Guided waves - Lecture 11

1 Wave equations in a rectangular wave guide

Suppose EM waves are contained within the cavity of a long conducting pipe. To simplify the geometry, consider a pipe of rectangular cross section with perfectly conducting walls, and oriented so that the long dimension of the cavity lies along the $\hat{z}$ axis. In general, the cavity is filled with a material of dielectric, ϵ, and permitivity, μ, which of course could also be free space, Figure 1. Then solve Maxwell's equations for the fields contained within the cavity.

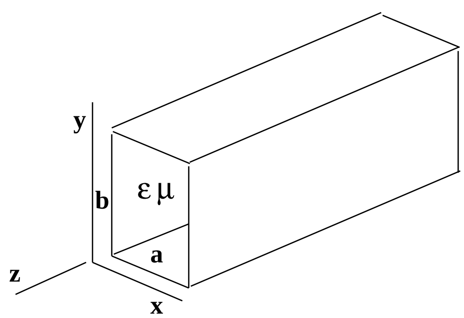

Figure 1: The geometry of a rectangular wave guide

For a perfect conductor, the E field must be perpendicular to the conducting surfaces and the magnetic field must be tangential to these surfaces. Apply the boundary conditions using the equations for the fields evaluated on the interior surface of the pipe.

$$\vec{B} \cdot \hat{n} = 0$$

$$\vec{E} \times \hat{n} = 0$$

In the above, $\hat{n}$ is the unit normal to a conducting surface. There is no free charge or currents within the dielectric of the cavity. Choose a harmonic time dependence.

$$\vec{E}(\vec{x}, t) \rightarrow \vec{E}(\vec{x})\, e^{-i\omega t}$$

$$\vec{B}(\vec{x}, t) \rightarrow \vec{B}(\vec{x})\, e^{-i\omega t}$$

Maxwell's equations then reduce to;

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = i\omega \vec{B}$$

$$\vec{\nabla} \times \vec{B} = -i\mu\epsilon\omega \vec{E}$$

The above equations are combined to obtain the differential equations for E and B.

$$[\nabla^2 + \mu\epsilon\omega^2] \begin{bmatrix} \vec{E} \\ \vec{B} \end{bmatrix} = 0$$

The z axis is the unique direction for wave propagation. The geometry of the pipe is Cartesian so solve the equations in Cartesian coordinates. The Laplacian operator in this case is the scalar Laplacian. Write;

$$\nabla^2 = \nabla_T^2 + \nabla_z^2$$

where $\nabla_z^2 = \frac{\partial^2}{\partial z^2}$. The operator, ∇_T^2, has the obvious partial derivatives with respect to x and y. Then look for a harmonic wave propagating in the $\hat{z}$ direction.

$$\vec{E}(\vec{x}, t) = \vec{E}(x, y)\, e^{i(kz - \omega t)}$$

$$\vec{B}(\vec{x}, t) = \vec{E}(x, y)\, e^{i(kz - \omega t)}$$

Substitute into the wave equation to obtain;

$$[\nabla_T^2 + (\mu\epsilon\omega^2 - k^2)] \begin{bmatrix} \vec{E} \\ \vec{B} \end{bmatrix} = 0$$

To proceed, the fields are separated into a longitudinal and a transverse component, for example;

$$\vec{E} = \vec{E}_T + E_z \hat{z}$$

The wave equation for $\vec{E}$ can be obtained using Maxwell's equations written in the transverse and longitudinal forms. For example, Faraday's law, $\vec{\nabla} \times \vec{E}/, = /, -\frac{\partial vecB}{\partial t}$ is;

$$\vec{\nabla}_T \times \vec{E}_T + \vec{\nabla}_T \times E_z \hat{z} + \frac{\partial}{\partial z}(\hat{z} \times \vec{E}_T) =$$

$$i\omega \vec{B}_T + i\omega B_z \hat{z}$$

with the condition that;

$$\vec{\nabla} \cdot \vec{E} \,=\, \vec{\nabla}_T \cdot \vec{E}_T \,+\, \frac{\partial E_z}{\partial z} \,=\, 0$$

