

# Transformations

## 1 The Lorentz Transformation

In the last lecture the relativistic transformations for space/time between inertial frames was obtained. These transformations essentially follow from the postulate that there is a limiting velocity for the propagation of information,  $c$ . The Lorentz transformation is;

$$x' = x$$

$$y' = y$$

$$z' = \gamma(z - v_0 t)$$

$$t' = \gamma(t - (v_0/c^2)z)$$

with  $\gamma = \sqrt{\frac{1}{1 - \beta^2}}$  and  $\beta = V_0/c$

The above equations have an inverse because a one-to-one map between the inertial systems is required. The inverse can be obtained by solving the equation set for the unprimed variables, or just noting that a symmetry operation would change the sign of the velocity. Both of these equation sets will be needed in the next section.

$$x = x'$$

$$y = y'$$

$$z = \gamma(z' + v_0 t')$$

$$t = \gamma(t' + (v_0/c^2)z')$$

These transformations are the simplest set of equations that preserve the postulates of relativity and have been shown to be consistent with experiment.

## 2 Velocity Transformation

The velocity transformation can be obtained in several ways. The most straightforward is to differentiate the spatial positions by time. We expect that the velocity transformations preserves the maximum velocity,  $c$ , as required when the Lorentz transformations were derived. Thus;

$$U'_x = \frac{dx'}{dt'} = \frac{dx'}{dt} \frac{dt}{dt'}$$

The form of the transform for  $V'_y$  should be similar as the equations are symmetric perpendicular to the boost direction  $z$ . Here we need to evaluate the derivative forms using the Lorentz transformations;

$$\frac{dt}{dt'} = \gamma(1 + (V_0/c)U'_z)$$

Then as  $x' = x$ , the velocity transformation is;

$$U_x = \frac{U'_x}{\gamma(1 + (V_0/c^2)U'_z)}$$

The transformation for the velocity,  $U_y$ , has the same form. For the velocity in the boost direction;

$$U'_z = \frac{dz'}{dt'} = \frac{dz'}{dt} \frac{dt}{dt'}$$

Using the following  $\frac{dt}{dt'}$ , evaluate;

$$\frac{dz'}{dt} = \gamma(U_z - V_0)$$

Combine this with the above expression and solve for  $U'_z$ . The transformation is

$$U'_z = \frac{U_z - V_0}{1 - (V_0/c^2)U_z}$$

Then collecting the velocity transformation equations one has;

$$U'_x = \frac{U_x}{\gamma(1 - (V_0/c^2)U_z)}$$

$$U'_y = \frac{U_y}{\gamma(1 - (V_0/c^2)U_z)}$$

$$U'_z = \frac{U_z - V_0}{1 - (V_0/c^2)U_z}$$

These equations are valid when the boost (velocity transformation) is in the same direction as the coordinate velocity. The velocity transformation for a boost in an arbitrary direction is more complicated and will be discussed later.

### 3 Acceleration Transformation

To find the transformation of an acceleration, take the derivative of the velocities.

$$a'_x = \frac{dU'_x}{dt'} = \frac{dU_x}{dt} \frac{dt}{dt'}$$

and;

$$a'_z = \frac{dU'_z}{dt'} = \frac{dU_z}{dt} \frac{dt}{dt'}$$

As previously, the appropriate derivatives of the velocity transformation equations are used. This results in;

$$a'_x = (1/\gamma^2) \left[ \frac{a^x}{(1 - (V_0 U_z)/c^2)^2} + \frac{(U_x V_0/c^2) a_z}{(1 - (V_0 U_z)/c^2)^3} \right]$$

$$a'_y = (1/\gamma^2) \left[ \frac{a^y}{(1 - (V_0 U_z)/c^2)^2} + \frac{(U_y V_0/c^2) a_z}{(1 - (V_0 U_z)/c^2)^3} \right]$$

$$a'_z = (1/\gamma^3) \left[ \frac{a_z}{(1 - (V_0 U_z)/c^2)^3} \right]$$

In a frame instantaneously at rest, place  $U_x = U_y = U_z = 0$ . Then;

$$a'_x = a_x/\gamma^2$$

$$a'_y = a_y/\gamma^2$$

$$a'_z = a_z/\gamma^3$$

Obviously, acceleration is not a Lorentz invariant. Thus Newtonian mechanics cannot be a physical law.

