Potentials - Lecture 13

1 Gauge invarience

A gauge transformation does not change the value of the fields which determine the Lorentz force and the field energy. Remember, energy and momentum (force) are the observables which determine the physics. Thus the physics remains unchanged under a gauge transformation. Note that since;

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

where $\vec{A}$ is the vector potential. Now add a gradient term to A. Thus;

$$\vec{A} \rightarrow \vec{A} + \vec{\nabla}\Lambda$$

Since the curl of the gradient vanishes the magnetic field does not change so the vector potential is not uniquely defined by the field equations. Then the electric field is determined from Faraday's law.

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \frac{\partial \vec{A}}{\partial t}$$

$$\vec{\nabla} \times [\vec{E} + \frac{\partial \vec{A}}{\partial t}] = 0$$

Therefore the term in the brackets can be obtained from the gradient of a potential, V.

$$[\vec{E} + \frac{\partial \vec{A}}{\partial t}] = -\vec{\nabla} V.$$

To keep $\vec{E}$ from changing when $\vec{A} \rightarrow \vec{A} + \vec{\nabla}\Lambda$ change the scalar potential as follows.

$$V \rightarrow V - \frac{\partial \Lambda}{\partial t}$$

This freedom to choose various values of the potentials is called a gauge transformation. Previously a particular gauge imposed the Lorentz condition which allowed separation of the coupled potential equations. The form of the Lorentz condition is;

$$\vec{\nabla} \cdot \vec{A} + (1/c^2)\frac{\partial V}{\partial t} = 0$$

In covarient form this is;

$$\partial_\alpha A^\alpha \, = \, 0$$

Under a gauge transformation this equation takes the form;

$$\vec{\nabla} \cdot \vec{A} \, + \, (1/c^2) \, \frac{\partial V}{\partial t} \, + \nabla^2 \Lambda \, - \, (1/c^2) \, \frac{\partial^2 \Lambda}{\partial t^2} \, = \, 0$$

So that the equation is;

$$\nabla^2 \Lambda \, - \, (1/c^2) \, \frac{\partial^2 \Lambda}{\partial t^2} \, = \, 0$$

Maxwell's equations in terms of the 4-potentials are;

$$\vec{\nabla} \cdot \vec{E} \, = \, -\nabla^2 V \, - \, \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{A} \, = \, \rho/\epsilon$$

$$\nabla^2 \vec{A} \, - \, \epsilon\mu \frac{\partial^2 \vec{A}}{\partial t^2} \, - \, \vec{\nabla}(\vec{\nabla} \cdot \vec{A} \, + \, \epsilon\mu \frac{\partial V}{\partial t}) \, = \, -\mu \vec{J}$$

Then all transformations which satisfy the Lorentz condition belong to the Lorentz gauge resulting in the equations for the 4-potentials;

$$\nabla^2 V \, - \epsilon\mu \frac{\partial^2 V}{\partial t^2} \, = \, -\rho/\epsilon$$

$$\nabla^2 \vec{A} \, - \epsilon\mu \frac{\partial^2 \vec{A}}{\partial t^2} \, = \, -\mu \vec{J}$$

However, it is possible to choose a gauge in which $\vec{\nabla} \cdot \vec{A} \, = \, 0$. This is the Coulomb gauge, and in this gauge;

$$\nabla^2 V \, = \, -\rho/\epsilon$$

with solution;

$$V \, = \, \frac{1}{4\pi\epsilon} \int \, d\tau' \, \frac{\rho(\vec{x}', t)}{|\vec{x}' - \vec{x}|}$$

In this gauge the scalar potential is the instantaneous Coulomb potential, but the vector potential is much more complicated, as it will be the solution of the pde obtained previously after substituting, $\vec{\nabla} \cdot \vec{A} = 0$

$$\nabla^2 \vec{A} \, - \, \epsilon\mu \, \frac{\partial^2 \vec{A}}{\partial t^2} \, = \, \mu \vec{J} \, + \, \epsilon\mu \, \vec{\nabla}(\frac{\partial V}{\partial t})$$

Gauge symmetry forms the basis of the fundamental equations governing the electromagnetic, weak, and strong (QCD) interactions.

