Inductance

1 Further Discussion of Faraday's Law

In Lecture 2 Faraday's law was developed using the Lorentz force on a charge within a conducting loop in a frame where the loop is moving and the magnetic field is at rest. The result was the expression;

$$\text{EMF} = \oint \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{\sigma} = -\frac{d\phi}{dt}$$

In the above, the magnetic flux through the area, σ, is enclosed by the line, $\vec{s}$. The equation in differential form is obtained by the application of Stoke's theorem.

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Previously it was noted that the partial rather than the total derivative is used in this equation, and this needs justification and development. Consider the convective deravitive which is used for a system moving with velocity, $\vec{V}$.

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{V} \cdot \vec{\nabla}$$

For constant velocity, use the identity;

$$\vec{\nabla} \times (\vec{B} \times \vec{V}) = (\vec{V} \cdot \vec{\nabla})\vec{B} - (\vec{\nabla} \cdot \vec{B})\vec{V} = (\vec{V} \cdot \vec{\nabla})\vec{B}$$

Using the above;

$$\int (\vec{V} \cdot \vec{\nabla})\vec{B} \cdot d\vec{a} = \int \vec{\nabla} \times (\vec{B} \times \vec{V}) \cdot d\vec{a} = \oint (\vec{B} \times \vec{V}) \cdot d\vec{s}$$

Collecting terms;

$$\oint (\vec{E}' - \vec{V} \times \vec{B}) \cdot d\vec{s}' = \int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}'$$

In the above $\vec{E}'$ is in the moving system. Thus write;

$$\vec{E}'_{moving} = \vec{E}_{rest} + \vec{V} \times \vec{B}$$

This means that $\vec{E}$, $\vec{B}$, when expressed in the same reference frame, results in Faraday's law in the form;

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

2 Definition of Inductance

When electric potentials are placed on a system of conductors, charges move to cancel the electric field parallel to the conducting surfaces and also cancelling the field inside the conductors. The amount of charge on a conducting surface is proportional to the applied potential, and the constant of proportionality is called the capacitance. The capacitance depends only on the geometry of the conducting system, not on the potential. Now we are interested in the current which flows in a conductor when induced by an EMF. First suppose that a current loop (1) creates a magnetic flux through another current loop (2), as illustrated in Figure 1. This is written using Stokes theorem with $\vec{A} = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{l_1}}{r_{12}}$. Here, $\vec{A}$ is the vector potential for the magnetic field, $\vec{B}_1$.

$$\phi_{12} = \int \vec{B}_1 \cdot d\vec{a}_2 = \oint \vec{A}_1 \cdot d\vec{l}_2$$

$$\phi_{12} = \oint [\frac{\mu_0}{4\pi} I_1 \oint \frac{d\vec{l_1}}{r_{12}}] \cdot d\vec{l}_2$$

$$\phi_{12} \approx M_{12} I_1$$

In the above, $M_{12} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\vec{I_1} \cdot d\vec{I_2}}{r_{12}}$. This is the constant of mutual inductance between the loops. As with capacitance, the inductance depends only on geometry. Thus, the current induced in a circuit by the magnetic field due to the current in another circuit is proportional to the flux connecting the circuits. Also it is clear from the equation for M_{12} that, $M_{12} = M_{21}$.

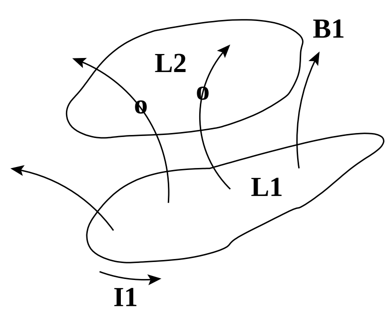

Figure 1: The interaction of 2 current loops to produce a mutual inductance

The inductance is always taken as a positive quantity, so the sign is not important. The time derivative of the flux is the EMF, which is related to the inductance in the above equation.