### 4 Images of a Lorentz Contraction

Consider the figure of a rod passing a distant observer as shown in Figure 1, and take a picture of the rod as it goes by. To do this, light from all points on the rod must reach the observer at the same time. Because some points are further away, light from these points must have been emitted earlier than points closer to the observer. We know that the length of the rod is contracted by  $\gamma$  as shown. Draw in the dotted figure of the rod. If the length of a side of the dotted figure is  $L$ , the angle between the rod and the dotted line is  $\cos(\alpha) = 1/\gamma$ . Light from the far corner of the rod travels a distance  $d$  in a time  $d/c$ . During this time the rod moves a distance  $(d/c)V$ . This makes the perpendicular side have a length  $\beta d/\sin(\alpha) = d$ . Therefore the rod appears as if it has rotated. Note the side of the rod perpendicular to the direction of motion becomes observable.

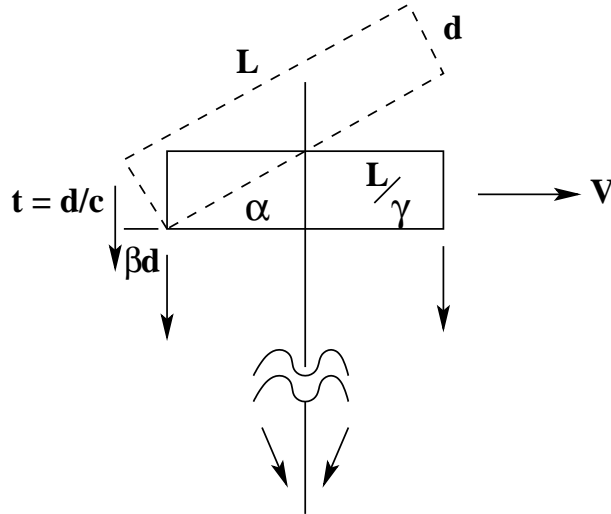


Figure 1: Rod traveling by a distant observer. Assume parallel light rays reach the observer at the same time.

Suppose we look at a hoop moving perpendicular to an axis as shown in Figure. 2. As with the analysis above, we expect to see the back side of the hoop. In a similar way to the analysis of the rod, the motion of the hoop appears to contract and rotate in the same way as the rod when simultaneous light rays are observed.

## 5 General velocity addition

Assume a velocity boost as illustrated in Figure 3. The velocity,  $\vec{V}$ , has an angle  $\theta$  with respect to a velocity,  $U_x \hat{x}$ . Deriving the transformation equations, is beyond the scope of this class as it requires a spatial rotation as well as a boost. Here, I just write the result.

$$S_x = \frac{U'_x + V' \cos(\theta)}{(1 + U'_x V' \cos(\theta)/c^2)}$$

$$S_y = \frac{V' \sin(\theta)}{\gamma(1 + U'_x V' \cos(\theta)/c^2)}$$

$$\gamma = \sqrt{\frac{1}{1 - U_x'^2/c^2}}$$

By symmetry,  $S_z$  has the same form as  $S_y$ . Then,

$$S^2 = S_x^2 + S_y^2 =$$

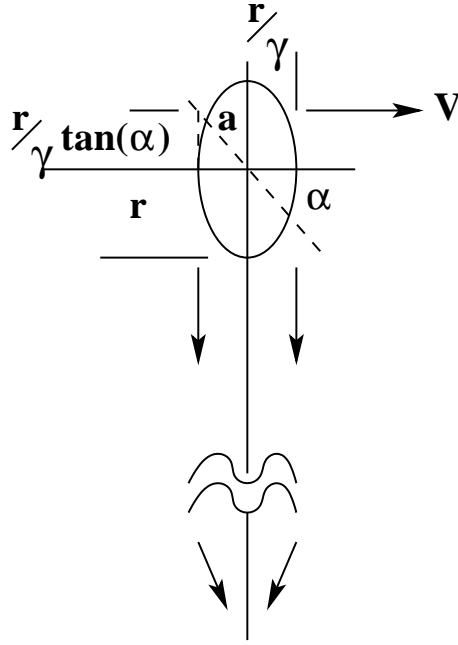


Figure 2: A hoop traveling by a distant observer. Light from all points on the surface must reach an observer at the same time.

$$\frac{U_x'^2 + V'^2 + 2U_x'V' \cos(\theta) - U_x'^2 V'^2 \sin^2(\theta)}{U_x'V' \cos(\theta)/c^2 + 1}$$

For  $\theta = \pi/2$

$$S_x = U_x'$$

$$S_y = V'/\gamma$$

$$S_2 = U_x'^2 + V'^2 - U_x'^2 V'^2/c^2$$

## 6 Analysis of a bar passing through a loop

In this example a bar, which has a length,  $L$ , is larger than the diameter of a loop,  $D$ , when measured at rest. It is boosted to a high velocity. The length is contracted sufficiently so that it is now measured to be less than the diameter of the loop. The loop is boosted in a direction perpendicular to that of the bar and allows the bar to pass through the loop. On the other hand, in the rest frame of the bar the diameter of the loop is contracted which makes its diameter even smaller than it was at rest. How does one explain this paradox?

