Gravitational Radiation Lecture 23

1 Introduction

One year after his paper on general relativity was published, Einstein made the first calculation of gravitational radiation (GR) using his then new theory. However, he made in his words a "marred error in the calculation". Still, this was the first calculation which treated the quadrupole behavior of gravitational waves. Remember that the dipole is the lowest order multipole in electromagnetic radiation as the wave is represented by a 4-vector potential with positive and negative charge oscillating against each other. A gravitational wave will be described in terms of a metric tensor of rank 2. However, radiation in either of these cases case is simply due to the causal behavior of the physics. Indeed, a not unreasonable result can be obtained by using the Newtonian force, $F = G\frac{mM}{r^2}\hat{r}$, (or scalar potential), and simply applying causality. This is illustrated in the next section as explained in figures and 1 and 2. Note how a change in position of the radiation source produces a transverse component fo the field.

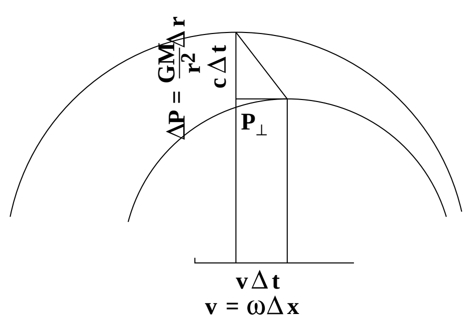


Figure 1: Example of how causality produces a tangential component of a radial field

Because the gravitational coupling constant, G, is weak and the lowest order multipole of a gravitational wave is the quadrupole, very large masses and velocities are required to generate detectable waves. One source of GR are binary Neutron stars. Some 40 years ago the energy loss of a decaying neutron star binary was observed and shown to be consistent with a GR. Other such systems offer additional possibilities for study, and there is hope that direct detection of GR will be forthcoming as detectors become more sensitive. Note that detection depends on the **radiated power** (energy per unit time). For example the earth-sun system radiates approximately 200w. This should be compared to the sun's electromagnetic radiation of some 3.9×10^{26} w.

2 Electromagnetic radiation

Examples of the electric field generated by an accelerating charge are shown in Figure 2. Note the concentration of the transverse components of the field at various positions due to acceleration of the charge and causality. Remember the transverse field is a radiation component while the longitudinal field is a static one. The transverse components correlate with the value of acceleration. Note that the pattern of transverse waves is due to the causality of the fields as described in class.

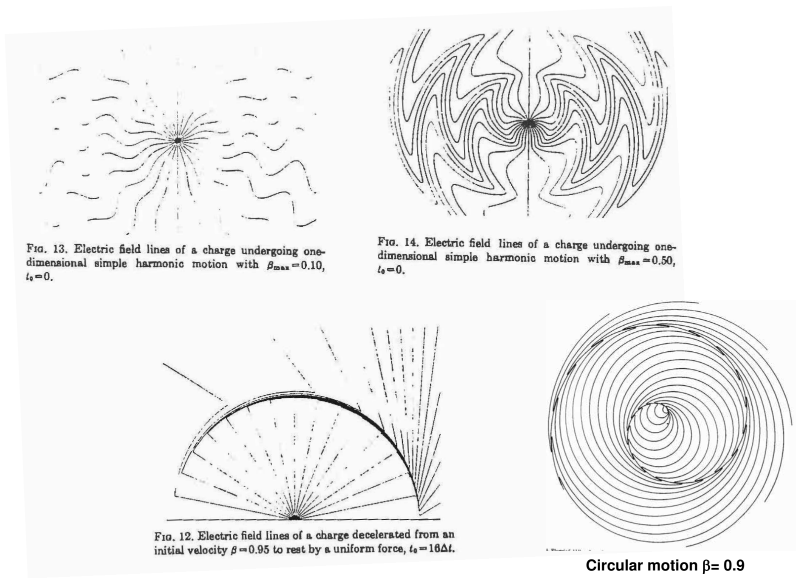


Figure 2: Examples of the electric field for an accelerated charge

3 Mathematical development of a linear theory

Finding a general relativistic metric which allows a time dependent propagation of energy is not possible. Thus to proceed, linearize a simple metric in order to study GW. Introduce a small perturbation to the Minkowski metric. After all the gravitational coupling constant is small so a perturbative solution is reasonable. The Minkowski metric has the form;

$$g^0_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Then add a small perturbation, $h_{\mu\nu}$, with coordinates $q = (q_1, q_2, q_3, q_4)$, so that $g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}$. The Minkowski metric is the tangent plane to the curved space. Keep only terms to 1^{st} order in $h_{\mu\nu}$ when constructing the Ricci tensor. The affine connection is;

$$\Gamma^{\mu}_{\lambda\nu} = (1/2) g^{\mu\tau} [\frac{\partial h_{\tau\nu}}{\partial q^{\lambda}} + \frac{\partial h_{\tau\lambda}}{\partial q^{\nu}} - \frac{\partial h_{\lambda\nu}}{\partial q^{\tau}}]$$