There are similar equations for $\vec{B}$. Now note that;

$$\hat{z} \times \vec{\nabla}_T \times \hat{z} \,=\, \vec{\nabla}_T$$

$$\hat{z} \times \hat{z} \times \vec{E}_T \,=\, -\vec{E}_T$$

$$\hat{z} \cdot (\vec{\nabla}_T \times \hat{z}) \,=\, 0$$

$$\hat{z} \times \vec{\nabla}_T \times \vec{E}_T \,=\, 0$$

Working through the algebra, the resulting equations for the transverse fields are;

$$\vec{E}_T \,=\, -\frac{i}{[k^2 - \omega^2 \mu\epsilon]}\,[k\vec{\nabla}_T E_z \,-\, \omega\hat{z} \times \vec{\nabla}_T B_z]$$
$$\vec{B}_T \,=\, -\frac{i}{[k^2 - \omega^2 \mu\epsilon]}\,[k\vec{\nabla}_T B_z \,-\, \omega\hat{z} \times \vec{\nabla}_T E_z]$$

The above equations express the solution for all transverse components of the fields in terms of the longitudinal field components E_z and B_z. Solutions to the above equations must also satisfy the divergence equations, $\vec{\nabla} \cdot \vec{E} = 0$ and $\vec{\nabla} \cdot \vec{B} = 0$

2 Investigation of the longitudinal and transverse fields

Rembering that the EM wave is transverse, a wave in the $\hat{z}$ directon must have $E_z \,=\, B_z \,=\, 0$. In the above equations for $\vec{E}_T$ and $\vec{B}_T$ a non vanishing solution can occur only when $k^2 \,=\, \omega^2(\mu\epsilon)$, respectively. In this case the wave travels in the $\hat{z}$ direction with the velocity of an EM wave in a material having dielectric constant and magnetic permitivity ϵ and μ. However, Faraday's law shows;

$$\vec{\nabla}_T \times \vec{E}_T \,=\, -\frac{\partial B_z}{\partial t}\hat{z} \,=\, 0$$

This is because $E_z \,=\, B_z \,=\, 0$. Therefore, the solutions for this geometry are represented by the 2-D static field.

$$\vec{\nabla}_T \times \vec{E}_T \,=\, 0$$

and;

$$\vec{\nabla}_T \cdot \vec{E}_T = 0$$

This means that $\vec{E}_T$ can be obtained from a potential function, V_T, such that;

$$\vec{E}_T = -\vec{\nabla}_T V_T$$

and;

$$\nabla_T^2 V_T = 0$$

Obviously, this is Laplace's equation in 2-D. However, the electric field vanishes inside the hollow of a conductor when all surfaces are held at a constant potential. Thus there can be no solution to the above equations when both $E_z = B_z = 0$ unless there is another conductor inside the pipe. That is, a coaxial cable type of geometry is required. On the other hand in the next sections, solutions are obtained if either E_T or B_T vanishes, but both cannot simultaneously vanish.

3 Transverse Electric (TE) Modes

Now look at the case where $E_z = 0$. The geometry is a rectangular pipe with conducting walls, as shown in cross section in Figure 2. In the above equations for $\vec{E}_T$ and $\vec{B}_T$, substitute $E_T = 0$ and define $\gamma^2 = \mu\epsilon\omega^2 - k^2$. The following equations result.

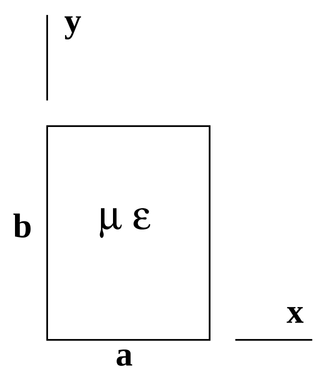

Figure 2: A cross section of the geometry of a rectangular pipe with conducting walls

$$\vec{B}_T = \frac{ik\vec{\nabla}_T B_z}{\gamma}$$

$$\vec{E}_T = -\frac{i\omega\,\hat{z} \times \vec{\nabla}_T B_z}{\gamma}$$

$$\nabla_T^2 B_z + \gamma^2 B_z = 0$$

The boundary condition requires that $E_T = 0$ when $x = 0, a$ or $y = 0, b$. In this case, E_T is obtained from $\vec{\nabla}_T B_z$ as is seen in the above equations. That is, require $\frac{\partial}{\partial n} B_z = 0$ at the surfaces with $\hat{n}$ the surface normal. Then the solution can be obtained by separation of variables in 2-D Cartesian coordinates.

