Models
Lecture 18

1 Introduction

In the next series of lectures we discuss various models, in particluar models that are used to describe strong interaction problems. We introduce this by discussing the hadronic force in terms of the exchange of mesons. Mesonic exchange basically gives the long range behavior of the interaction, $\sim$ 1 fm or greater. Shorter range forces would be described by gluon exchange between the quarks (QCD) but as we know QCD cannot be easily solved. Thus models can help develop physical intuition of these process, and good models contain the power to predict system behavior. Therefore instead of QCD introduce models using the generic term of Quantum Hadro-Dynamics (QHD). Some QHD models have been used in one form or another since the foundation of hadronic physics.

2 The hadronic many body problem

In general, the description of interacting hadrons via QHD is also a complex problem. Even when describing the interaction of only 2 hadrons, the strength of the interaction is sufficient to prevent pertubation calculations in most cases, and due to the interaction of virtual mesons, the description requires a multi-body system of strongly interacting particles. However, models can be used to simplify the system, hopefully capturing the essence of the physics.

Begin looking at the interaction of nucleons at non-relativistic energies. The nucleon mass is approximately 939 MeV and a nucleon should be treated relativisticaly if its kinetic energy is close to its rest mass. Binding energies of nucleons are of the order of a few to a few 10's of MeV, and a nucleon-nucleon potential formulation has a well depth of approximately 50 MeV. Thus nuclear systems can be treated to good approximation by non-relativisitic kinematics. However, the nuclear potential is obtained from the difference between vector and scalar meson exchange, both of which are large. In addition, spin is a relativistic effect, so that relativity cannot be completely ignored.

To expand on this subject, nucleons are composed of massless quarks which interact by gluon exchange. At short range (high energy) the interaction is weak and at long range, (low energy and comparable to the range of the nuclear force -$\sim$ 1 fm) the interaction is very strong. Conversely, QHD is mediated by meson exchange ($q\overline{q}$ pairs) and meson exchange can be used to model the long range, low density interactions. In a non-relativistic interaction, the interaction force can be described in terms of the deriviatve of a potential function. Simple, non-relativistic potentials in common use are shown in Figure 1.

An idealized QHD system is nuclear matter, which is formed of equal numbers of protons and neutrons without the Coulomb interaction. The baryon number of this system $\rightarrow \infty$, *i.e.* surface effects are neglected. A solution using a realistic potential provides an av-

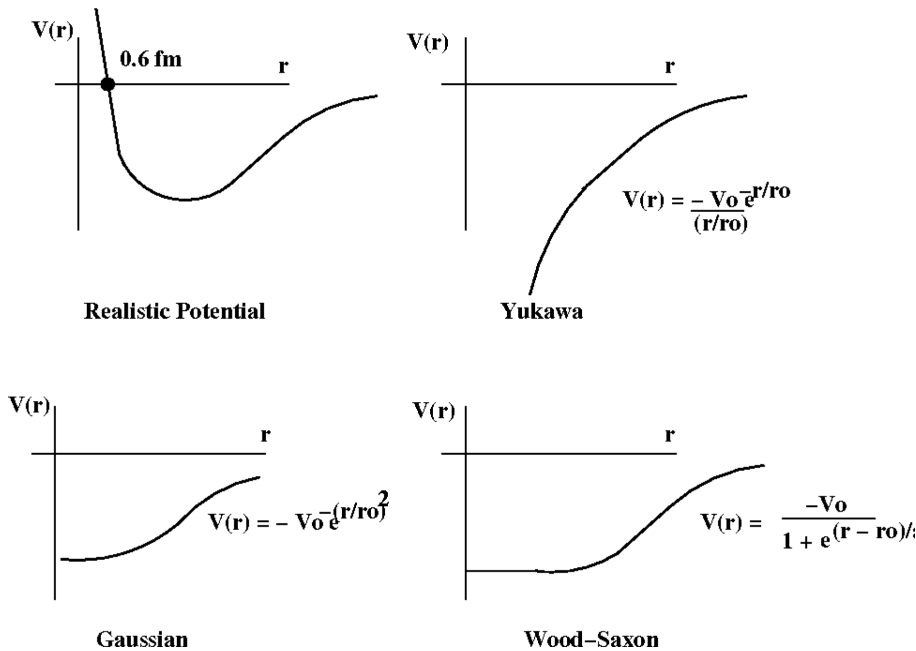

Figure 1: Common potentials used to describe the hadron-hadron interaction

erage binding energy and a nuclear density for ∞ nuclear matter. Values of these variables can be compared to those found in a heavy nucleus such as Pb.