2 Advanced and retarded potentials

Now develop the potential formulation for an arbitrary, time-dependent charge distribution. Begin using the covarient form of Maxwell's equations written in terms of the field tensor. (ignore covarient and contravarient notation)

$$\frac{\partial F_{ik}}{\partial x_k} = (1/c) j_i$$

Remember that this equation reproduces Ampere's law and Gauss's Law. The other 2 equations are automaticaly satisfied by the definition of the form for the field strength tensor.

$$F_{ik} = \frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k}$$

Substitute this expression into Maxwell's equations above and apply the Lorentz condition;

$$\frac{\partial A_i}{\partial x_i} = 0$$

The result is;

$$\frac{\partial^2 A_i}{\partial x_k^2} = (1/c) j_i$$

This can be expanded in terms of the 4-vector components;

$$\nabla^2 \vec{A} - (1/c^2)\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}$$

$$\nabla^2 V - (1/c^2)\frac{\partial^2 V}{\partial t^2} = -\rho/\epsilon$$

These are the inhomogeneous wave equations obtained previously, and confirm that the potentials form a Lorentz 4-vector. For an elegant construction of a solution to these equations see the appendix to this lecture. However, a solution here is obtained using symmetry and analogy. Considering a spherically symmetric charge distribution. This is not really a restriction as a solution can be constructed for the general case by superposition (note the equations are linear). The source (charge or a vector component of the current density) is assumed to be confined to a small volume $r < a$ about the origin. Choose a time dependent source.

$$s = s(t)$$

For a source confined to a small volume, the wave equation for $r > a$ is independent of angle.

$$[\nabla^2 - (1/c^2)\frac{\partial^2}{\partial t^2}]\lambda = 0$$

Then $\lambda = \lambda(r,t)$. Therefore in spherical coordinates;

$$[(1/r)\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) - (1/c^2)\frac{\partial^2}{\partial t^2}]\lambda = 0$$

Let $\lambda = \eta/r$ and substitute;

$$\frac{\partial^2\eta}{\partial r^2} - (1/c^2)\frac{\partial^2\eta}{\partial t^2} = 0$$

The solutions must have the form, $\eta = f(r \pm ct)$. Because of causality, choose only the outgoing wave.

$$\lambda = \frac{f(r - ct)}{r} \quad r > a$$

For a sufficiently small volume about $r < a$ a solution for a point charge having the form below occurs.

$$\lambda = \frac{s(t)}{r} \qquad\qquad r < a \to 0$$

Compare solutions for $r < a$ and $r > a$;

$$f(-ct) = s(t)$$

Then the form of the general solution is expected to be;

$$\lambda(t) = \frac{s(t - r/c)}{r}$$

This indicates that the time at the position of the potential is retarded by the velocity of the signal, c. By superposition, combine solutions from different points in the source distribution using $r \to |\vec{r} - \vec{r}'|$. The integral is over d$\tau'$ which is the differential 4-volume (space,time).

$$\lambda(\vec{r}, t) = \int d\tau' \, \frac{s(\vec{r}', t')}{|\vec{r} - \vec{r}'|}$$

Causality is imposed by;

$$t' \, = \, t \, - \, |\vec{r} - \vec{r}'|/c$$

This results in the following retarded potential forms. The solution has imposed the retarded time, by applying causality.

$$V \, = \, \frac{1}{4\pi\epsilon} \int \, d^3x' \, \frac{\rho(\vec{x}', \, t' = t - R/c)}{R}$$

$$\vec{A} \, = \, \frac{\mu}{4\pi} \int \, d^3x' \, \frac{\vec{J}(\vec{x}', t' = t - R/c)}{R}$$