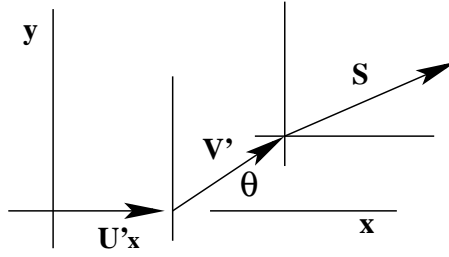


Figure 3: The geometry of a general velocity transformation

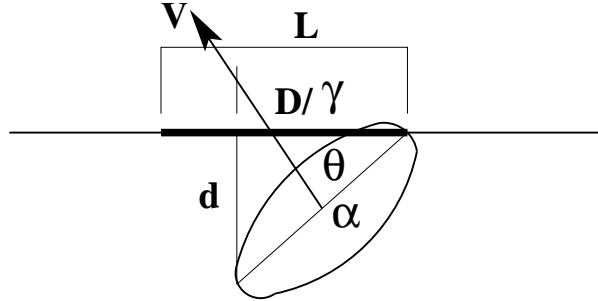


Figure 4: The geometry of a bar and a loop in motion

To analyze this problem, first note that to determine the length of the bar compared to that of the loop, one must have both in the same rest frame. Then in order to boost to a moving frame, an acceleration must be applied. Accelerating frames are not inertial frames, so how the measurements transform is complicated. Ignoring these issues, place an observer in the rest frame of the bar at the point where the bar and loop just begin to cross. The geometry is illustrated in Figure 4. The figure shows the loop diameter projected onto the plane containing the bar,  $D/\gamma$ . The observer sees the other end of the loop at an earlier time because it takes time for the signal (light) to travel to the observer's position. The position of this end is a distance  $Vt$  from the plane of the bar. The light from this point must travel the distance  $\alpha = ct$  to get to the observer. Note that this means that the bar is rotated by an angle,  $\theta$ . We find that;

$$\sin(\theta) = \beta$$

$$\cos(\theta) = 1/\gamma = D/\gamma\alpha$$

Which means that the length of the bar in its rest frame is  $\alpha = D$ . The loop is tilted and it now passes over the bar.

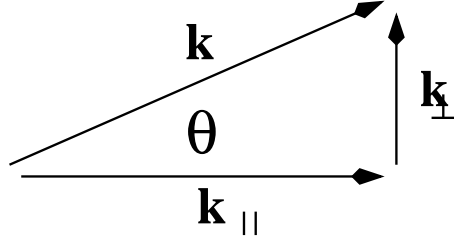


Figure 5: The wave vectors involved in the doppler shift.

## 7 Doppler Shift

Consider a plane wave propagating with wave vector  $\vec{k}$ . The phase of the wave has the form;

$$\phi = \vec{k} \cdot \vec{x} - \omega t$$

The phase of the wave is a scalar quantity as it is a vector contraction (dot product) of two 4-vectors. This will be demonstrated later, but since the phase is a scalar it has the same form in any inertial frame. Transform the phase between inertial frames.

$$\phi = \phi' = \vec{k}' \cdot \vec{x}' - \omega' t'$$

Substitute the Lorentz transformations to get the equation;

$$\vec{k} \cdot \vec{x} - \omega t = \gamma(V/c)k_z(z - Vt) + k_x x + k_y y - \gamma\omega'(t - (v/c^2)z)$$

Collect terms in  $x, y, z, t$ , which because of linear independence, each must be equal zero.

$$\gamma(\vec{K}' \cdot \vec{V} + \omega') = \omega$$

$$k'_{\perp} = k_{\perp}$$

$$\gamma(k'_{\parallel} + (\omega'/c^2)V) = k_{\parallel}$$

Because of the constancy of the light velocity

$$k'/\omega' = k/\omega = 1/c$$

As  $k_{\parallel} = k \cos(\theta)$  in the Figure 5, the Doppler shift is obtained.