$$\langle B_E \rangle / A \approx -15.75 \text{ MeV}$$

$$\langle \rho \rangle \approx \frac{2k_F^3}{3\pi^2} \; k_k \approx 1.36 \pm 0.06 \text{ fm}^{-1}$$

Here ρ is the density in terms of the Fermi Momentum. From this calculation we see that the nuclear force is relatively independent of the number of interacting nucleons. Thus we think of nuclear matter as an incompressible liquid, perhaps similar to a condensed gas.

Because the energy per nucleon is approximately stable, the nu-

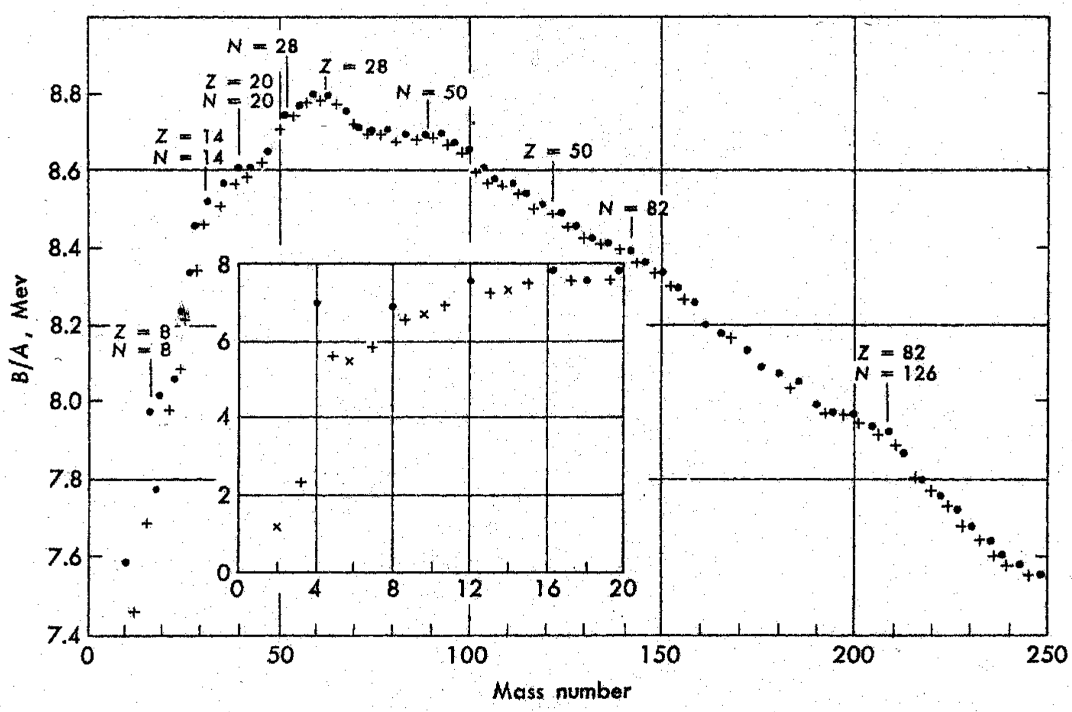

Figure 2: Common potentials used to describe the hadron-hadron interaction

clear force saturates. If each nucleon interacts with another through an attractive force due to meson exchange, the potential energy would be proportional to the number of interacting pairs $[A(A-1)]/2 \approx A^2$. But B/A is approximately constant, so that each nucleon interacts with only a few others. From the figure, saturation occurs at about 10 particles. In addition, if the force were entirely attractive, the nucleus would collapse, so there must be a repulsive force in addition to Pauli exclusion. This short range behavior is called the repulsive core of the potential.