3 Lienard-Wiechert potentials

Now apply the above solutions to a point charge moving in an arbitrary direction. Note that at any instant of time the potential is due to a charge at a previous point in space time, *ie* a retarded point on the world line. The time t' corresponding to this point is determined by;

$$t' = t - R/c$$

To determine the potential solution, assume that the charge has finite dimensions. Although not obvious, the spatial dimension causes a shift in the time due to the distance between the differing positions of the charges. This time delay does not vanish in the limiting case when the size of the distribution is reduced to zero, so it must be included in the source. The source term integrand for the scalar potential uses the charge density $\rho(\vec{r}', t_r)$ where t_r is the retarded time point. Note that an integral over the spatial dimensions of the source, $\int \, d^3x' \, \rho$, does not result in the charge of the particle. To get the charge one must integrate over the distribution at the same instant of time, but the potential due to ρ is evaluated over the world line in which the time changes as the particle moves. Thus a point charge must be regarded as a limit of an extended charge. In general the apparent volume is related to the actual volume by the factor;

$$d^3x' \, = \, d^3x/[1 - \hat{r} \cdot \vec{\beta}]$$

This includes the Lorentz contraction as observed from the angle θ and is shown in the Figure 1. Note that the above expression is obtained by observing that the time for the light to travel the extra distance from the trailing edge is L'/c while the volume moves a distance $L - L'$ with a velocity v. Thus;

$$L'/c = (L' - L)/v$$

Observed from an angle;

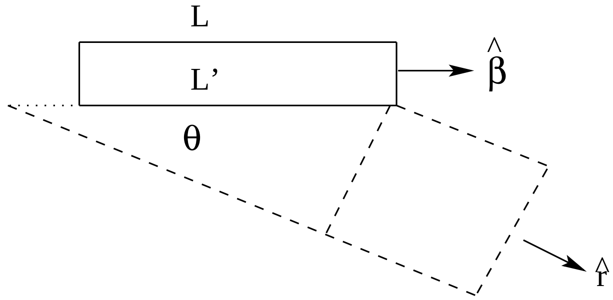


Figure 1: The length element for the volume as viewed at the angle θ

$$L' \, cos(\theta)/c \;=\; (L' - L)/v$$

$$L \;=\; L'[1 \;-\; \beta \, cos(\theta)]$$

As the volume shrinks to a differential point, the charge must be divided by a factor of $\kappa \;=\; 1 - \hat{n} \cdot \vec{\beta}$.

$$q = \rho \, dx \, dy \, dz \;\rightarrow\; \rho dx' \, dy' \, dz'/\kappa \;=\; q'/\kappa$$

It then follows that the scalar potential is obtained from the expression;

$$V \;=\; \frac{1}{4\pi\epsilon} \frac{q}{R - \vec{R} \cdot \vec{\beta}}$$

The vector potential is obtained in a similar way for each component.

$$\vec{A} \;=\; \frac{1}{4\pi\epsilon c} \frac{q\vec{\beta}}{[R - \vec{R} \cdot \vec{\beta}]}$$

These are the Lienard-Wiechert potentials for a point charge.

4 Fields from the Lienard-Wiechert potentials

The fields are obtained using the equation below where the geometry is shown in Figure 2.

$$\vec{E} \;=\; -\vec{\nabla} V \;-\; \frac{\partial \vec{A}}{\partial t} \;\; \vec{B} \;=\; \vec{\nabla} \;\times\; \vec{A}$$

In the figure $R = |\vec{r} - \vec{x}| \;=\; c(t - t_r)$, where t_r is the retarded time. The subscript r indicates that the retarded solution is used. After some careful algebra;

