Neutron Star)
Lecture 22

1 Neutron star

A neutron star is a stellar object held together by gravity but kept from collapsing by electromagnetic (atomic) and strong (nuclear including Pauli exclusion) forces. As the radiation pressure in a star decreases, the star begins to collapase increasing its matter density. Star collapse may result in a supernova, and if the radius of the supernova remnant reaches the Schwarzschild limit, it becomes a black hole. If not, the gravitational pressure still becomes so large that a lower energy state can be obtained by applying gravitational energy, changing some protons into neutrons using the inverse Beta decay process, $p + e^- \rightarrow n + \nu_e$ and $p \rightarrow n + e^+ + \nu_e$ and decreasing the star radius. The later process is suppressed by Fermi pressure. Proton decay provides release from electromagnetic repulsion between the protons and Pauli exclusion (Fermi pressure) which would require the protons to occupy higher momentum states. Depending on the mass, the remnant may reach an equilibrated radius above the Schwarzchild limit to form a neutron star. The properties of a neutron star are described by the neutron matter equation of state which is obtained from a relativistic mean-field theory.

2 Equation of state

The hadronic equation of state in relativistic mechanics is obtained from a thermodynamic model. The energy-momentum tensor, $T^{\mu\nu}$, for an isotropic fluid without shear forces and moving with velocity, u^{μ}, is;

$$T^{\mu\nu} \; = \; Pg^{\mu\nu} + (\rho + P/c^2)u^{\mu}u^{\nu}$$

In the above, P is the pressure, ρ is the fluid density, and u^{μ} is the 4-vector velocity of the fluid flow. The system is assumed to lie in a spherically symmetric, static, coordinate system. The Schwarzchild metric describes this geometry.

$$g_{\mu\nu} \; = \; \begin{bmatrix} A(r) & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\, sin^2(\theta) & 0 \\ 0 & 0 & 0 & -B(r) \end{bmatrix}$$

The components of the Einstein equation are then given by;

$$R^{\mu\nu} = \kappa(T^{mu\nu} - (1/2)T^{\lambda}_{\lambda}g^{\mu\nu})$$

Inside the spherical source, the Schwarzchild metric gives;

$$S^{\mu\nu} \; = \; (T^{mu\nu} - (1/2)T^{\lambda}_{\lambda}g^{\mu\nu}) \; = \; 0$$

$$S^{rr} \; = \; (1/2)(\rho c^2 - P)A(r)$$

$$S^{\theta\theta} = (1/2)(\rho c^2 - P)r^2$$

$$S^{\phi\phi} = (1/2)(\rho c^2 - P)r^2\, sin^2(\theta)$$

$$S^{44} = (1/2)(\rho c^2 + 3P)B(r)$$

The above equations are valid for a static, spherically symmetric fluid with no shear forces. The equation of state is $P = P(\rho)$ with P the presssure and ρ the density. Energy and momentum are conserved, and there is a conserved baryon number obtained by taking the co-varient derivative of the 4-vector current; $N^\mu = (n(x)/c)u^\mu(x)$ with $n(x)$ the particle density.

3 Tolman-Oppenheimer-Volkoff (TOF) equation

Using the equations with the metric of the last section, the Einstein equations are obtained. The equation of state, $P = P(\rho)$, $P = P(\rho)$ is then added to these equations. This results in 3 linearly independent equations with the unknowns $A(r)$, $B(r)$, and $\rho(r)$. The solution outside the source is clearly the Schwarzchild metric, since in this case $\rho = 0$.

After much algebra, the result of combining these equations is;

$$\frac{dP}{dr} = -\frac{G\mathcal{M}(r)\rho(r)}{r^2}[\frac{1 + P(r)/\rho(r)c^2}{1 - 2G\mathcal{M}(r)/c^2 r}][1 + \frac{4\pi P(r)r^3}{\mathcal{M}(r)c^2}]$$

$$\mathcal{M} = \int_0^r ds\, 4\pi s^2 \rho(s)$$

The above is the TOV equation which is a non-linear integro-differntial equation for the pressure in terms of the mass density, $\mathcal{M}$. As previously the Newtonian limit is obtained by letting $c \to \infty$. These equations apply to static, condensed objects such as neutron stars. The equations must be solved numerically, given a set of initial conditions. The equation of state for nuclear matter is then developed.