3 Averaged nuclear parameters

For finite nuclei, the surface is important, and in addition to the Coulomb force, a few other properties of the interaction must be included in any realistic model. Recall that the nuclear radius is given by $r = 1.2A^{1/3}$. The proton radius is approximately 0.77 fm, so that a nucleus is almost a close-packed array. Obviously the hard core is "very" hard so that the nucleus is nearly incompressible. The original nuclear model was that of an incompressible fluid, and this model is still used in a slightly different form, as we will later see.

4 Liquid drop model

The first model of a system of interacting nucleons was that of an incompressible fluid. The model can only describe average properties of the nuclear system, such as its geometry, density and binding energy. It has features that are easily understood from classical physics. One of the most important successes is the development of the semi-empirical mass formula, Figure 3.

5 The semi-empirical mass formula

In a finite nucleus, the Bethe-Weisäcker semi-empirical mass formula is written;

$$B_E = -a_1 A + a_2 A^{2/3} + a_3 \frac{Z^2}{A^{1/3}} + a_4 \frac{(A-2Z)^2}{A} + a_5 \frac{\lambda}{A^{3/4}}$$

In this equation, A is the number of nucleons, and Z is the number of protons. The various terms are ;

$a_1 = 15.75$ MeV The volume term

$a_2 = 17.8$ MeV The surface term

$a_3 = 0.71$ MeV The Coulomb term

$a_3 = 23.7$ MeV The symmetry term

$a_5 = 34.$ MeV The paring term

Each term in the formula is justified by the liquid drop model. The volume energy was described in the previous section, and the Coulomb energy is obvious. The surface energy can be viewed as a

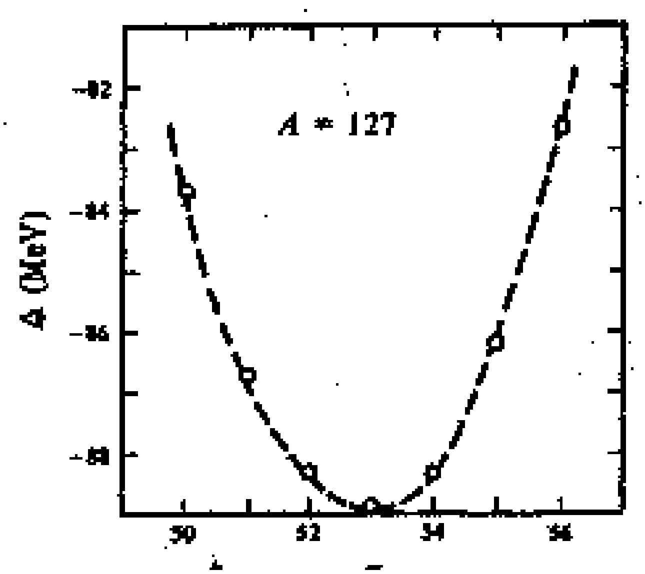

Figure 3: A plot of the semi-empirical mass for A = 127 showing the most stable system

surface tension resulting from attraction of the interior nucleons. The symmetry energy attempts to make the number of neutrons equal to the number of protons, and the paring energy term has $\lambda = +1$ for odd-odd nuclei, 0 for odd-even nuclei, and -1 for even-even nuclei (*i.e* it recognizes that like nucleons wish to form pairs). The coefficients of each of these terms is an average of vaues obtained by fitting to a wide range of masses.

6 Fermi momentum

Nucleons within the nucleus must be in motion, as we know from the uncertainty principle. Thus if the nucleons are bound within a potential well, they must have a momentum, $\delta p = \hbar/\delta x$. To find the Fermi momentum, consider a 1-D wave function;

$$\psi = A\, cos(kx)$$

Then constrain the wave function to dimensions $-a \leq x \leq a$ so that $\psi = 0$ when $x = \pm a$. This means that

$$k = \frac{(2n+1)\pi}{2a}$$

auli principle allows 1 particle per quantum number. Write;

$$\Delta k = \frac{2\pi}{2a}\, \Delta n$$

In a similar way one can obtain the change in k for each dimension. Thus for 3-D space;