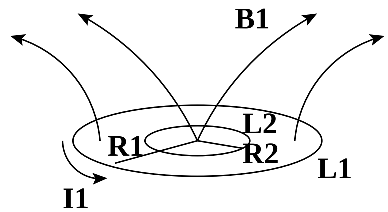

Figure 2: The geometry to find the mutual inductance between 2 concentric current loops

The unit of inductance is the Henry, and a 1 Hy inductance produces 1 volt with a change in the current of 1 ampere/s.

$$\frac{\partial\phi}{\partial t} = -\int \vec{E}\cdot d\vec{l} = -(EMF)_2 = M_{12}\,\frac{dI_1}{dt}$$

A current loop also generates a flux through its own loop. A change in this flux generates a current tending to keep the flux constant in the loop. The self inductance is defined by;

$$\phi = L\,I$$

As in the case of the mutual inductance, the self inductance depends only on geometry. The induced EMF is;

$$EMF = -L\,\frac{dI}{dt}$$

3 Examples

In general the magnetic flux can be obtained by integrating the magnetic field over the area of the current loop, using for exaxmple the Biot-Savart law to find the field. However, in most cases, the calculation of inductance is difficult, and it is easier to find $\vec{E}$ from Faraday's law and to determine the sign of the EMF using Lenz's law.

3.1 Concentric current loops

Assume two concentric loops of current as shown in Figure 2. If $R_2 \ll R_1$, approximate the magnetic field through the loop R_2 as a constant given by the field at the center of the loop.

$$B_1(r=0) = \frac{\mu_0}{4\pi}\int I_1\,dl_1/R_1^2 = \frac{\mu_0}{4\pi}(2\pi)\frac{I_1}{R_1} = \frac{\mu_0 I_1}{2R_1}$$

The flux through the loop R_2 is then approximately, $\pi R_2^2[\frac{\mu_0 I_1}{2R_1}]$. The mutual inductance becomes;

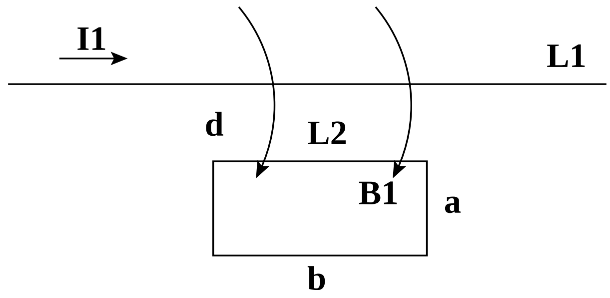

Figure 3: The geometry to find the mutual inductance between a long wire and a current loop

$$M_{12} \approx \frac{\mu_0 \pi R_2^2}{2R_1}$$

3.2 Long wire near a current loop

This example places a long wire in the plane and parallel to the edge of a loop as shown in Figure 3. The magnetic field around a long wire can be obtained from Ampere's law.

$$B = \frac{\mu_0 I_1}{2\pi r}$$

The flux through loop 2 is then;

$$\phi = \int \vec{B} \cdot d\vec{\sigma} = \frac{\mu_0 I}{2\pi} \int_d^{a+d} dr/r \int_0^b dl$$

$$\phi = \frac{\mu_0 b}{2\pi} ln[\frac{a+d}{d}]\, I$$

$$M = \phi/I = \frac{\mu_0 b}{2\pi} ln[\frac{a+d}{d}]$$

4 Energy and Inductance

The power applied to overcome an induced EMF is;

$$\frac{dW}{dt} = -(EMF)I = (\frac{d\phi}{dt})I = [L\frac{dI}{dt}]I$$

From this write;

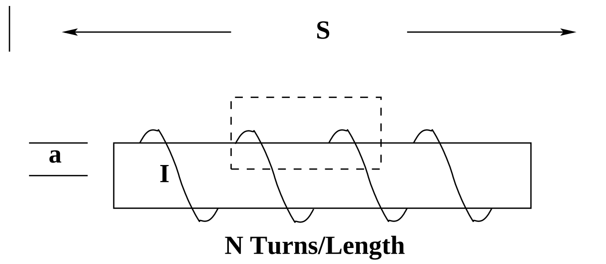

Figure 4: The geometry to find the self inductance of a long solenoid

$$dW = (LI)\, dI$$

Which when integrated gives;

$$W = (1/2)LI^2$$

Since the current, I, is linearly related to the magnetic field we expect the energy to be related to B^2, as indeed we have previously found for the energy density , $\mathcal{W} = (1/2\mu_0)B^2$.