Insert this into the Ricci tensor to obtain the perturbation to first order in $h_{\mu\nu}$;

$$R_{\mu\nu} = \frac{\partial}{\partial q^{\lambda}} \Gamma^{\lambda}_{\mu\lambda} - \frac{\partial}{\partial q^{\nu}} \Gamma^{\lambda}_{\mu\lambda}$$

Note that the Ricci tensor has the form;

$$R_{\mu\nu} = -(1/2)[\nabla^2 - (1/c^2)\frac{\partial^2}{\partial t^2}] + (1/2)[\frac{\partial}{\partial q^{\mu}}(\frac{\partial \Omega^{\lambda}_{\nu}}{\partial q^{\lambda}}) + \frac{\partial}{\partial q^{\mu}}(\frac{\partial \Omega^{\lambda}_{\nu}}{\partial q^{\lambda}})]$$

The metric is symmetric and can be contracted to obtain the perturbation for the scalar curvature. This is then used to obtain the perturbation of the Einstein tensor. Define a new tensor;

$$\Omega^{\nu}_{\mu} = \Omega^{\nu}_{\mu} = h^{\nu}_{\mu} - 1/2 h \delta^{\mu}_{\mu}$$

where $h = h^{\lambda}_{\lambda}$. When observing the above equations, it would appear extremely useful if $\frac{\partial}{\partial q^{\lambda}} \Omega^{\mu}_{\nu} = 0$. The general relativity relation represents 4 coupled equations. In electrodynamics, there are 2 coupled equations obtained from Maxwell's equations and these are separated by choice of a gauge transformation; (*ie* a coordinate transformation which introduces a relation between the components of the 4-vector potential. Thus a similar procedure is sought for the general relativity equations.

4 Gauge invariance

Remember Maxwell's equations can be combined into 2 coupled, differential equations involving the scalar and vector potentials. A gauge transformation is used to separate the equations. This transformation does not change the value of the fields which determine the Lorentz force and the field energy. Remember, energy and momentum (force) are the observables which determine the physics. Thus the physics remains unchanged under a gauge transformation since;

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

where $\vec{A}$ is the vector potential. Now add a gradient term to A. Thus;

$$\vec{A} \rightarrow \vec{A} + \vec{\nabla}\Lambda$$

Thus the curl of the gradient vanishes and the magnetic field does not change so the vector potential is not uniquely defined by the field equations. Then the electric field is determined from Faraday's law.

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \frac{\partial \vec{A}}{\partial t}$$

$$\vec{\nabla} \times [\vec{E} + \frac{\partial \vec{A}}{\partial t}] = 0$$

The term in the brackets can be obtained from the gradient of a potential, V.

$$[\vec{E} + \frac{\partial \vec{A}}{\partial t}] = -\vec{\nabla} V.$$

To keep $\vec{E}$ from changing when $\vec{A} \rightarrow \vec{A} + \vec{\nabla}\Lambda$ change the scalar potential as follows.

$$V \rightarrow V - \frac{\partial \Lambda}{\partial t}$$

This freedom to choose various values of the potentials is called a gauge transformation. A particular gauge imposed the Lorentz condition which allows separation of the coupled potential equations. The form of the Lorentz condition is;

$$\vec{\nabla} \cdot \vec{A} + (1/c^2)\,\frac{\partial V}{\partial t} = 0$$

In covariant form this is;

$$\partial_\alpha A^\alpha = 0$$

Under a gauge transformation this equation takes the form;

$$\vec{\nabla} \cdot \vec{A} + (1/c^2)\,\frac{\partial V}{\partial t} + \nabla^2 \Lambda - (1/c^2)\,\frac{\partial^2 \Lambda}{\partial t^2} = 0$$

So that the equation is;

$$\nabla^2\Lambda \,-\, (1/c^2)\,\frac{\partial^2\Lambda}{\partial t^2} \,=\, 0$$

Maxwell's equations in terms of the 4-potentials are;

$$\vec{\nabla}\cdot\vec{E} \,=\, -\nabla^2 V \,-\, \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} \,=\, \rho/\epsilon$$

$$\nabla^2\vec{A} \,-\, \epsilon\mu\frac{\partial^2\vec{A}}{\partial t^2} \,-\, \vec{\nabla}(\vec{\nabla}\cdot\vec{A} \,+\, \epsilon\mu\frac{\partial V}{\partial t}) \,=\, -\mu\vec{J}$$