4 Relativistic mean field for nuclear matter

Neutron stars are gravitationally held massive objects in β-equilibrium (no weak decay). They have radii of about 12 km and masses of $(1-2)M_\odot$ to $2.5M_\odot$. Here $M_\odot$ stands for a solar mass. Their composition is dominated by neutrons. A relativistic mean field (RMF) calculation provides an equation of state to use in the TOV equation, developed in the last section. RMF is an effective field theory which has been used in applications of many-body interactions in nuclear physics.

The RMF equations use a two component baryon field involving protons (p) and neutrons (n);

$$\psi = \begin{pmatrix} p \\ n \end{pmatrix}$$

To this field a neutral scalar field, ϕ, couples with coupling constant,

g_s, to the baryon density $\bar{\psi}\psi$ (as for example a pion), and a neutral vector field, V_μ, couples to the conserved baryon current, $i\bar{\psi}\gamma^\mu\psi$, with coupling constant, g_v. Scalar coupling provides long range nuclear attraction and vector coupling provides hard-core repulsion in the interaction potential. This model has the following set of simplified equations.

$$\frac{\partial}{\partial x_\nu} V_{\mu\nu} + m_{mu}^2 \; = \; i g_\nu \bar{\psi} \gamma_\mu \psi$$

$$[\nabla^2 - \frac{\partial^2}{\partial (ct)^2}]\phi \; = \; -g_s \bar{\psi}\psi$$

$$[\gamma_\mu(\frac{\partial}{\partial x_\mu} - i g_\nu V_\mu) + (M_b - g_s\phi)]\psi = 0$$

The vector field tensor in the above is $V_{\mu\nu} = \dfrac{\partial V_\nu}{\partial x_\mu} - \dfrac{\partial V_\mu}{\partial x_\nu}$.

Note the first equation is Maxwell's equations with the vector potential replaced by a massive vector meson field. The second equation is the Klein-Grodon equation or a scalar field with the scalar baryon density as the source. The third equation is the Dirac equation for the baryon field with simple, minimal couplings. The equations need to be quantized and solved, but this is much too complex to describe here. In essence the spacial volume is uniform, but the density is time dependent. Thus average over spacial dimensions to obtain the expectation values and allow the densities to be spacially independent constants. In addition, define an effective mass by $M_{ef} = M_b - g_s\phi_0$ and allow a degeneracy factor $n = 4$, which applies Fermi exclusion only allowing 2 protons and 2 neutrons, each with opposing spins, to occupy a given state of the system. The density is therefore given by;

$$\rho_B \;=\; \frac{n}{(2\pi)^3}\int_F^k dk^3 \;=\; \frac{nk_F^3}{6\pi^2}$$

$$\mathcal{M} \;=\; \int^R ds\, 4\pi s^2 \rho(s)$$

$$V_o \;=\; \frac{g_v}{m_v^2}\,\rho_B$$

The energy density is the expectation value of the Hamiltonian (spacial average) and the system is allowed to minimize the energy so the system is in equilibrium.

$$\frac{\partial\epsilon}{\partial\phi_0}|_{\rho_m} \;=\; 0$$

The result is;

$$(\frac{m_s}{g_s})^2(M_b - M_{ef}) \;=\; \frac{n}{(2\pi)^3}\int^{k_F} \frac{M_{ef}}{(\vec{k}^2 + M_{ef})^{1/2}}\, d^3k$$

The above is an integral equation for ϕ_o. Thus M_{ef} must be numerically solved for each k_F. The equation of state for the pressure is then;

$$P(\rho_B, \phi_0) \;=\; (1/2)(g_v/m_v)^2\rho_B^2 - (1/2)(m_s/g_s)^2(M_b - M_{ef})^2 + \frac{n}{3(2\pi)^2}\int^{k_F} \frac{\vec{k}^2}{(\vec{k}^2 + M_{ef}^2)^{1/2}}\, dk^3$$