Now suppose a long solenoid of length, S, and radius, a, as shown in Figure 4. The magnetic field is found by Ampere's law, $BS = \mu_0 NIS$

$$B = \mu_0 NI$$

Then the energy stored is obtained by integration over the energy density;

$$W = \frac{1}{2\mu_0} \int B^2\, d\tau$$

$$W = \frac{1}{2\mu_0} [\mu_0 NI]^2 \pi a^2 S$$

This must be equal to $(1/2)LI^2$ so that the inductance per unit length of the solenoid is;

$$L/S \approx \mu_0 \pi a^2 N^2$$

5 Coaxial Cable

The geometry of the problem is shown in Figure 5. For sufficiently high frequencies, currents can be assumed to flow on the surfaces of the conductors. The field between the inner and outer conductor is obtained by Ampere's law.

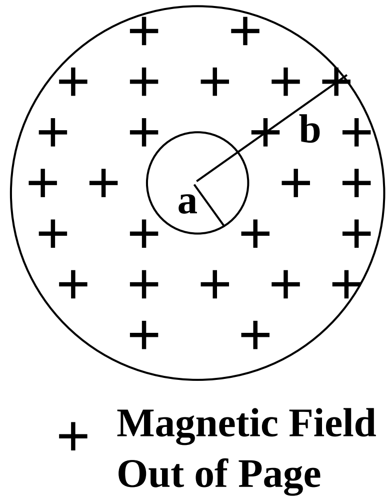

Figure 5: The geometry to find the self-inductance of a coaxial cable

$$B = \frac{\mu_0 I}{2\pi r}$$

Calculate the stored energy in a length S of the cable using the energy density.

$$W = \frac{1}{2\mu_0} \int d\tau \, B^2 = \frac{S}{2\mu_0} \int_a^b \frac{\mu_0^2 I^2}{4\pi^2 r^2} \, 2\pi r \, dr$$

$$W = \frac{S\mu_0 I^2}{4\pi} \, ln[b/a] = (1/2)LI^2$$

$$L/S \approx \frac{\mu_0}{2\pi} \, ln[b/a]$$

6 Eddy currents and induction motors

As a magnetic field moves through, and changes intensity in, a conductor, currents are induced in the conductor by Faraday's law. This is most easily seen by applying Lenz's law to the flux through some surface area of the conductor. If the conductivity were perfect, then the flux through any surface in the conductor would not change at all. However, in the case of a superconductor there is an additional effect (Meisner effect) which will be described later. However, in a physical conductor, there will be a resistance to the current flow, and thus energy due to the changing field strength is lost in ohmic heating in the conductor. The induced currents are called eddy currents and are important components which one must consider when designing magnetic devices. Conservation of energy requires that a moving magnet near a conductor exerts a force on the conductor (and likewise the conductor exerts a force back on the magnet). Consider a spinning magnetic as shown in Figure 6a. As the

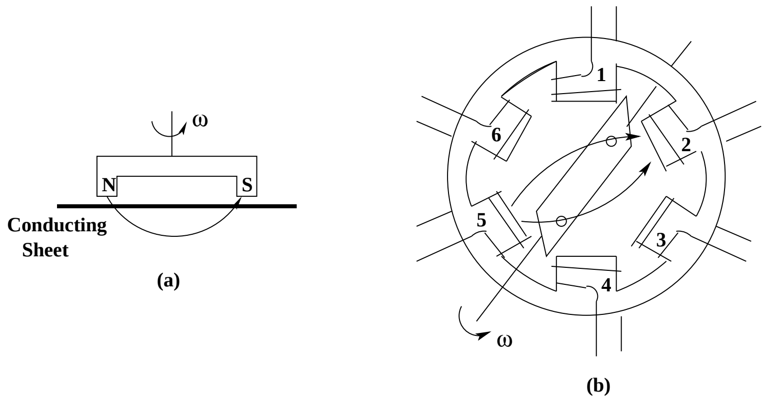

Figure 6: (a) A simple design of a system that induces a torque on a conducting sheet due to a rotating permanent magnet. (b) The geometry of an induction motor using a 3-phase alternating current