All transformations which satisfy the Lorentz condition belong to the Lorentz gauge resulting in the equations for the 4-potentials;

$$\nabla^2 V \,-\epsilon\mu\frac{\partial^2 V}{\partial t^2} \,=\, -\rho/\epsilon$$

$$\nabla^2\vec{A} \,-\epsilon\mu\frac{\partial^2\vec{A}}{\partial t^2} \,=\, -\mu\vec{J}$$

However, it is possible to choose a gauge in which $\vec{\nabla}\cdot\vec{A} \,=\, 0$. This is the Coulomb gauge, and in this gauge;

$$\nabla^2 V \,=\, -\rho/\epsilon$$

with solution;

$$V \,=\, \frac{1}{4\pi\epsilon}\int d\tau'\,\frac{\rho(\vec{x}',t)}{|\vec{x}'-\vec{x}|}$$

In this gauge the scalar potential is the instantaneous Coulomb potential, but the vector potential is much more complicated, as it is the solution of the pde obtained previously after substituting, $\vec{\nabla}\cdot\vec{A} = 0$

$$\nabla^2\vec{A} \,-\, \epsilon\mu\,\frac{\partial^2\vec{A}}{\partial t^2} \,=\, \mu\vec{J} \,+\, \epsilon\mu\,\vec{\nabla}(\frac{\partial V}{\partial t})$$

The Coulomb gauge produces potentials which are not relativistically invariant. They can be used to calculate the fields but only in the particular reference frame where the potentials are evaluated.

5 Gauge transformation of GW

In the case of general relativity, it can be demonstrated that to 1^{st} order in a perturbation expansion about a flat metric, one can find a transformation such that $\frac{\partial}{\partial q^\lambda}\Omega^\mu_\nu \,=\, 0$ and $g^{\mu\nu} = g^0_{\mu\nu} + \gamma_{\nu\mu}$. Here, $\gamma_{\nu\mu}$ is the transformed perturbation of the Minkowski metric. Using this transformation in a source-free space, the general relativity equations which involve a perturbation to 1^{st} order to the Minkowski metric, and where the stress-energy tensor vanishes are;

$$[\nabla^2 - (1/c^2)\frac{\partial^2}{\partial t^2}]\gamma_{\mu\nu} = 0$$

Obviously this results in the wave equation with waves traveling with velocity, c. Thus the solutions can be represented by a superposition of plane waves $A_{\mu\nu}\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$. The solution is subject to the constraint that $k^{\mu} A_{\mu\nu} = 0$. The 4-vectors $\vec{k}$ and $\vec{x}$ are obvious, and the constraints are a result of the GW gauge condition. The constraint also demonstrates that the wave is transverse.

6 Plane wave solution

Look for a GW moving with velocity c in the z direction $\hat{z}$ in the tangent plane, $q^{\mu}(x, y, z, ct)$.

$$\gamma_{ij}\, A_{ij}\, e^{ik(z-ct)}$$

Here $(i, j) \rightarrow (x, y)$ and $A_{ij} \rightarrow$Constant. The equations must also satisfy the gauge condition;

$$\frac{\partial \Omega^{\nu}_{\mu}}{q^{\nu}} = 0$$

In addition, assume a solution where only $\gamma_{xx}, \gamma_{xy}, \text{and} \gamma_{yy} \neq 0$ and all others vanish. Then recall that $\Omega^{\nu}_{\mu} = \gamma^{\nu}_{\mu} - 1/2\gamma\delta^{\mu}_{\mu}$. It can be shown that the auxiliary (gauge) condition is satisfied if;

$$\gamma_{xx} + \gamma_{yy} = 0$$

Therefore a plane wave solution for the metric which has the above relation between the components and satisfies the equations is;

$$(\nabla^2 - (1/c)^2\frac{\partial^2}{\partial t^2})\gamma_{\mu\nu} = 0$$

$$\frac{\partial \gamma_{\mu\nu}}{\partial x^{\nu}} = 0$$

The Ricci tensor, $R_{\mu\nu}$, also vanishes, and the perturbation has only 2 non-vanishing components;

$$\gamma_{xx} = -\gamma_{yy} \qquad \gamma_{xy} = \gamma_{yx}$$

Hold (z, t) constant so the metric is then;

$$(d\vec{s})^2 \; = \; (1+\gamma_{xx})dx^2 + (1+\gamma_{yy})dy^2 + 2\gamma_{xy}dx\,dy$$

This metric is a 2d surface on the tangent plane (x, y) so project the length distortions due to the wave onto the unit vectors, $\hat{x}$ and $\hat{y}$.