The result is shown in figure 1 for $n = 4$ (symmetric nuclear matter), and $n = 2$ neutron matter. The binding energy is plotted against

k_F with k_F the Fermi momentum. Note that there is a minimum in the curve. The coupling constants are evaluated using experimental values. The scalar potential provides a long range attraction and the vector potential provides short range repulsion. The pressure and energy density are dominated by the vector repulsion. Note also in the figure that neutron matter is not bound and the effective mass, $M_{ef} \rightarrow 0$ at high baryon density. The pressure as a function of the density (the equation of state) is used in the TOV equation to numerically determine the central density as a function of the mass of a neutron star. The result is plotted in figure 2. Neutron stars with central densities greater than the value given by the maximum in the curve are unstable and collapse to black holes because the slope of the curve is negative.

While some protons still exist in neutron stars, the Fermi pressure is partially relieved by using gravitational energy to convert protons to neutrons. This equilibrates the Fermi pressure between neutrons and electrons. Eventually pressure can also be relieved by converting neutrons into Λ or other particles composed of one or more strange quarks. Indeed, quark stars have also been suggested.

5 Black hole thermodynamics

Black hole thermodynamics can be quite confusing because of ambiguities in transferring classical thermodynamics to black holes. In particular, there is confusion in applying a statistical mechanical interpretation of heat, temperature, and entropy to relativistic systems.

First review the laws of classical thermodynamics.

6 Laws of classical thermodynamics

1. The zeroth law of thermodynamics states that if two objects have the same temperature there is no energy flow between the objects

2. The first law states that energy is conserved. In thermodynamics heat is a form of energy, so as objects increase or decrease their internal energy, the temperature will change

3. The second law states that entropy of an isolated system cannot decrease. Entropy is a measure of order or the number of occupied states. Entropy is also related to information.

4. The third law states in essence that an object has its minimum entropy when at a temperature of 0° K.

6.1 Application of Thermodynamics to black holes

Thermodynamics must apply to the universe in which we inhabit. Thus the change in the energy of a black hole comes partly from mechanical rotation and partly from, $T\delta S$ where T is a temperature and S the entropy. The temperature (energy) is identified with the gravitational potential, κ, at the surface, and the entropy with the surface area at the event horizon. Of course the problem here is that black holes absorb but do not radiate. The simple assumption is that the entropy of a black hole and its temperature must be $0°$ K.

Now consider when a star has just begun collapse, and swallows another object. The collapsed star classically has zero entropy, but the entropy would be reduced by the absorption. To amplify this process, suppose a box containing a mass, m_γ, of thermal radiation with a temperature, T is lowered from infinity toward a spherically symmetric black hole. Neglect the mass of the box and lowering mechanism. The energy at a distance r from the black hole is $\epsilon = m_o(1 - \frac{2m}{r})^{1/2}$. When the the box reaches the horizon at $r = 2m$ the energy vanishes thus an energy m_0 is released in the process of lowering the box (reseting the point where the potential energy is measured). Then open the box and let a mass, δm, enter the black hole. This does not change the mass of the black hole as seen from infinity (remember the black hole has assumed zero entropy). The box is now raised back to infinity with the expendature of energy $m_o - \delta m$. As a result of the cycle an amount of energy δm is convered into work at infinity. This violates the second law of thermodynanics as it has decreased the entropy of the world. The conclusion is that a black hole must have entropy and temperature. Thus a black hole is assigned a temperature; $T_B \propto \frac{hc^3}{GM\kappa}$ where for a Schwarzschild black hole κ is $\frac{c^4}{4GM}$.

7 Laws of black hole thermodynamics

The entropy of a black hole is proportional to the area of the black hole event horizon divided by the planck area, $S_B = \frac{\kappa A}{4l_p^2}$ where $l_p^2 = \frac{G\hbar}{c^3}$, where A is the horizon area and κ is the Boltzman constant.