$$B_z = B_0 cos(\tfrac{m\pi}{a} x)\, cos(\tfrac{n\pi}{b} y)$$

with $\gamma^2 = \pi^2([m/a]^2 + [n/b]^2)$. This choice of the general harmonic solution satisfies the boundary conditions at the surfaces. Various values of the integers, (m,and n), represent different TE modes. The dispersion relation is;

$$k^2 = \mu\epsilon\omega^2 - \gamma^2$$

For k to be real $\mu\epsilon\omega^2 \geq \gamma^2$. This means there is a frequency, ω_c, below which no wave will propagate in the wave guide. Remember that if k is imaginary, the wave decays exponentially instead of propagating with harmonic solution..

$$\omega_c = \gamma/\sqrt{\mu\epsilon} = \frac{\pi}{\sqrt{\mu\epsilon}}([m/a]^2 + [n/b]^2)^{1/2}$$

Almost always one chooses a dimension of the wave guide so that only one (lowest mode) propagates. Therefore if $b < a$, choose $m = 1$, $n = 0$, so that the lowest possible mode is then;.

$$\omega_c = \frac{\pi}{a\sqrt{\mu\epsilon}}$$

$$k = \sqrt{\mu\epsilon}\,[\omega^2 - \omega_c^2]^{1/2}$$

$$\gamma^2 = [\pi/a]^2$$

Define "free space" as unbounded space with μ and ϵ instead of μ_0 and ϵ_0. The wavelength of a wave in this space is $k = \sqrt{\mu\epsilon}\omega$ so that the wavelength in the guide is always less than the free space value. The phase velocity is;

$$V_p = \omega/k = \frac{1}{\sqrt{\mu\epsilon}}[\frac{1}{1 - (\omega_c/\omega)^2}]^{1/2}$$

This is greater than the phase velocity in free space. Note that as $\omega \rightarrow \omega_c$ $V_p \rightarrow \infty$. This must be further investigated, because it illustrates how the wave propagates within the guide. Before doing this, look at the other possible case, the TM mode, where $B_z = 0$ and $E_z \neq 0$. The diagram showing the fields for the TE_{01} and other simple TE and TM modes are shown in Figure 3.

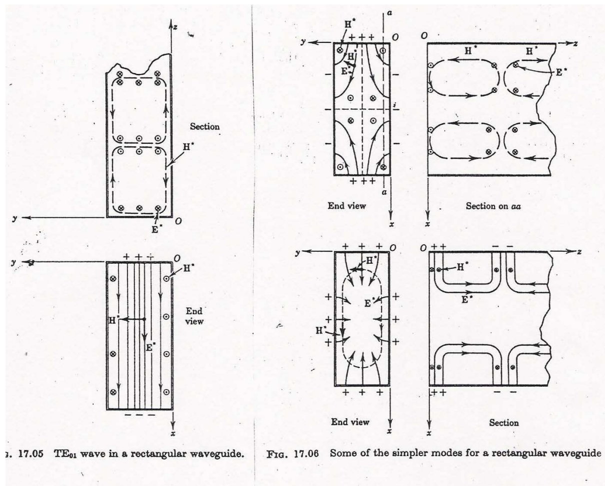

Figure 3: An example of the T_{01} mode and two other simple TE and TM modes

The group velocity must always be less that $1/\sqrt{\mu\epsilon}$. Evaluating;

$$V_g \,=\, \frac{dk}{d\omega} \,=\, \frac{[\omega^2 - \omega_c^2]^{1/2}}{\omega\,\sqrt{\epsilon\mu}}$$

