Special Relativity

1 The postulates of special relativity

Observational astronomy and precise measurements of physical phenonmena in the local environment of the earth show no difference in the behavior of systems due to their position in space. Thus one concludes that in free space (absence of massive objects for example) the postion in space/time has no affect on physical systems. Thus one assumes;

1) There exist a set of reference frames moving at constant velocity with respect to each other in which the desription of all phenonmena are indentically discribed. These are the inertial reference frames in which Newtonian mechanics holds, *eg* the law of inertia $\vec{F} = m\vec{a}$ is the same in all sich systems. Note that postulating a unique set of reference frames is unsatisfactory, but this can only be corrected in general relativity.

2) The second observation is that the velocity of EM radiation in free space is constant, independent of the source or reference frame.

Consider the first postulate. Suppose 2 inertial reference frames as shown in Fig. 1. When transforming between these frames;

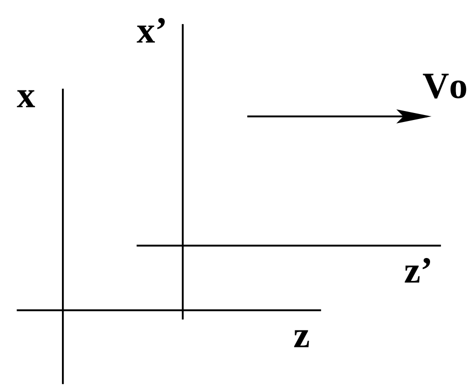

Figure 1: Reference frames for the Lorentz transformation

$$x' = x$$

$$y' = y$$

$$z' = z - V_0 t$$

$$t' = t$$

The above equations are the Galilean transformations between inertial reference frames. The velocity transformations between these frames are ;

$$V_x' = V_x$$

$$V_y' = V_y$$

$$V_z' = V_x - V_0$$

As previously seen, the acceleration transformations are ;

$$\vec{a}' = \vec{a}$$

The above equation demonstrates that the law of inertia is the same in all inertial frames of reference. However, Maxwell's equations are not invarient under these transformations. For example, put the Galilean transformations into the wave equation for the vector potential, V, as obtained from Maxwell's equations. The equation has the form;

$$[\frac{\partial^2}{\partial x^2} - (1/c^2)\frac{\partial^2}{\partial t^2}]V = 0$$

In the above c is the wave velocity. In a Galelian transformation the velocities add between moving frames is additive. However, the equation above requires that c is a constant.

2 Simultaneous events

Consider here the meaning of postulate 2 (constancy of the velocity of light). This postulate requires that there is a limiting velocity, c, and information cannot propagate faster than this velocity. From Maxwell's equations, EM energy travels at this limiting velocity. As an interesting aside, this requires that the photon (the quantum of EM energy) is massless.

Now study the impact on observations due to the consistancy of the speed of light. Suppose a relativistic train as shown in Figure 2. There are 3 marks on the train and train tracks as shown. Point C is halfway between the points A and B. Observers on the train and tracks are positioned at C' and C. There is a lightning flash striking the train and track at A'/A and B'/B at the same time. The observer on the tracks sees the light flash simultaneously and confirms that the lightning struck the tracks at A and B at the same time. However the observer on the train moves to meet the flash from A' and away from the flash at B'. Thus this observer believes that A' occured before B', and the observers cannot agree on whether

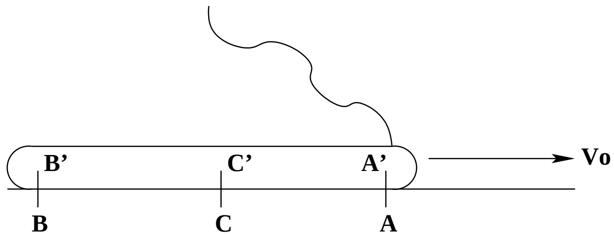

Figure 2: Simultaneous events and a moving train

the events are simultaneous (occur at the same time). In addition the observers cannot agree on whether the length AB or $A'B'$ is longer, since it is measured by determining when the light from A and B reached the center of the train. All of this is a direct result that information cannot travel through space with a velocity greater than c. Therefore, time and length depend on the frame of reference.We incorporate this into a coordinate transformation for motion in the $\hat{z}$ direction by writing;

$$x' = x$$

$$y' = y$$

$$z' = f(z,t)$$

$$t' = g(z,t)$$

Here f, g are functions to be determined. To determine these functions we assume 2 additional postulates.

3) Space time is isotropic and homogeneous. This means we must choose linear functions so that any point in space is weighted the same as any other.

4) The relations must reduce to the Galilean transformation in the limit of low velocities.

Applying these assumptions, the transformation equations have the form,

$$x' = x$$

$$y' = y$$

$$z' = \alpha(z - V_0 t)$$

$$t' = rz + st$$

Now use the fact that c is constant in all frames of reference, and find the constants α, r, and, s. This is done by using the constant distance-time equation; .

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2$$

Substitute into the above transformations and collect terms in x, y, z, t. Because the variables are linearly independent, each coefficient must vanish. This results in the solution defining the Lorentz transformation.

$$x' = x$$

$$y' = y$$

$$z' = \gamma(z - V_0 t)$$

$$t' = \gamma(t - (V_0/c^2)z)$$

with $\gamma = \sqrt{\frac{1}{1-\beta^2}}$ and $\beta = V_0/c$

From this analysis we find the following;

Simultaneity depends on the coordinate frame, so that time is dependent on a transformation between coordinates.

