

Scattering

1 Classical scattering of a charged particle (Rutherford Scattering)

Begin by considering radiation when charged particles collide. The classical scattering equation for this process is called Rutherford scattering, although the equation for the particle trajectory also describes the motion of any 2 particles interacting via a $1/r$ potential (eg bound states or even particles interacting gravitationally). To find the radiation, use the acceleration of the interacting charges, which means the particle trajectories accelerate subject to the Coulomb potential. Note that that radiation results in energy loss and therefore the trajectories change, although this energy loss is ignored in Rutherford scattering. The geometry is shown in figure 1. Here ignore the motion of the heavier particle, or use CM coordinates. (That is, assume a fixed scattering center.)

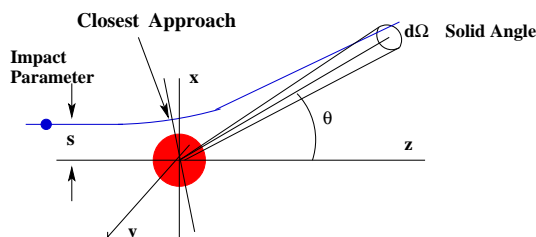


Figure 1: Motion of a particle in a central ($1/r$) potential.

The incident angular momentum is $L = mvs = s\sqrt{2mT_0}$, where m is the projectile mass, v its velocity, s its impact parameter, and T_0 its initial energy at at time $t = -\infty$. Let I equal the number of particles per unit area per second (incident flux). The number of particles in the ring between s and $s + ds$ are scattered into the ring between θ and $\theta + d\theta$. The definition of the cross section for scattering, σ , as the number of particles scattered per second divided by the incident flux. The cross section has units of area. In this case, the differential cross section is obtained by the following equation representing the fact that the number incident in the ring defined by the impact parameter, s , equals the number scattered into the ring defined by the solid angle, $d\Omega$.

$$[I 2\pi s ds] = -2\pi \sin(\theta) d\theta [I \frac{d\sigma}{d\Omega}]$$

Therefore the differential cross section is obtained by dividing by $\sin(\theta) d\theta$. The negative sign notes that as s increases the angle θ decreases.

$$\frac{d\sigma}{d\Omega} = -\frac{s}{\sin(\theta)} \frac{ds}{d\theta}$$

Assume a completely elastic collision (*ie* energy is conserved during the motion of the charge). Thus in spherical coordinates, the conservation of energy is given by;

$$(1/2)mr\dot{r}^2 + (1/2)\frac{L^2}{mr^2} + U = \text{constant} = T_0$$

The angular momentum, l for a central force (Coulomb) is also conserved. This is;

$$l = mr^2\omega$$

Exchange the variable of integration from t to θ .

$$d\theta' = \frac{l dr}{mr^2 \sqrt{(2/m)[t_0 - U_r - l^2/(2mr^2)]}}$$

This is integrated after using the Coulomb potential, $U = \frac{1}{4\pi\epsilon} \frac{zZe^2}{r}$. Here z and Z are the charge on the particle and scattering center, respectively, and e the electronic charge. The solution is;

$$(1/r) = -\frac{mqQ}{L^2}(1 + \eta \cos(\theta)')$$

This is the equation of a hyperbola or an ellipse depending on the sign of the energy. The excentricity, η , is

$$\eta = \sqrt{1 + \frac{8\pi\epsilon E_0 L^2}{mzZ(e)^2}}$$

Now change the angle θ' to the scattering angle θ as observed in Figure 2, where $\theta = \pi - 2\theta'$. This results in the equation;

$$\cot(\theta/2) = \frac{8\pi\epsilon T_0 s}{zZe^2}$$

$$s = \frac{zZe^2}{8\pi\epsilon T_0} \cot(\theta/2)$$

Substitution into the equation for the cross section gives;

$$\frac{d\sigma}{d\Omega} = \left[\frac{zZe^2}{4\pi\epsilon T_0}\right]^2 \left[\frac{1}{\sin^4(\theta/2)}\right]$$

In reality the scattering is not elastic as the charge is accelerated and loses energy. This loss could be determined by a perturbation series as it is small. However, to first order the trajectory does not change so the velocity and acceleration of the charge q can be found using the acceleration as determined from the equation of motion to get the radiated power.