Then note that;

$$V_g V_p \,=\, \frac{1}{\epsilon\mu}$$

4 Transverse magnetic (TM) modes

For the TM modes, $B_z = 0$ and solve the equation;

$$(\nabla_T^2 \,+\, \gamma^2)E_z \,=\, 0$$

$$\gamma^2 \,=\, \mu\epsilon\omega^2 - k^2$$

The boundary conditions are $E_z = 0$ for $x = 0, a$ and $y = 0, b$. Separation of variables provides the solution which satisfies the boundary conditions;

$$E_z \,=\, E_0\, sin(\tfrac{m\pi}{a}x)\, sin(\tfrac{n\pi}{b}y)$$

and $\gamma^2 \,=\, \pi^2([\frac{m}{a}]^2 \,+\, [\frac{n}{b}]^2)$

5 The coaxial transmission line

Consider the solution when $E_z = B_z = 0$. As pointed out earlier, this requires a non-vanishing solution to the static 2-D potential equation. The solution takes the form;

$$\vec{E} = \vec{E}(x, y)e^{i[kz-\omega t]}$$

$$\vec{B} = \vec{B}(x, y)e^{i[kz-\omega t]}$$

and;

$$\nabla_T^2 V_T = 0$$

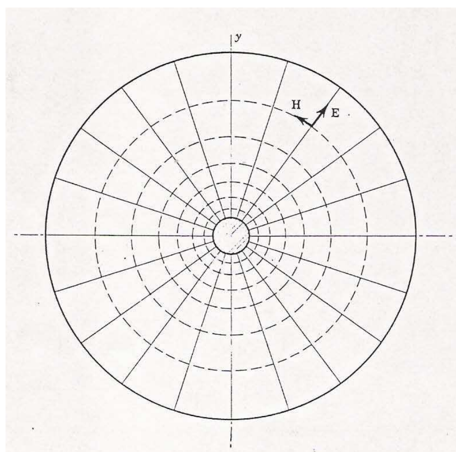

FIG. 17.11 TEM wave in a coaxial transmission line.

Figure 4: The transverse fields of a TEM mode of a coaxial line

Figure 4 shows the transverse fields for a coaxial transmission line. These fields are easily obtained from the application of Gauss' law, Ampere's law, and symmetries as previously demonstrated. The coaxial wave travels with phase velocity, $V_p = \omega/k = \frac{1}{\sqrt{\mu\epsilon}}$, in the $\hat{z}$ direction. Because $E_z = B_z = 0$, this propagation mode is called the Transverse Electric-Magnetic (TEM) mode. There are other geometries in which a TEM mode can propagate, and a few of these are illustrated in Figure 5. In each case, the fields are found by solving the static Laplace equation in 2-D with conducting boundary conditions in the appropriate geometry.

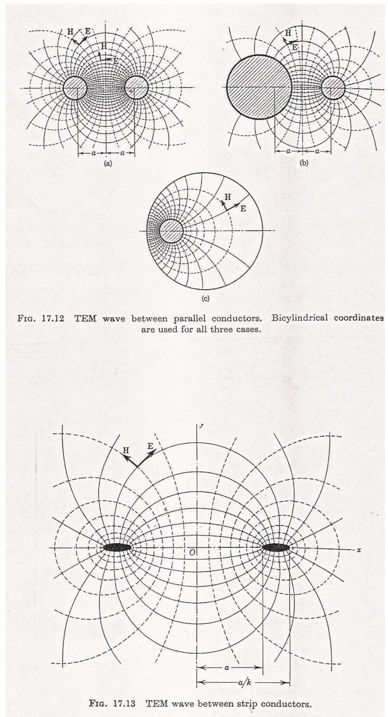

Fig. 17.12 TEM wave between parallel conductors. Bicylindrical coordinates are used for all three cases.

Fig. 17.13 TEM wave between strip conductors.

Figure 5: Examples of a few geometries in which TEM modes are possible.

6 Other geometries

Although detail studies of other geometries are not pursued, recognize that one could solve for the power flow in conductors by considering either the current (for example the transmission line developed in the last lecture) or the fields (as developed here). Solution in certain cases, particularly at high frequencies where transmission occurs over a distances of many wavelengths, are more easily obtained by considering the fields. As a representatives of other guided waves in other geometries, Figure 6, shows some propagation modes in cylindrical pipes. All solutions satisfy the pde with appropriate boundary conditions.

$$[\vec{\nabla}_T + \gamma^2] \begin{bmatrix} E_z \\ B_z \end{bmatrix}$$

7 The phase and group velocities in wave guides

It was previously found that the TEM mode propagates in the $\hat{z}$ direction with velocity $\sqrt{1/\mu\epsilon}$. Now return to the lowest TE mode obtained previously.