magnetic field changes through the conductor, eddy currents are induced in the conductor. These interact with the field imparting a torque on the conductor in the same direction as the angular velocity of the magnet.

Figure 6b shows an example of a simple induction motor. Alternating currents between pairs of magnets (1,4), (2,5) and (3,6) can produce a rotating magnetic field which exerts a torque on the current loop. causing it to turn. In this case the rotating field could be excited by a 3-phase current, but there are other designs which use normal, single-phase alternating current.

7 Meissner Effect

A superconducting material is found to expell a magnetic field from its interior. This effect was discovered by Meissner and Ochsenfeld who observed that once the temperature of a superconducting material is lowered below the critical temperature and the material becomes superconducting, magnetic fields are pushed to the outside of the material. Thus the material not only has zero resistance to current flow, but also becomes a perfect diamagnetic material *ie* a material where atomic electron currents perfectly flow to cancel any applied field. The BCS theory, which explains superconductivity, does not completely explain the transient behavior this effect.

8 Circuits

Suppose the simple series circuit shown in Figure 7. The applied voltage is harmonic at a fixed frequency. By Ampere's circuit equations (conservation of charge and linear voltage sums) we obtain for the sum of the voltage elements;

$$IR + Q/C = V - L\frac{dI}{dt}$$

Note that the inductance is applied to the EMF (V) side of the equation. Write the above in terms of the current;

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + I/C = \frac{dV}{dt}$$

Now apply a Fourier transform. This is done by multiplying each term by $e^{i\omega t}$ and integrating over the time. The transformation converts the current as a function of time to a function of frequency (time space to frequency space). The transformations are defined by;

$$\mathcal{I}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, I(t)$$

$$I(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, \mathcal{I}(\omega)$$

Here let the frequency range $-\infty \leq \omega \leq \infty$ although the frequency is defined for $\omega > 0$. This can be handled by symmetry relations letting $\omega = -\omega$, but ignore this problem here. Apply the Fourier transformation and assume one can set I, $\frac{dI}{dt} = 0$ at $\pm\infty$. After multiplication of each term in the equation by $e^{i\omega t}$ integrate the differentials by parts and use the above boundary conditions. This results in the equation;

$$-L\omega^2\mathcal{I} + -i\omega R\mathcal{I} + \mathcal{I}/C = -i\omega\mathcal{V}$$

The solution is;

$$\mathcal{I}(\omega) = \frac{i\omega\mathcal{V}}{L[(-\omega^2 + 1/LC)i + R\omega/L]}$$

Then find the transform, $\mathcal{V}(\omega)$, for the voltage, $V(t)$, and insert it in the above, and take the inverse transform of $\mathcal{I}$ to get the current. For the moment proceed by identifying a complex impedance, Z, for the circuit components defined in Table 8.

Then apply the above to the circuit to obtain;

$$i\omega LI + IR - iI/\omega C = V$$

Table 8 The Complex Impedance for Circuit Components

Impedance	Component	Reactance	Voltage
	Resistance	R	IR
Z	Capacitance	-i / ω C	-i I /ω C
	Inductance	i ω L	i I ω L