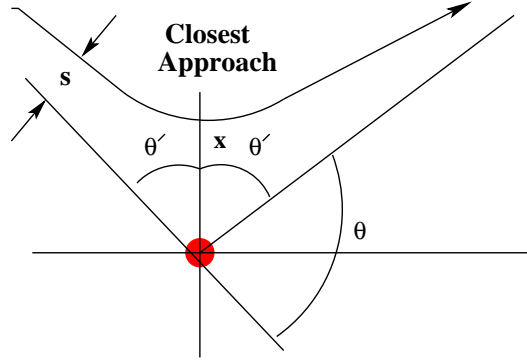


Figure 2: The geometry to convert the angle used to solve the equation of motion to the scattering angle

2 Solution to the scattering equation

In the above section the scattering of a particle from another particle when the interaction was the Coulomb force was obtained (represented as will be seen later, by the exchange of a virtual photon). In this section a general description of the scattering of a classical, real photon from a charge is obtained. Such scattering can be considered as a solution to the inhomogeneous wave equation. Although the scalar wave equation is used, recognize that a vector such as an EM field can be considered as a set of scalar components. The wave equation has the form;

$$[\nabla^2 - (1/c^2)\frac{\partial^2}{\partial t^2}]\psi = S(\vec{r}, t)$$

In the above, ψ is the wave amplitude and S the scattering center, or in this case the source of the scattered wave. In the case of EM scattering, the source of the scattered wave depends on the strength of the incident wave so rewrite this term as $S \rightarrow S\psi$. In the scattering region (*ie* as $r \rightarrow \infty$) the solution, ψ , consists of an incident wave plus an outgoing spherical wave. Scattering assumes that the the source term is localized so that sufficiently far away this term $\rightarrow 0$ and waves in this region are solutions to the homogeneous wave equation. Now also assume that the source and solution are harmonic in time, $\psi(\vec{r}, t) \rightarrow \psi(\vec{r})e^{i\omega t}$ which removes the time dependence of the equation. The time independent wave equation is then;

$$[\nabla^2 - k^2]\psi = S(\vec{r})\psi$$

The incident wave has the form of a plane wave solution;

$$\psi_{in} = e^{ikz} = e^{ikr \cos(\theta)}.$$

The scattered wave has the form of an outgoing spherical wave;

$$\psi_{out} \rightarrow f(\theta, \phi) \frac{e^{ikr}}{r}$$

Then the asymptotic form of the solution as $r \rightarrow \infty$ of the scattering equation is

$$\psi = \psi_{in} + \psi_{out}$$

with $R = |\vec{r} - \vec{r}'|$ and $\mathcal{L} = \nabla^2 - k^2$ operating on the wave amplitude, ψ , to obtain the source. Then formally define an operator \mathcal{L}^{-1} . The solution has the form at a time $t \rightarrow \infty$;

$$\psi = \psi_{in} + \psi_{out}$$

where ψ_{in} is the form of the incoming wave and ψ_{out} is the outgoing spherical wave. The wave equation can be written in operator form as;

$$\psi_{out} = \mathcal{L}^{-1} S \psi.$$

Formally write;

$$\psi_{out} = \frac{1}{1 - (\mathcal{L}^{-1} S)} \mathcal{L}^{-1} (S \psi_{in})$$

Then substitute for $\psi = \psi_{in} + \psi_{out}$ and collect terms. The formal solution is then written as;

$$\psi_{out} = [1 + \mathcal{L}^{-1} S + \mathcal{L}^{-1} S \mathcal{L}^{-1} S + \dots] \mathcal{L}^{-1} S \psi_{in}$$

The series converges if the source term is sufficiently small. Generally the scattered EM wave is much less than the incident wave so only use the first term, called the Born term, as the approximate solution. The differential scattering cross section is then defined by the scattered power into a solid angle $d\omega$ divided by the incident flux. The incident flux is obtained from the Poynting vector of the incident plane wave.

$$S_{in} = |E|^2 / \mu c$$

Thus the cross section in the Born approximation can be written;

$$\frac{d\sigma}{d\Omega} = \frac{(1/c\mu) |f(\theta, \phi)|^2}{(1/c\mu) |e^{ikz}|^2} = |f(\theta, \phi)|^2$$