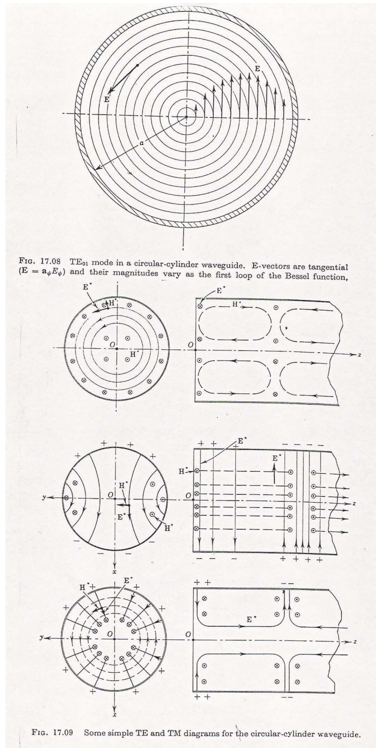

Figure 6: An infinitesmal element of a delay line

$$B_z = B_0 \, cos(\tfrac{\pi x}{a}) \, e^{i(kz - \omega t)}$$

$$B_x = \frac{ika}{\pi} \, B_0 \, sin(\tfrac{\pi x}{a}) \, e^{i(kz - \omega t)}$$

$$E_y = -\frac{i\omega a}{\pi} \, B_0 \, sin(\tfrac{\pi x}{a}) \, e^{i(kz - \omega t)}$$

Note that $\vec{B}$ has components in both the $\hat{z}$ and $\hat{x}$ directions. The Poynting vector (instantaneous), $\vec{S} = \vec{E} \times \vec{H}$ does not point along the $\hat{z}$ axis, and the wave moves diagonally as shown in Figure 7. Thus the wave reflects when it hits the perfectly conducting walls, so rewrite the E_y field in the form;

$$E_y = \frac{E_0}{2\imath} \, [e^{i[\pi x/a]} - e^{-i[\pi x/a]}] \, e^{i(kz - \omega t)}$$

This represents a superposition of two waves traveling in opposite directions in $\hat{x}$. When combined with the motion in $\hat{z}$, the wave travels diagonally with respect to the $\hat{z}$ direction. The wave vector along the diagonal is;

$$k^2 + (\pi/a)^2 = k_f^2 = (\omega/V_f)^2$$

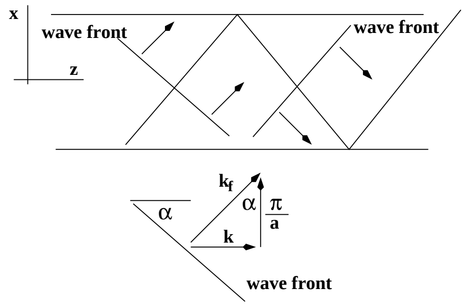

Figure 7: The wave fronts and propagating direction of the lowest TE wave in a rectasngular guide

$$k = \sqrt{\mu\epsilon}\,[\omega^2 - (\pi/a)^2 \frac{1}{\mu\epsilon}]^{1/2}$$

This is the dispersion relation obtained earlier, and gives the value of the cut off frequency. Thus each wave in the guide travels with phase velocity ω/k diagonally along the wave guide at an angle α with respect to the $\hat{z}$ axis. On reflection at the wall, the velocity in the $\hat{x}$ direction is reversed.

$$tan(\alpha) = \frac{ka}{\pi}$$

$$sin(\alpha) = k/k_f$$

$$cos(\alpha) = \frac{\pi}{ak_f}$$

The phase velocity in the $\hat{z}$ direction represents a wave crest traveling in the $\hat{z}$ direction;

$$V_p = \frac{\omega}{k_f\, sin(\alpha)} = \omega/k$$

This becomes ∞ as $\alpha \rightarrow 0$. For this mode, $\omega_c = \frac{\pi}{a\sqrt{\mu\epsilon}}$. Since $\omega/k = \sqrt{1/\mu\epsilon}$ then;

$$k_f = k_c = \pi/a$$

Since $\omega \geq \omega_c$ then $k \geq k_c$. This means that the wavelength must be longer than the dimension of the guide in this direction. Because the tangential component of E must vanish at the wall, a propagating mode must at least fit a wavelength within the wall spacing. The group velocity is $V_g = \frac{d\omega}{dk}$. Applied to the dispersion relation, this gives;