Length measurement depends on the coordinate frame since length is measured relative to a reference.

The above results are directly due to limiting the velocity of information propagation to c.

3 Length transformation

A length measurement occurs by comparing the position of the ends of two rods at the same time. Thus compare the length of a moving system (rod) to a rod at rest at the same instant of time as shown in Figure 3. Set up a coordinate system for a Lorentz transformation as previously.

$$
\begin{aligned}
X' &= X & (1)\\
Y' &= Y & (2)\\
Z' &= \gamma[Z - V_0\, t] & (3)\\
t' &= \gamma[t - (V_0/c^2)Z] & (4)\\
\gamma &= \sqrt{\frac{1}{1 - \beta}} & (5)\\
\beta &= V_0/c & (6)\\
& & (7)
\end{aligned}
$$

In the primed and unprimed frames frame the lengths are;

$$L' = Z_2' - Z_1'$$

$$L = Z_2 - Z1$$

The comparison of the end points is made at the same instant of time in the rest frame, t. Substitute in the Lorentz transformation to obtain;

$$(Z_2' - Z_1') = L' = \gamma[Z_2 - V_0 t] - \gamma[Z_1 - V_0 t]$$

$$L' = \gamma\, L$$

Since $\gamma \geq 1$, the length of a moving rod appears contracted compared to a rod at rest.

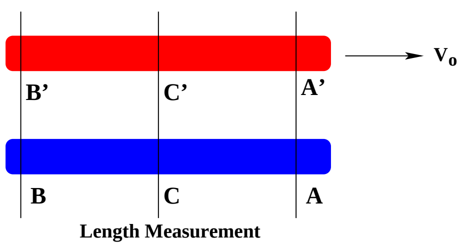

Figure 3: Reference frames for a length measurement

4 Time transformation

To measure time, compare the rate of two clocks. We arrange the clocks so that their times are synchronized to $t = 0$ at the position $z = 0$. At some later time the moving clock has traveled a distance $d = V_0 t$. The lateral difference between the clocks requires that the time information travel back to the clock at rest from its new position. At the time measurement the clocks are a distance $Z = V_0 t$ apart, Figure 4. Then the time difference measured at the poistion of the clock at rest is;

$$\tau' = \gamma(t_2 - v_0/c^2 z_2) - 0$$

$$\tau' = \gamma(t_2 - v_0/c^2 V_0 t)$$

$$\tau' = \tau/\gamma$$

Apply the Lorentz transformation;

$$
\begin{aligned}
X' &= X && (8)\\
Y' &= Y && (9)\\
Z' &= \gamma[Z - V_0\, t] && (10)\\
t' &= \gamma[t - (V_0/c^2)Z] && (11)\\
\gamma &= \sqrt{\frac{1}{1 - \beta}} && (12)\\
\beta &= V_0/c && (13)\\
& && (14)
\end{aligned}
$$

$$\tau = t_2 - t_1$$

$$\tau' = t_2' - t_1'$$

$$\tau' = \gamma[t_2 - (V_0/c)^2 t_2] - \gamma[0 - (V_0/c)^2 0]$$

$$\tau' = (1/\gamma)\,\tau$$

Therefore the clocks in the moving frame run slower than in the rest frame.

5 Example

Muons are produced by the collision of cosmic ray protons with atoms in the upper atmosphere of the earth. Muons are unstable particles and decay with a meanlife of approximately 2.2×10^{-6} s when at rest. These muons, however, are moving at relativistic speed after the collision. Consider two observers, one moving with the muons and one on the earth's surface.

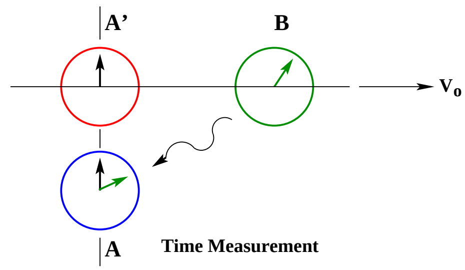

Figure 4: Reference frames for a time measurement

To the observer on the Earth's surface, the number of muons reaching the surface can be obtained by the decay equation for muons at rest.

$$N = N_0 e^{-\lambda_{moving} t}$$

where N is the number reaching the surface, N_0 is the number produced at altitude, λ is the meanlife in the rest frame, and t is the time in the earth-observed's frame of reference. The time in the moving frame is slower so that $t' = t/\gamma$. The distance traveled is D at velocity V, so the time in the earth frame is $t = D/V$. Thus the number observed at the surface is;

$$N = N_0 e^{-\lambda D/(\gamma V)}$$

On the other hand when traveling in the muon frame, the muon travels a distance that is contracted with respect to the distance in the earth frame due to the muon velocity, $D' = D/\gamma$. The time to reach the surface is $t' = D/(\gamma v)$. Thus in this frame the number observed at the surface is;

$$N = N_0 e^{-\lambda D/(\gamma V)}$$

The number of decays is the same as in the earth frame but the interpretation differs. In the earth frame the time is dialated by the muon velocity. In the muon frame the distance is contracted.