Now a properly normalized outgoing Green's function for a point source can be written as

$\frac{e^{ikR}}{R}$ (ie see the Green function in a previous lecture). Therefore the solution in practical rather than in the above formal formal terms can be written ;

$$\psi_{out} = \int d\vec{r}' \frac{e^{ikR}}{R} S(\vec{r}') \psi_{in} + \int d\vec{r}' \frac{e^{ikR}}{R} S(\vec{r}') \psi_{out}$$

This is not a solution but an integral equation because ψ is not known until the solution is obtained, ie ψ_{out} term in the intergrand is unknown.

3 Scattering of the EM wave

Scattering generally occurs when the EM wavelength is large compared to the scattering source. EM Scattering occurs when some of the incident wave is absorbed and re-radiated. As previously, keep the lowest terms in the multipole expansion of the scattered fields. Almost always this results in dipole radiation as the incident wave polarizes the charges in the scattering center. If the incident wave has large amplitude, then it is possible that the charges do not move linearly with the incident E field, but bend due to the $\vec{v} \times \vec{B}$ force of the magnetic component in the incident wave. An intense incident wave would probably also require terms of higher order than the Born term in the perturbation expansion for the cross section. The scattering formulation occurs as follows (always assume a time dependence of $e^{i\omega t}$)

A plane, monochromatic, linearly polarized wave is incident on a scattering center. The incident wave is polarized in the \hat{x} direction and moves in the \hat{z} direction.

$$\vec{E}_{in} = \hat{x} E_0 e^{ik_{in} r \cos(\theta)}$$

$$\vec{B}_{in} = \hat{k}_{in} / c \times \vec{E}_{in}$$

These fields induce dipole moments (\vec{p}, \vec{m}) in the scattering center. The dipoles radiate energy. This radiation can be calculated in the multipole approximation using the static fields obtained from the multipole moments. The dipole fields are;

$$\vec{E}_{sc} = \frac{k^2}{4\pi\epsilon} \frac{e^{ikr}}{r} [\hat{n} \times \vec{p} \times \hat{n}] - \frac{z_0 k^2}{4\pi} \frac{e^{ikr}}{r} [\hat{n} \times \vec{m}]$$

$$\vec{B}_{sc} = (1/c) \hat{n} \times \vec{E}_{sc}$$

In the above, \vec{p} and \vec{m} are the induced electric and magnetic dipole moments.

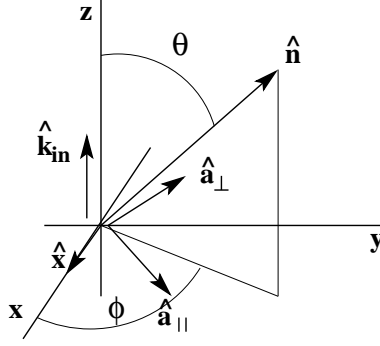


Figure 3: The geometry used to obtain the polarization vectors for scattering from a dielectric sphere

4 Dipole cross section for a dielectric sphere

Using the Poynting vector and the dipole fields obtained previously. The differential radiated power for dipole can be obtained.

$$\frac{d\sigma}{d\Omega} = \frac{r^2 |\hat{a} \cdot \vec{E}_{sc}|^2}{\hat{z} \cdot |\vec{E}_{in}|^2}$$

The unit vector \hat{a} defines directions perpendicular to the observation direction \hat{n} . Substitution for the field gives

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon)^2} |\hat{a} \cdot \vec{p} + (1/c)(\hat{n} \times \hat{a}) \cdot \vec{m}|^2$$

For a small dielectric sphere of radius, η , the electric dipole moment of the sphere is;

$$\vec{p} = 4\pi\epsilon_0 \left(\frac{\epsilon - 1}{\epsilon + 2}\right) \eta^3 \vec{E}_{in}$$