The laws of black hole thermodynamics are;

1. The Zeroth law states that the temerature of the black hole is constant on the horizon, because the surface gravity is constant on the horizon

2. The first law states that the energy of a black hope is given by $dm = T_h dS_h + \omega\, dJ$. Here ω is the rotational velocity and J the moment of inertia.

3. The second law if transferred from classical thermodynamics, states that the area of the black hole and thus its entropy cannot decrease. However the black hole can radiate so the correct statement is that the entropy of the external universe cannot decrease.

4. The third law cannot be transferred from classical thermodynamics. Quantum gravity needs to be invoked to understand

how, and if, this law is to be applied.

8 Hawking radiation

Radiation from a black hole is expected but the mechanism is not completely understood. Hawking showed that quantum effects allow black holes to radiate electromagnetic radiation inversely proportional to the mass of the black hole. Suppose a particle-antiparticle pair is created by the strong gravitational field near the horizon. Then the escape of one of the particles means the black hole loses energy (mass). Thermal radiation is also possible since the black hole has a finite temperature.

The temperature of a black hole increases as it loses mass. Note that temperature is proportional to surface gravity. The rate of temperature increase is exponential and probably results in an explosion of the black hole in a burst of gamma rays. A description of this process requires quantum gravity as this effect occurs near the Planck radius. There remians an information paradox, which appears to state that the information content (entropy) of a black hole is lost as it dissipates.

The lifetime of a black hole of mass, M is $t_h \propto 8.3\times10^{19}(M_0/10^{12})^3$ s so a primordial black hole of mass 10^{11} should have expload.

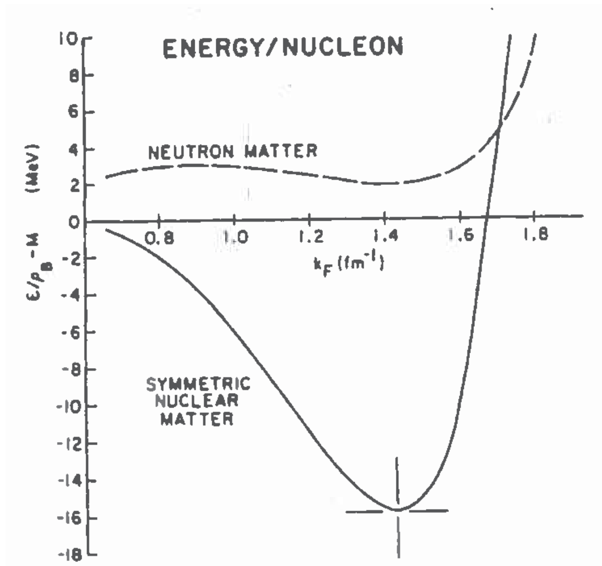

Figure 1: A representation of the binding energy vs total mass of symmetric and neutron nuclear matter using an effective field theory

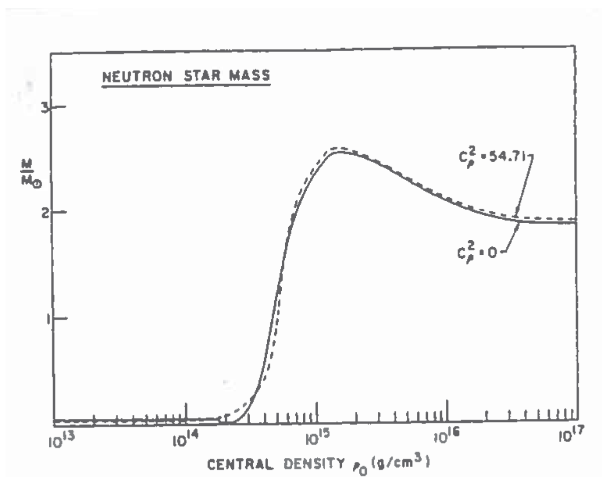

Figure 2: The total mass of a neutron star as a function of its central density