$$\omega = \gamma\omega'(1 + \beta \cos(\theta))$$

There is then a relation between the angles in the inertial frames between the wave vector and the propagation direction;

$$\tan(\theta) = k_{\perp}/k_{\parallel} = \frac{\sin(\theta')}{\gamma(\cos(\theta') + \beta)}$$

$$\tan(\theta') = k'_{\perp}/k'_{\parallel} = \frac{\sin(\theta)}{\gamma(\cos(\theta) - \beta)}$$

Note there is also a transverse Doppler shift when the angle is not 0 or  $\pi$ .

## 8 Boosted Light Cone

Now assume a light source in its rest frame which emits light isotropically with total power,  $P_0$ . Thus the intensity is defined as the power per unit area onto a surface. The intensity,  $I$ , into a solid angle must be constant independent of the reference frame. Therefore;

$$I(\theta) = I(\theta') \frac{d\Omega'}{d\Omega} \propto \frac{\sin(\theta') d\theta'}{\sin(\theta) d\theta}$$

From the Doppler transformation;

$$\sin(\theta') = \frac{\sin(\theta)}{\gamma[1 + (V/c) \cos(\theta)]}$$

This leads to;

$$I(\theta) \propto \frac{1 - (V/c)^2}{[1 - (V/c) \cos(\theta)]^2}$$

As  $(V/c) \rightarrow 1$  the light becomes forward peaked about  $\theta = 0$  with an angular width of  $\sqrt{1 - (V/c)^2}$ , Figure 6.

## 9 Twin Paradox

Suppose 2 clocks are synchronized and one placed on a rocket while the other remains at rest. The rocket travels a distance  $L$  at a velocity  $V$  and returns back to the starting point with velocity  $V$  where it is brought back to rest. The clock at rest,  $A$ , sends signals at a frequency  $f'$  to the clock on the rocket,  $B$ . The clock on the rocket sends signals to the clock  $A$  with the same rest frequency. Now determine the number of signals sent and received by each clock. To do this the doppler shifted frequencies are required.

$$f_{rest} = f'_{moving} \sqrt{\frac{1 \pm \beta}{1 \mp \beta}}$$



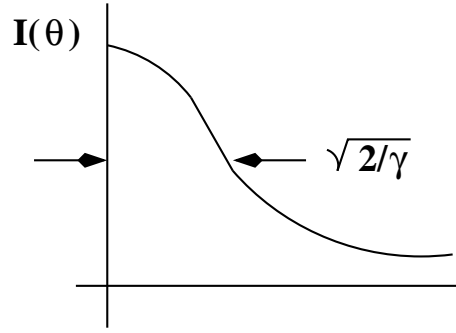


Figure 6: The searchlight effect for a uniformly emitted light source boosted to a reference frame moving with  $V/c \approx 1$

Fill out the table below to find that both clocks agree that the number of signals sent is the number received by the other clock. However, the number of signals sent by the clock in motion is less than the one which remains at rest.

Table 1: Times and counts related to the twin paradox

Name	A (earth rest)	B (onboard rocket)
Time	$t = 2L/V$	$t' = 2L/(\gamma V)$
No. Signals sent	$ft$	$ft'$
Time	$t = 2L/V$	$t' = 2L/(\gamma V)$
Time to B's Turn	$t = L/V + L/c$	$t' = L/(\gamma V)$
No. Signals Received	$f_{-}t = (fL/V)(1 + \beta)\sqrt{\frac{1 - \beta}{1 + \beta}}$	$f_{-}t' = (fL/V)\sqrt{1 - \beta^2}\sqrt{\frac{1 - \beta}{1 + \beta}}$
Remaining Time	$t = L/V - L/c$	$t' = L/(\gamma V)$
No. Signals Received	$f_{+}t = (fL/V)(1 - \beta)\sqrt{\frac{1 + \beta}{1 - \beta}}$	$f_{+}t' = (fL/V)\sqrt{1 - \beta^2}\sqrt{\frac{1 + \beta}{1 - \beta}}$
Total signals received	$2fL/\gamma V$	$2fL/V$

The reason that the problem is not symmetric, *ie* one cannot swap  $A$  and  $B$  and arrive at a symmetric answer, is that  $B$  must be accelerated to move away and return while  $A$  remains at rest and the Lorentz transformations are valid only between inertial frames.