Use the geometry shown in Figure. 3. The cross section is obtained for the scattered polarization perpendicular \hat{a}_\perp and parallel, \hat{a}_\parallel to the scattering plane as defined by the plane containing the incident and scattering vector directions. From the figure;

$$\hat{a}_\parallel \cdot \hat{x} = \cos(\theta) \cos(\phi)$$

$$\hat{a}_\perp \cdot \hat{x} = -\sin(\phi)$$

Averaging over all possible initial polarizations (*ie* averaging over ϕ) the following result is obtained.

$$\frac{d\sigma_\parallel}{d\Omega} = \frac{k^4 \eta^6}{2} \left| \frac{\epsilon - 1}{\epsilon + 2} \right|^2 \cos^2(\theta)$$

$$\frac{d\sigma_{\perp}}{d\Omega} = \frac{k^4\eta^6}{2} \left| \frac{\epsilon - 1}{\epsilon + 2} \right|^2$$

The total differential cross section is;

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma_{\parallel}}{d\Omega} + \frac{d\sigma_{\perp}}{d\Omega}$$

The total cross section is obtained by integration over $d\Omega$.

$$\sigma_T = \frac{8\pi k^4\eta^6}{3} \left| \frac{\epsilon - 1}{\epsilon + 2} \right|^2$$

The polarization of the scattered wave is;

$$P(\theta) = \frac{d\sigma_{\perp}/d\Omega - d\sigma_{\parallel}/d\Omega}{d\sigma_{\perp}/d\Omega + d\sigma_{\parallel}/d\Omega} = \frac{\sin^2(\theta)}{1 + \cos^2(\theta)}$$

5 Form factor

Suppose the scatterers are a system of charges with fixed locations in space. The result is a superposition of scattering amplitudes from all the point scattering centers. As previously, coherence must be included when one adds the contributions from each scattering point. Assume that the positions of the scatters are located by the vectors, \vec{x}_j . The incident wave will have the phase factor, $e^{ik_0\hat{n}_0\cdot\vec{x}_j}$, and the phase for the scattered wave at large distances will be $e^{ik\hat{n}\cdot\vec{x}_j}$. Here \hat{n}_0 is the direction of the scattered wave. Assume dipole scattering and write;

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon)^2} \left| \sum_j [\hat{a} \cdot \vec{p}_j + (1/c)(\hat{n} \times \hat{a}) \cdot \vec{m}_j] e^{i\vec{q}\cdot\vec{x}_j} \right|^2$$

In the above write $\vec{q} = k\hat{n}_0 - k\hat{n} = k(\hat{n}_0 - \hat{n})$ for elastic scattering from massive points, (*ie* no recoil). Then suppose that the scatterers are identical so that $\vec{p}_j = \vec{p}$ and $\vec{m}_j = \vec{m}$. Rewrite the above as;

$$\frac{d\sigma}{d\Omega} = \left[\frac{d\sigma_s}{d\Omega} \right] \mathcal{F}(q)$$

where;

$$\frac{d\sigma_s}{d\Omega} = \frac{k^4}{(4\pi\epsilon)^2} [\hat{a} \cdot \vec{p}_j + (1/c)(\hat{n} \times \hat{a}) \cdot \vec{m}_j]^2$$

$$\mathcal{F}(q) = \left| \sum_j e^{i\vec{q}\cdot\vec{x}_j} \right|^2 = \sum_{jn} e^{i\vec{q}\cdot(\vec{x}_j - \vec{x}_n)}$$

For a random distribution of a large number of scattering centers the phase difference between the points causes the structure factor, $\mathcal{F}(q)$, to average to zero unless $j = n$. When

$j = n$, the sum results in $\mathcal{F}(q) = N$, the number of scattering centers. On the other hand if the scattering centers are distributed in an ordered array then $\mathcal{F}(q)$ is small unless $\vec{q} \approx 0$. This is coherent scattering. Thus consider the form in 1-D;

$$\sum_{jn} e^{i\vec{q} \cdot (\vec{x}_j - \vec{x}_n)} \rightarrow \sum_{jn} e^{iqx_j} e^{iqx_n}$$

Now let $x_j = jd$ where d is the spacing of the centers. The sum is converted to an integral by;

$$1 = [(j+1) - j] = (\delta j) = [x_{j+1} - x_j]/d = \Delta x/d$$

$$\sum_{j=-\infty}^{\infty} e^{iqx_k} = \sum (\delta j) e^{iqx_k} = (1/d) \int_{-\infty}^{\infty} dx e^{iqx_k} = \frac{\delta(q)}{2\pi d}$$

