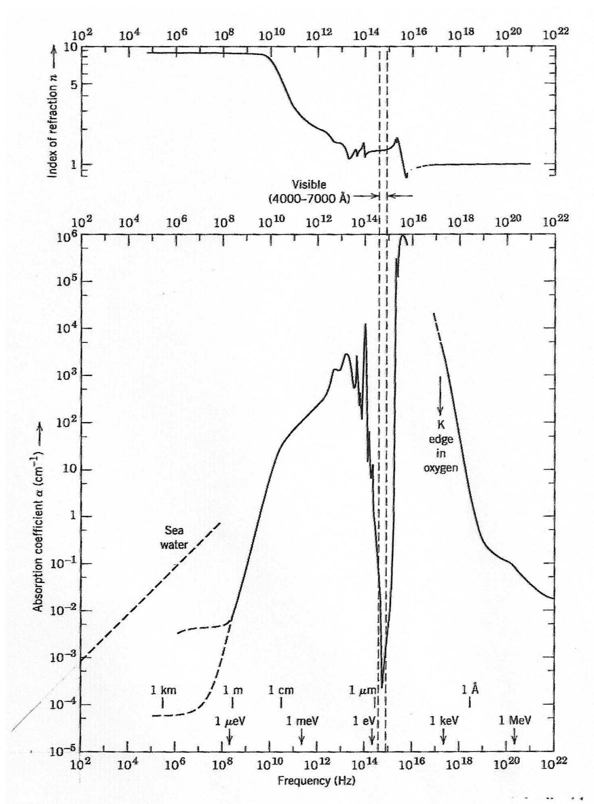

Figure 4: The index of refraction and the absorption coefficient for water. Note the value for sea water which is important for EM oil exploration/logging and ground penetration radar. The absorption coefficient in the plot, α (not the α in the text), is β in the above. Recall that the index of refraction $n = \sqrt{\epsilon\mu}/\sqrt{\epsilon_0\mu_0}$ and for σ large, $\beta = \sqrt{\mu\sigma\omega/2}$