$$\Delta n_x \Delta n_y \Delta n_z = \frac{(4\pi)k^2 dk}{(2\pi)^3}\, (Volume)$$

The volume comes from multiplying the box sides $2a$ together for each dimension. Assume 2 protons and 2 neutrons occupy a bin (spin up and down for each nucleon). The number density is therefore;

$$number/Volume = \frac{2k^2 dk}{\pi^2}$$

Integration over the total momentum k from zero up to the Fermi momentum level is;

$$\rho = \frac{2}{\pi^2} \int_0^{k_F} dk\, k^2 = \frac{2}{3\pi^2}\, k_F^3$$

Now the number of particles per volume is $\frac{A}{(4\pi/3)(1.2A^{1/3})^3} = 1.4 \times 10^{38}$ nucleons/cm^3 The energy density (Energy/Volume with 939 MeV the nucleon mass)

$$Energy/Volume = (1.4{\times}10^{38})(939)(10^{-13})^3 = 130\,\mathrm{MeV/fm^3}$$

This must be related to the number density previously obtained

$$Energy\,Density = \frac{2}{3\pi^2}k_F^3(939)$$

This gives $k_F = 1.3$ fm^{-1}

7 Fermi energy

Because we consider a system at zero temperature, the lowest energy states are filled and the Pauli principle is applied so that only 2 protons and 2 neutrons (one for each spin state) occupy one bin in momentum space. The Fermi energy is the energy of the highest occupied energy level after all nucleons have been placed in the lowest available energy states. In the case of nucleons in a nucleus this is approximately;

$$E_F = [\frac{(\hbar\pi)^2}{2mV^{2/3}}]\,n_F \approx 30MeV$$

Here n_F is the number of nucleons per unit volume. The Fermi energy can be related to the Fermi temperature of the system given by

$T_F \; = \; E_F/k$

Here k is the Boltzman constant and T_F the Fermi temperature.

8 Fermi gas model

The above description leads to another model, somewhat connected to the liquid drop model. This model of a many nucleon system is composed of a degenerate gas of non interacting nucleons. It is a useful concept for electrons in a metal or in a neutron star. The model is also applied to nuclear interactions at energies much higher than the Fermi energy (quasi-free scattering). The kinetic energy of the nucleons in a degenerate gas at zero temperature, is the sum of the proton and neutron energies. Fermi levels for neutrons and protons in a nucleus are shown in the figure.

For the proton;

$$E_0 \; = \; \frac{2Volume}{(2\pi\hbar)^3} \int_0^{p_F} dp \, 4\pi (\frac{p^2}{2m}) \, p^2 \; = \; \frac{\pi^{4/3} 3^{5/3} \hbar^2}{10m} \, (\frac{Z}{V})^{2/3} \, Z \; = \; (3/5) E_F Z$$

The zero energy value for the neutron is similar. The total energy of the nucleus is then;

$$E_T - E_0 \; = \; 0.08 A^{2/3} (Z^{1/3} + (A - Z)^{1/3}) (kT)^2$$

For $A \; = \; 100$, $Z \; = \; 44$ and $E_T \; = \; 11(kT)^2$ then $kT \; = \; 1 MeV$