Finally suppose the centers are distributed by some function $f(\vec{r})$ so that $\vec{p}f(\vec{r})$ represents the electric dipole moment per unit volume (use the electric moment here but it is obvious how to include the magnetic moment). The scattering due to this distribution is ;

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon)^2} \left| \int d^3x' (\hat{a} \cdot \vec{p}) f(\vec{r}') e^{i\vec{q} \cdot \vec{x}'} \right|^2$$

which can be re-written;

$$\frac{d\sigma_s}{d\Omega} = \frac{k^4}{(4\pi\epsilon)^2} |[\hat{a} \cdot \vec{p}]|^2$$

$$\mathcal{F}(q) = \left| \int d^3x' f(\vec{r}') e^{i\vec{q} \cdot \vec{x}'} \right|^2$$

and

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma_s}{d\Omega} \mathcal{F}(q)$$

The structure function (form factor) is the Fourier transform of the spatial distribution of the scattering centers.

6 Expansion of a plane wave in spherical harmonics

The scattering problem is solved naturally in spherical coordinates, as the center of the coordinate system is chosen to lie at the center of the scattering source. Also the scattering solution is obtained in a multipole series as the scattering amplitude is observed as the distance from the scattering center $\rightarrow \infty$. However, a plane wave incident on the scattering center is formulated in Cartesian coordinates. Thus one must formulate the incident wave in a spherical system, written in terms of spherical harmonics. Write;

$$e^{ikz} = e^{ikr \cos(\theta)} = \sum C_n Y_n^0$$

Use the orthonormal properties of the spherical harmonic to write;

$$C_n = 2\pi \int d(\cos(\theta)) Y_n^0 e^{ikr \cos(\theta)}$$

Note that an integral representation of the spherical Bessel function, $j_l(kr)$, is

$$j_l(kr) = (-i)^l / 2 \int_0^\pi d(\cos(\theta)) e^{ikr \cos(\theta)} P_n(\cos(\theta))$$

Also use the addition theorem which has the form;

$$P_l(\cos(\theta)) = \frac{4\pi}{2l+1} \sum_m Y_l^m(\theta_1, \phi_1) Y_l^{*m}(\theta_2, \phi_2)$$

where the angles (1) and (2) refer to the angles of vectors \vec{k} and \vec{r} in the specified coordinate system. This gives the required form for the plane wave;

$$e^{ikr \cos(\theta)} = 4\pi \sum_{l,m} i^l j_l(kr) Y_l^m(\hat{k}) Y_l^{*m}(\hat{r})$$

This form for the incident wave can then be matched at the scattering surface to the field equations inside and outside the scattering center by matching the boundary conditions at the surface. Recall that a similar technique was used to obtain the reflection (scattering) and transmission coefficients of an EM wave incident on a dielectric and conducting surface planer surfaces. For example, this technique can be used to develop Mie scattering (light scattering from small dielectric particles).

7 Thompson scattering (*e.g.* Classical EM scattering from atomic electrons)

The power radiated by an accelerated charge is obtained from the Poynting vector.

$$\vec{S} = \vec{E} \times \vec{H} = \epsilon c |E|^2 \hat{n}$$

The radiation field of an accelerated charge when $\beta \ll c$ is ;

$$\vec{E} = \frac{q}{4\pi\epsilon} \left[\frac{\hat{n} \times \hat{n} \times \vec{\beta}}{R} \right]_{ret}$$

Thus the radiated power into solid angle $d\Omega$ is ;

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\epsilon c} [\hat{n} \times \hat{n} \times \vec{\beta}]^2$$

If a charged particle is accelerated by the \vec{E} field of an EM wave then;

$$\vec{E}_I = E_{0I} e^{i(\vec{k}\cdot\vec{r}-\omega t)} \vec{\eta}_0$$

$$\vec{F}(\text{force}) = m\vec{a} = mc\vec{\beta} = q\vec{E}_I$$

$$\vec{\beta} = (q/mc)E_{0I} e^{i(\vec{k}\cdot\vec{r}-\omega t)} \hat{\eta}_0$$

Ignore phase differences (take the instantaneous time) and let ($\hat{\eta}$ be the polarization direction, Figure 4.