$$V_g = \frac{k/(\mu\epsilon)}{\omega} = \frac{1}{\omega\epsilon V_p}$$

Then $V_p V_g \, = \, c$

8 Power flow in a wave guide

The Poynting vector determines the power flow. The time averaged power flow as calculated from the fields is ;

$$\langle \vec{S} \rangle \, = \, (1/2\mu) Re(\vec{E} \times \vec{B}^*)$$

Now work this out for the TE mode, but the result for the TM mode follows easily and produces a similar result. The TE fields are;

$$\vec{B} \, = \, \vec{B}_T \, + \, B_z \hat{z}$$

$$\vec{B}_T \, = \, -\frac{i}{[k^2 - \omega^2 \epsilon \mu]} [k \vec{\nabla}_T B_z]$$

$$\vec{E}_T \, = \, -\frac{i\omega}{[k^2 - \omega^2 \epsilon \mu]} [\hat{z} \times \vec{\nabla}_T B_z]$$

Substitute into the expression for the Poynting vector, and expand the double cross product.

$$\langle \vec{S} \rangle \, = \, \frac{\omega k}{2\mu [k^2 - \omega^2 \epsilon \mu]^2} |\vec{B}_T|^2 \hat{z}$$

Thus even though the actual waves travel at an angle with respect to the $\hat{z}$ direction, the **TIME AVERAGE** power flows in the $\hat{z}$ direction. Instantaneously however, this is not the case.

9 Power loss in a wave guide

The power loss in the guide obviously must be due to fields which penetrate the walls of the guide. Previously perfectly conducting walls were assumed. By the way, a wave guide with superconducting walls also has power loss. This effect is discussed after developing radiation which results from accelerating charge. For the present, if there is a tangential component of $\vec{B}$ at a conducting surface, there must be a current which has a sheet of approximate thickness, $\delta \, = \, \sqrt{\frac{2}{\mu\omega\sigma}}$, where δ is the skin depth, as developed previously. Neglect the displacement current in Ampere's law (assume excellent but not infinite conductivity with $\vec{J} \, = \, \sigma \vec{E}$).

$$\vec{\nabla} \, \times \, \vec{H} \, = \, \vec{J} \, = \, \sigma \vec{E}$$

Assume harmonic time dependence, so that Faraday's law is;

$$\vec{\nabla} \times \vec{E} = i\omega \vec{B}$$

Now to simplify, choose the TE mode of a rectangular wave guide with transmission in the $\hat{z}$ direction. Consider the wall perpendicular to the $\hat{y}$ axis. For the TE mode as obtained previously;

$$B_T = -\frac{i}{[k^2 - \omega^2 \mu\epsilon]} [k\, \vec{\nabla}_T B_z]$$
$$\vec{E}_T = -\frac{i\omega}{[k^2 - \omega^2 \mu\epsilon]} [\hat{z} \times \vec{\nabla}_T B_z]$$

Consider a wave penetrating into the conductor in the $\hat{x}$ direction. The surface current is expected to be in the (y, z) plane. Then for the TE mode, E_y is the only non-zero component. Faraday's law gives;

$$\frac{\partial E_y}{\partial x} = i\omega B_z$$

From Ampere's law;

$$\frac{\partial B_z}{\partial x} = \sigma\mu E_y$$

Decoupling these equations;

$$\frac{\partial^2 B_z}{\partial x^2} = i\mu\sigma\omega\, B_z$$

Use the skin depth $\delta^2 = \frac{2}{\mu\sigma\omega}$ to write the solution in the conductor as;

$$B_z = B_0\, e^{-[1-i]x/\delta}$$

From this obtain;

$$E_y = -\frac{[1-i]}{\mu\sigma\delta} B_z$$

The time averaged Poynting vector is then;

$$\langle S \rangle = (1/2\mu) Re|\vec{E} \times \vec{B}^*| = (2\mu)^{-3/2} \sqrt{\omega/\sigma} |B_z|^2$$

which points into the conductor. The power loss is obtained by integration over the surface area of the walls. Note that B_z is the tangential magnetic field in the guide hollow at the surface of the wall. The impedance of the wave at the wall is;

$$Z = E/H = \sqrt{2\mu\omega/\sigma}$$

The above also shows that $E << H$.