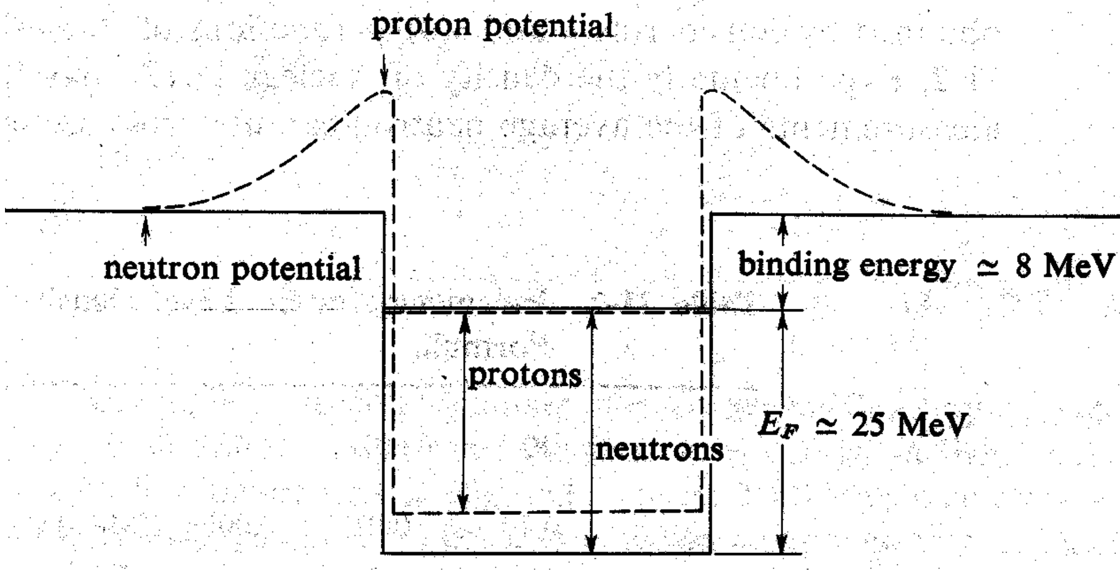

Figure 4: Fermi levels for neutrons and protons in a heavy nucleus

corresponds to $E_T - E_0 = 11$ MeV. Each degree of freedom of the gas would have a kinetic energy of $1/2kT$. For $A = 100$ there are $3 \times 100 = 300$ degrees of freedom so that $E_T - E_0 = 300 MeV$.

9 Mean Field

From comparison of data to the above models we learn that;

1. The nucleons are held together in the nucleus by a long range force generated by pion (and other meson) exchange.
2. The repulsive core of this potential and the Pauli principle keeps the nucleus from collapsing

3. The repulsive core and a sufficiently weak attractive well allow the nucleons to have equilibrium positions providing a density of approximately 1 nucleon/fm.
4. The binding energy saturates indicating that a nucleon interacts with only a few neighboring nucleons.

We then presume that the nucleus is defined by a non-relativistic potential acting between pairs of nucleons. The Hamiltonian is ;

$$H = \sum_i T_i + \sum_{i \neq j} V_{ij}$$

with solution ψ. Obviously this is an equation depending on 3 spatial coordinates for each of the nucleons, and is usually much too complicated to solve. We attempt to find a solution by the self-consistent Hartree-Foch method is used to solve the many-electron atom problem. Thus we assume that the solution has the form;

$$\psi = \prod^i \psi_i$$

The above form must be appropriately anti-symmetrized but we leave it in this form for convenience. Then we average the two body potential for particle, i, over all the other particles, j, to obtain a meanfield potential for the particle i. Thus the mean field potential is written;

$$V_i = \sum_{j \neq i} \langle \psi_j | V_{ij} | \psi_j \rangle$$

We then solve the Hamiltonian;

$$H_i = T_i + V_i$$

for each wave function ψ_i and iterate this set of equations until self consistancy is obtained. In general the idea is to obtain a mean-field potential in which the nucleons move. The solution can then be further refined by using these wave functions to diagonalize a full potential Hamiltonian. Below we simply use the extreme single particle model, where a nucleon moves in a central potential field created by all of the other nucleons. A set of energy levels for the single particle model is given in Figure 5 below. It shows solutions for the extreme cases of a harmonic oscillator and square well potentials, as well as a diagonalization for a spin-orbit potential.

10 Magic numbers and the spin-orbit term

In the case of atomic structure, shell closure is responsible for unusual stability or reactivity of the nuclei near certain atomic numbers. In the case of nucleons, there appears binding energy gaps (separation energy) near mass numbers 2, 8, 20, 28, 50, 82, 126 for both neutrons and protons. These represent the closure of shells for these nucleons, but this closure does not generally correspond to the shell closure calculated by a common potential forms. To account for this experimental difference it is necessary to introduce a strong spin-orbit coupling, ($V_{s0} = -A(\vec{s} \cdot \vec{L})$ where $\vec{s}$ is the nucleon spin and $\vec{L}$ is its angular momentum). Of course the Coulomb potential also affects proton binding. Adjusting the strength of the spin orbit potential can reproduce the experimental numbers. While the spin orbit term was introduced here by hand, this term would come naturally in a relativistic formulation.