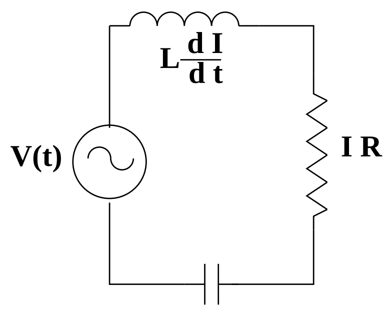

Figure 7: A simple circuit composed of elemental circuit components. The applied voltage is assumed harmonic so that the impedance of the components in terms of reactance can be obtained

$$I \;=\; \frac{\omega V}{L[(\omega^2 - 1/LC)i \,+\, R\omega/L]}$$

The assumption in the above is that $V \;=\; V_0 e^{i\omega_0 t}$. Apply the Fourier transform to obtain;

$$\mathcal{V}(\omega) \;=\; \sqrt{2\pi} V_0 \,\delta(\omega + \omega_0)$$

Substitute this expression into the equation for $\mathcal{I}$ above and take the inverse transform. This results in the above expression for I in terms of V.

Now apply the impedance expressions in a circuit, combining the elements using the equation forms for resistance as developed in undergraduate classes. The Fourier transformation is appropriate for steady-state solutions. For time dependent, transient solutions one chould apply a Laplace transformation. In lieu of introducing the Laplace transformation, solve the problem illustrated in Figure 8 using a direct solution to the differential equation for the current.

The two circuit equations as obtained from the figure are;

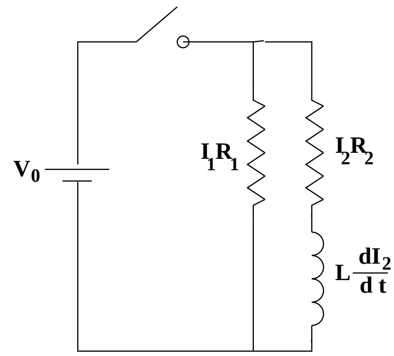

Figure 8: A circuit to illustrate the combination of circuit components and a solution to the time dependent differential equation for the current.

$$V_0 \, = \, I_1 R_1 \, = \, I_2 R_2 \, + \, L\frac{dI_2}{dt}$$

$$I \, = \, I_1 \, + \, I_2$$

Combine the equations into a differential equation for I and use the fact that V_0 is constant in time.

$$L\frac{dI}{dt} \, + \, R_2 I \, = \, V_0 \frac{R_1 + R_2}{R_1}$$

The solution is then;

$$I \, = \, \frac{V_0(R_1 + R_2)}{R_1 R_2} \, + A\, e^{-(R_2/L)t}$$

The constant A is obtained from the initial conditions. Assume that at $t \, = \, 0$ the switch closes and initially all current flows through the resistance, R_1. Thus $I(0) \, = \, I_1 \, = \, V_0/R1$ for $t \, = \, 0$. Therefore;

$$A \, = \, -\frac{V_0}{R_2}$$

$$I \, = \, (V_0/R_2)[\frac{R_1 + R_2}{R_1} \, - e^{-(R_2/L)t}]$$

For $t > 0$ the current through the inductor is ;

$$I_2(t) \, = \, I \, - \, I_1 \, = \, I \, - \, V_0/R_1$$

$$I_2(t) \, = \, (V_0/R_2)(1 \, - \, e^{-(R_2/L)t})$$

At this time the energy stored in the inductor is $(1/2)LI_2^2$

9 Laplace Transform

It was pointed out that the Fourier transforms provides a solution for the steady-state problem. However, many cases require transient solutions, so a different technique is required. In addition, the Fourier transformation of $f(x)$ requires that $\int_{-\infty}^{\infty} dx\,|f(x)|$ converge. Thus, suppose a condition that $f(x) = 0$ for $x < 0$, and use;

$$f(x) \rightarrow \begin{bmatrix} e^{-\gamma x}\,f(x) & x > 0 \\ 0 & x < 0 \end{bmatrix}$$

In the above $\gamma > 0$. Then look at the Fourier transformation.