$$\hat{x} = (1+\gamma_{xx})^{1/2} cos(\xi)\hat{e}_x + sin\xi\hat{e}_y$$

$$\hat{y} = (1+\gamma_{xx})^{1/2} sin(\xi)\hat{e}_x + cos(\xi)\hat{e}_y$$

Then the matrix elements are;

$$g_{xx} = \hat{x}\cdot\hat{x} = 1+\gamma_{xx}$$

$$g_{yy} = \hat{y}\cdot\hat{y} = 1+\gamma_{yy}$$

$$g_{xy} = \hat{x}\cdot\hat{y} = (1+\gamma_{xx})^{1/2}(1+\gamma_{xx})^{1/2}\, sin(2\xi)$$

The metric is then ;

$$g_a \; = \; \begin{pmatrix} (1+\gamma_{xx})^{1/2}\, cos(\xi) & (1+\gamma_{yy})^{1/2}\, sin(\xi) & 0 & 0 \\ (1+\gamma_{xx})^{1/2}\, sin(\xi) & (1+\gamma_{yy})^{1/2}\, cos(\xi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The wave generates a time dependent Lorentz contraction and expansion in the direction transverse to the wave velocity with the $\hat{x}$ and $\hat{y}$ oscillating out of phase by 90°.

See movie of the wave as it crosses a plane perpendicular to the wave velocity and the movie of spiral waves from a star binary.

7 Detection

A number of gravitational wave detectors have been constructed over a period of more than 30 years. These attempt to measure the displacement strain on detector components produced by GR. The main detection problem is the very low amplitude of a gravitational wave and its frequency. Binary stars are prime candidates for GR sources, but the effects on a detector on earth would result in a strain of less than 1 in 10^{20}. The radiated power of a binary of masses, m_1 and m_2 orbiting at a separation distance, r, is;

$$P \; = \; (32G^4/5c^5)\frac{(m_1 m_2)(m_1+m_2)}{r^5}$$

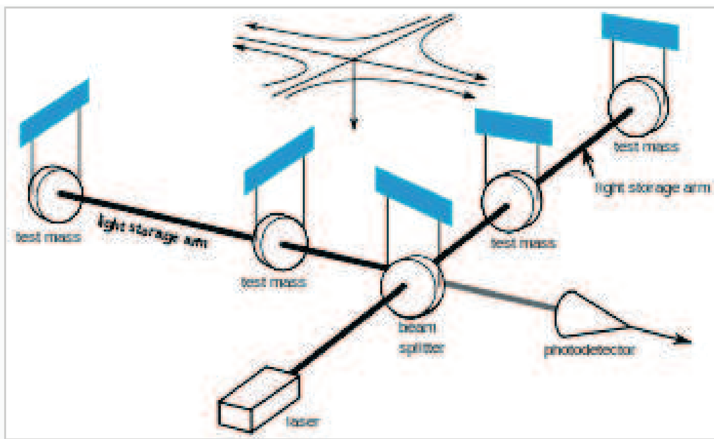
Extreme sensitivity to signal and not noise is required.

8 LIGO

A pair of solar mass neutron stars in a circular orbit a distance of about 2×10^8m apart live for about 4×10^5 years and have an orbital period of 10^3 s (frequency 10^{-3} s). As these stars radiate energy, their orbital radius, period, and lifetime decreases, but in general the frequencies are too low. Gravitational waves cover the frequency range from 10^{-16} to 10^4 Hz. Obviously only the intermediate frequencies can be experimentally addressed. There are ground based interferometers such as LIGO and satellite based interferometers, LISA.

The Laser Interferometer Gravitational-Wave Observatory, LIGO, has a sensitivity of about 5×10^{22} in a frequency band from 30-7000 Hz. LIGO consists of two gravitational wave observatories, one in Livingston, Louisiana , and the other at Hanford, Washington. The sites are separated by 3,002 kilometers. Since gravitational waves are expected to travel at the speed of light, this distance corresponds to a difference in gravitational wave arrival times of up to ten milliseconds, and the difference in arrival times can determine the direction of the wave source. Each observatory has an L-shaped ultra high vacuum system, measuring 4 kilometers long with up to five interferometers in each vacuum system, figure 3. This system has active vibration isolation providing a factor of 10 reduction in noise in the frequency range between 0.1 to 5 Hz.

When a gravitational wave passes through the interferometer, the local space-time is altered. Depending on the source of the wave and its polarization, this results in an effective change in length of one or both of the cavities. The effective length change between the beams causes the light moving in the cavity to become slightly out of phase and slightly out of resonance. The beams, which are tuned to destructively interfere at the detector, then produces a signal.



A schematic diagram of a laser interferometer

Figure 3: A sketch of a laser interferometer system such as employed by LIGO