11 Collective model

The treatment of the nucleus as a continuium structure of nuclear matter was extended to include a quantum description of the classical motion of a liquid. This allowed the description of excited states in terms of rotations and vibrations. Any quantum drop can experience vibrational excitations, but only a deformed nucleus can have rotational excitations. Note above that nuclei near the so-called magic numbers are spherical. However, nuclei can have deformed ground state structure if the number of nucleons is far from a magic number. A schematic potential well is shown in Figure 6, illustrating a minimum in the potential (binding energy) as the deformation parameter is varied.

Rotation of the deformed system adds an energy that is classically equal to $\mathcal{I}\omega^2/2$ where $\mathcal{I}$ is the rotational inertia, and ω the rotational velocity. This can be connected to the angular momentum of the rotation, so that quantum mechanically the excitation energy is;

$$E_R = (\hbar^2/2\mathcal{I})L(L+1)$$

The rotational inertia $\mathcal{I}$ is only due to the component of the nucleus that is rotating, and this is just the small component of the nucleus that is involved with the deformation, Figure 7.

This raises the issue of reconciling the single particle model which describes excitations of a nucleus as due to a nucleon moving in a mean field, with the collective model which describes excitations in terms of the collection of nucleons. The connection is made by understanding that the collective model actually describes the motion of only a few nucleons around a nuclear core.

A representative set of nuclear excitations is shown in Figure 8. Note the energy scale. Core excitations are due to resonant motion of neutrons against protons.

Radial vibrations (breathing modes) measure nuclear compressibility.