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-\gamma x}\, f(x)\, e^{-i\alpha x}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\alpha\, e^{\gamma x}\, F(\alpha)\, e^{i\alpha x}$$

Use these equations to write;

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha\, e^{\gamma x}\, e^{i\alpha x} \int_{0}^{\infty} dx'\, e^{-\gamma x'}\, f(x')\, e^{-i\alpha x'}$$

Define;

$$p = \gamma + i\alpha \quad dp = i d\alpha$$

$$\phi(p) = \int_{0}^{\infty} dx'\, f(x')\, e^{-px'}$$

$$f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} dp\, \phi(p)\, e^{px}$$

The function $\phi(p)$ is the Laplace transform of the function $f(x)$. In almost cases the transform will now converge. However, convergence problems have been pushed into obtaining an inverse transform which requires integration in the complex plane. In some cases, the inverse transform can be handled by the convolution theorem. Thus supppose the integrand in the inverse transform is a product of 2 known Laplace transforms.

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} dp\, \phi(p)\, \Psi(p) e^{px} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} dp\, \phi(p)\, e^{px} \int_{0}^{\infty} g(y)\, e^{-py}$$

Or;

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} dp\,\phi(p)\,\Psi(p)e^{px} = \frac{1}{2\pi i}\int_0^{\infty} g(y) \int_{\gamma-i\infty}^{\gamma+i\infty} dp\,\phi(p)\,e^{p(x-y)}$$

Since $f(x-y) = 0$ if $(x-y) < 0$, the above is written;

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} dp\,\phi(p)\,\Psi(p)e^{px} = \int_0^{\infty} dy\,g(y)\,f(x-y)$$

Finally in many cases one can use tables of common Laplac transforms in the same way as one uses integral tables to obtain integral functions. Laplace transforms are not pursued further here given the required mathematical development.

10 The Displacement Current

As discussed previously, the equations of electrodynamics as developed to this point, still have an inconsistancy in Ampere's law. To understand this in more detail, take the divergence of the present form for Ampere's law, $\vec{\nabla} \times \vec{B} = \mu\vec{J}$.

$$\vec{\nabla}\cdot(\vec{\nabla} \times \vec{B}) = \mu\vec{\nabla}\cdot\vec{J} = 0$$

However, from the equation of continuity (conservation of charge);

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$$

where ρ is the charge density. Since $\vec{\nabla}\cdot\vec{J}$ represents a outflow of charge from a small volume, the charge density in the volume must decrease. Now use Gauss' Law $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon$. Then we must have the form;

$$\vec{\nabla}\cdot(\vec{J} + \epsilon\frac{\partial\vec{E}}{\partial t})$$

Therefore Ampere's Law must be modified so that;

$$\vec{\nabla} \times \vec{B} = \mu\vec{J} + \mu\epsilon\,\frac{\partial\vec{E}}{\partial t}$$

The above equation also seems symmetric with Faraday's Law, as we previously found a connection between $\vec{B}$ and $\vec{E}$ in transformations between moving coordinate frames. The term $\mu\epsilon\,\frac{\partial\vec{E}}{\partial t}$ is sometimes called the displacement current, but it is really an induction effect similar to that discussed earlier for the magnetic field. We will later see that the displacement current is required by special relativity. Maxwell's equations now have the forms;

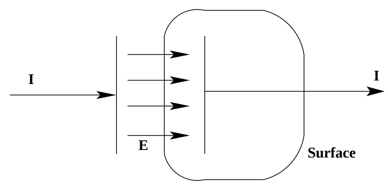

Figure 9: An example of the displacement current in the discharge of a capacitor

$$\vec{\nabla} \cdot \vec{D} = \rho$$
$$\vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$
$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$
$$\vec{H} = \vec{B}/\mu - \vec{M}$$

The force equation remains;

$$\vec{F} = q[\vec{E} + \vec{V} \times \vec{B}]$$

11 A second look at displacement current

Upon examination, Figure 9 shows that if we apply Ampere's law in integral form to the surface shown, the result is;

$$\oint \vec{B} \cdot d\vec{l} = \int (\vec{\nabla} \times \vec{B}) \cdot d\vec{a} = \mu_0 I$$

Thus, there is a current out of the surface due to the charge flowing from the capacitor, but there are no magnetic fields, so the left side of the equation vanishes, and something is missing. The introduction of the displacement current shows that the decreasing E field flowing through the surface equals the current flow out of the surface. Thus $\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ is the required displacement current.