$$\langle \frac{dP}{d\Omega} \rangle = \frac{c}{\epsilon} \left(\frac{q^2}{4\pi mc^2} \right)^2 |\hat{\eta} \cdot \hat{\eta}_0|^2$$

The incident flux is the time-averaged Poynting vector of the incident wave so the cross section is;

$$\frac{d\sigma}{d\Omega} = \frac{\langle (dP/d\Omega) \rangle}{\langle S_{in} \rangle}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{q^2}{4\pi\epsilon mc^2} \right)^2 |\hat{\eta} \cdot \hat{\eta}_0|^2$$

The evaluation of the factor $|\hat{\eta} \cdot \hat{\eta}_0|^2$ is obtained from Figure 4.

$$\hat{\eta}_1 = \cos(\theta)[\hat{x} \cos(\phi) + \hat{y} \sin(\phi)] - \hat{z} \sin(\theta)$$

$$\hat{\eta}_2 = -\hat{x} \sin(\phi) + \hat{y} \cos(\phi)$$

Choose $\hat{\eta}_0 = \hat{x}$ and sum over the final polarizations to obtain;

$$|\hat{\eta} \cdot \hat{\eta}_0|^2 = \cos^2(\theta) \cos^2(\phi) + \sin^2(\phi)$$

This result can be substituted above to obtain the scattered differential power. Note we have neglected the effect of the magnetic field on the motion of the accelerated charge, but this effect was worked out to first order in an earlier example.

8 Scattering of virtual quanta

This technique is an approximation to the interaction of charged particles, by viewing the field as a collection of virtual photons. The approximation works for photons that are almost

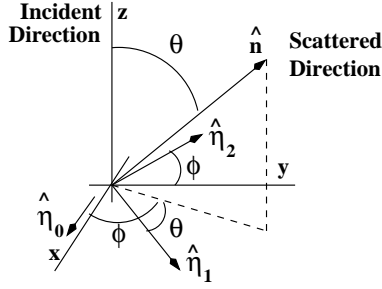


Figure 4: The geometry used to evaluate the angles in Thomson scattering

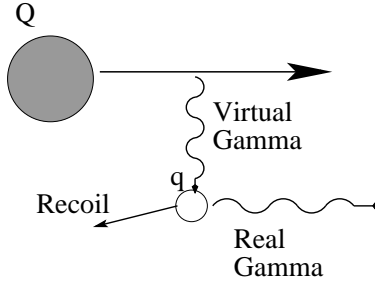


Figure 5: A schematic of the scattering of a photon from the field of a charge Q

“on the mass shell”, *ie* virtual photons that have mass nearly zero.

Consider the scattering of a charged particle from another charge system. The figure 5 is a schematic of the model to be described. Indeed the figure illustrates how bremsstrahlung could be described as the scattering of a virtual photon into one of real mass. Generally we would have a light particle scattering from a heavier one, but we use a frame in which the heavier mass is in motion and the lighter mass is at rest so that the motion of the heavier mass can be assumed to move with constant velocity during the collision. The lighter particle recoils and emits radiation. The relativistic factor for the movement of the charge Q is $\gamma \gg 1$. Now if $\gamma \gg 1$ the E and B fields are almost transverse. The fields at point P are;

$$E_1 = \frac{Q}{4\pi\epsilon} \frac{\gamma vt}{(b^2 + (\gamma vt)^2)^{3/2}}$$

$$E_2 = \frac{Q}{4\pi\epsilon} \frac{\gamma b}{(b^2 + (\gamma vt)^2)^{3/2}}$$

$$\vec{B} = \vec{V} \times \vec{E}$$

The equation for the \vec{B} is due to the fact there is no magnetic field in the rest frame of Q . The E field is just the static field of a charge Q a distance r' away from the point, Q . Apply a Fourier transform to the wave equations for the vector and scalar potentials;