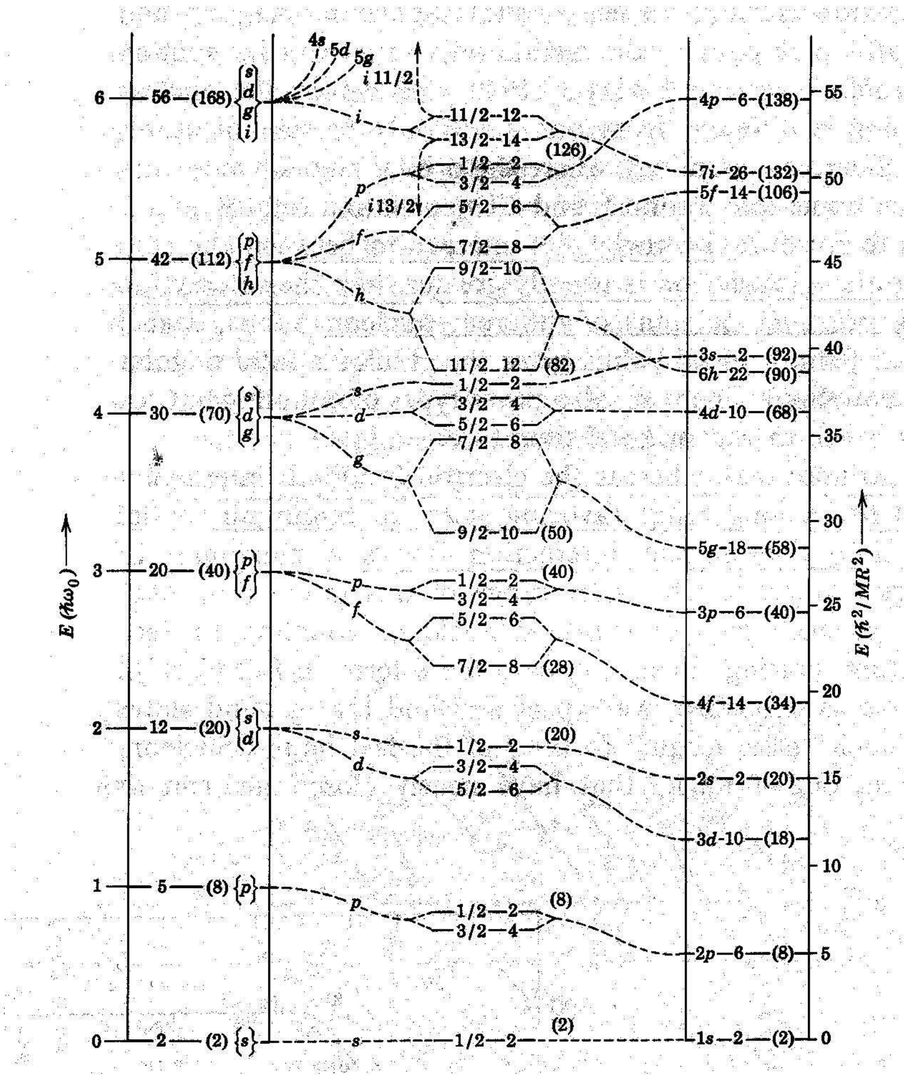

Figure 5: Single particle shell model states. Left is harmonic oscillator, right is square well, enter shows splitting due to a spin-orbit term

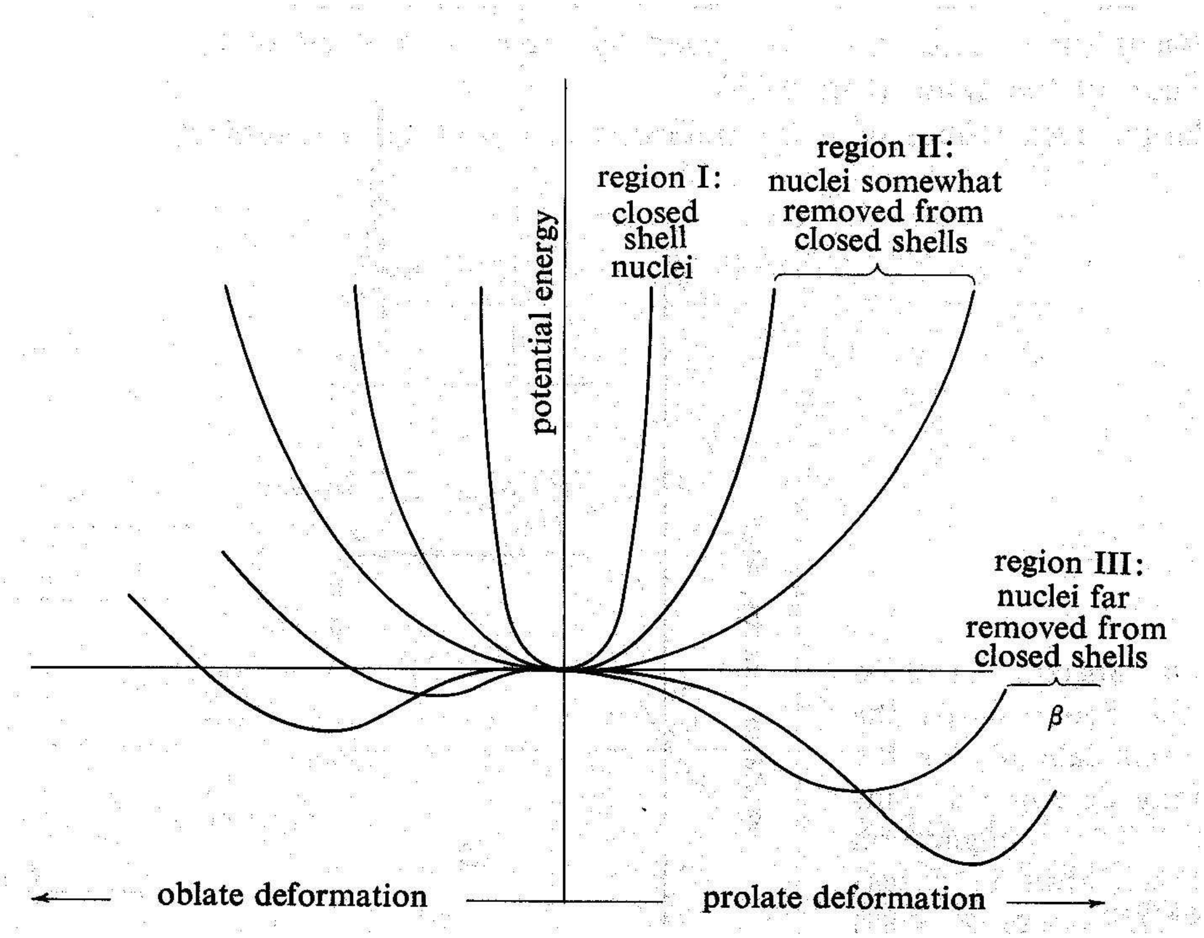

Figure 6: Potential energy as a function of deformation

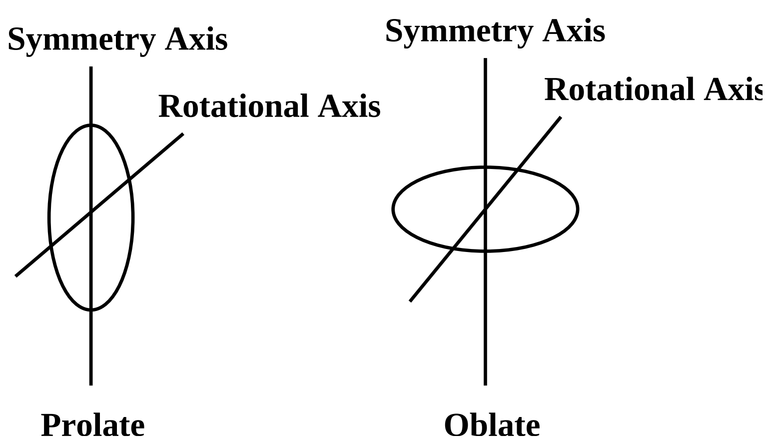

Figure 7: Geometric shapes and rotation of nuclei in the collective model

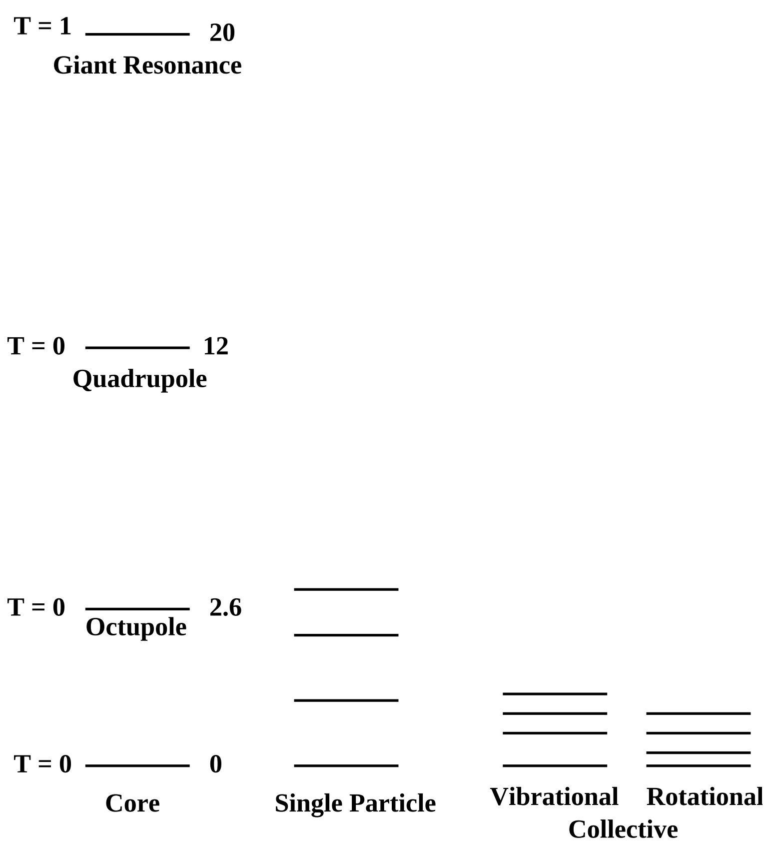

Figure 8: A schematic of models and energy levels predicted by various models