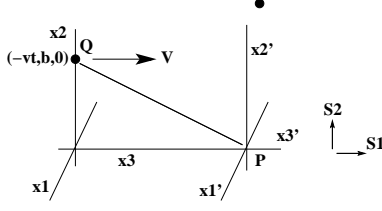


Figure 6: The geometry uses for calculating the EM impulse due to the EM field

$$[k^2 - \omega^2/c]V(k, \omega) = (-1/\epsilon)\rho(k, \omega)$$

$$[k^2 - \omega^2/c]\vec{A}(k, \omega) = (-\mu)\vec{J}(k, \omega)$$

Then let $\rho = Q\delta(\vec{x} - \vec{V}t)$ and $\vec{J} = \vec{V}\rho$. Apply a Fourier transform to obtain

$$\rho = \frac{Q}{2\pi} \delta(\omega - \vec{k} \cdot \vec{v})$$

The potentials then have the form;

$$V = -\frac{Q}{2\pi\epsilon} \frac{\delta(\omega - \vec{k} \cdot \vec{v})}{k^2 - \omega^2/c^2}$$

$$\vec{A} = \vec{v}/c^2 V$$

The fields are then obtained from $\vec{E} = \vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}$ and $\vec{B} = \vec{\nabla} \times \vec{A}$

This results in;

$$\vec{E}(\vec{k}, \omega) = i(\omega\vec{v}/c^2 - \vec{k})V$$

$$\vec{B}(k, \omega) = i/c^2(\vec{k} \times \vec{v})V$$

To get the fields at a distance perpendicular b to the particle path;

$$\vec{E}(b, \omega) = \frac{1}{(2\pi)^{3/2}} \int d^3k \vec{E}(k, \omega) e^{ik_x b}$$

Integration gives

$$E_z(\omega) = -\frac{i\omega Q}{V^2\epsilon} \sqrt{2/\pi}(1 - \beta^2)K_0(\lambda b)$$

$$E_x(\omega) = \frac{Q}{v^2\epsilon} \sqrt{2/\pi} (\lambda/\epsilon) K_1(\lambda b)$$

where $\lambda = (\omega^2/v)[1 - \beta^2]$, and K_i are modified Bessel functions. The fields have a spectrum of frequencies. We can calculate a Poynting vector;

$$\vec{S} = (1/\mu)\vec{E} \times \vec{B}$$

This can be written in terms of the Fourier transforms of E and B as ;

$$W/Area = \int_{-\infty}^{\infty} d\omega E(\omega) B^*(\omega)$$

However the frequencies for E and B must be the same, *ie* time correlated. This is true for E_2 and B_3 which gives an electromagnetic pulse of photons in the S_1 direction. However, E_1 and B_3 are not. We arbitrarily add a magnetic field so that $\vec{B}_3 \rightarrow [B_3 + E_1]\hat{x}_3$. This generates a pulse S_2 . Note that this B field has no effect on the static problem and in any event the pulse S_1 dominates the impulse.

There are then two photon flux terms;

$$\left(\frac{\frac{d^2 I_1}{d\omega d area}}{\frac{d^2 I_2}{d\omega d area}} \right) = \frac{Q^2 c}{4\pi^3 \epsilon v^2 b^2} \left(\frac{(\omega b/\gamma v)^2 K_1(\omega b/\gamma b)}{(\omega b/\gamma v)^2 K_0(\omega b/\gamma b)} \right)$$

Then integrate over the impact parameters;

$$\frac{dI}{d\omega} = 2\pi \int_{b_{min}}^{\infty} b db \frac{d^2 I_1}{d\omega d area}$$

In the above the total value of the Power spectrum $I_1 + I_2$ is used. This results in;

$$\frac{dI}{d\omega} = \frac{Q^2 c}{2\pi^2 \epsilon v} [\chi K_0(\chi) K_1(\chi) - (\beta^2/2)\chi^2 [K_1^2(\chi) - K_0^2(\chi)]]$$

with $\chi = \frac{\omega b_{min}}{\gamma v}$. The minimum impact parameter can be obtained from the uncertainty relation $b_{min} q_{max} = \hbar$ with $q_{max} = |\vec{P}_I - \vec{p}_F|$

$$\frac{d\sigma}{d\Omega} = \left(\frac{q^2}{(4\pi)^2 \epsilon m c^2} \right)^2 [1/2(1 + \cos^2(\theta))]$$