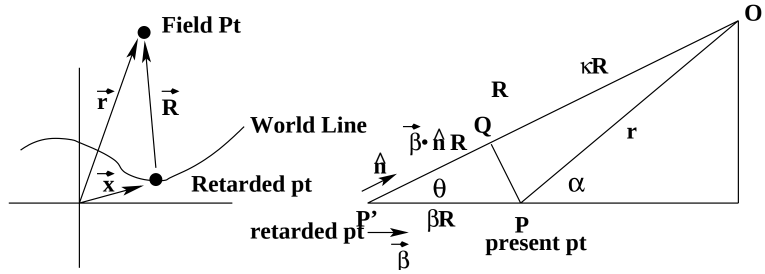


Figure 2: The geometry of the field and source points for a moving point charge

$$\vec{E} = \frac{q}{4\pi\epsilon}[\frac{\hat{n} - \vec{\beta}}{\gamma^2(1 - \vec{\beta}\cdot\hat{n})^3 R^2}]_r + \frac{q}{4\pi\epsilon c}[\frac{\hat{n}\times([\hat{n} - \vec{\beta}]\times\dot{\vec{\beta}})}{(1 - \vec{\beta}\cdot\hat{n})^3 R}]_r$$

The 2 terms represent static (first term) and radiation fields (second term). Note that the second term depends on the acceleration, $\dot{\vec{\beta}}$. Now look at the first term. It has a form that decreases with distance as $\frac{1}{r^2}$ as one would expect for the field of a static charge (including a field of a charge moving with constant velocity). Now use geometric relations obtained from Figure 2, and define $\kappa = (1 - \vec{\beta}\cdot\hat{n})$. The following identities are used.

$$(\kappa R)^2 r^2 = (PQ)^2 = R^2\beta^2 sin^2(\theta) = r^2 - \beta^2 r^2 sin^2(\alpha)$$

and;

$$(\hat{n} - \vec{\beta})R = \vec{r}$$

Substitution in the static field above yields;

$$\vec{E}_s = \frac{q}{4\pi\epsilon}\frac{(1 - \beta^2)\vec{r}}{\gamma^2(1 - \beta^2 sin^2(\theta))^{3/2} r^3}$$

This is the equation for the electric field of a charge moving with constant velocity which was obtained earlier by a Lorentz transformation on the field of a charge at rest.

Then look at the radiation field (second term). The field decreases with distance as $\frac{1}{r}$ as expected for a radiation field. In the radiation zone (far field) a wave solution of the form $e^{i(\vec{k}\cdot\vec{r}-\omega t)}/r$ is obtained. Placed into Faraday's law the magnetic field has the form; $\vec{B} = (1/c)\hat{k}\times\vec{E}$. Then write the Poynting vector;

$$\vec{S} = (1/\mu)\vec{E}\times\vec{B} = (1/\mu c)[\vec{E}\times\hat{k}\times\vec{E}] = \frac{\hat{k}E^2}{\mu c}$$

Collect terms so that in the MKS system the magnitude of the Poynting vector is ;

$$S \;=\; \frac{\mu c q^2}{16\pi^2}[\frac{(\hat{n}\times(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}})^2}{R^2|(1-\vec{\beta}\cdot\hat{n})^3|^2}]_r$$

Note the dependence on the acceleration, $\dot{\vec{\beta}}$. Recall that S is the power flowing through a surface area. Choose the infintesmal area $R^2 d\Omega$ and change the time derivative of the power $\frac{dP}{dt}$ from the present time to the retarded time t'. Thus from previous expressions;

$$t' \;=\; t \;-\; R(t')/c$$

$$\frac{dt}{dt'} \;= \kappa$$

Collect terms to write the poynting vector so that it represents power radiated into a solid angle $d\Omega$ in the particle's own time frame.

$$\frac{dP}{d\Omega} \;=\; \frac{\mu_0 q^2 c}{16\pi^2 \epsilon_0}\,[\frac{|\hat{n}\times(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}|^2}{(1-\vec{\beta}\cdot\hat{n})^5}]_r$$

5 Limiting cases

Now look at two limiting cases. The case where the velocity is parallel to the acceleration (linear acceleration), and the case where the velocity is perpendicular to the acceleration (circular acceleration).

5.1 Parallel velocity and acceleration

When $\vec{\beta}$ and $\dot{\vec{\beta}}$ are parallel;

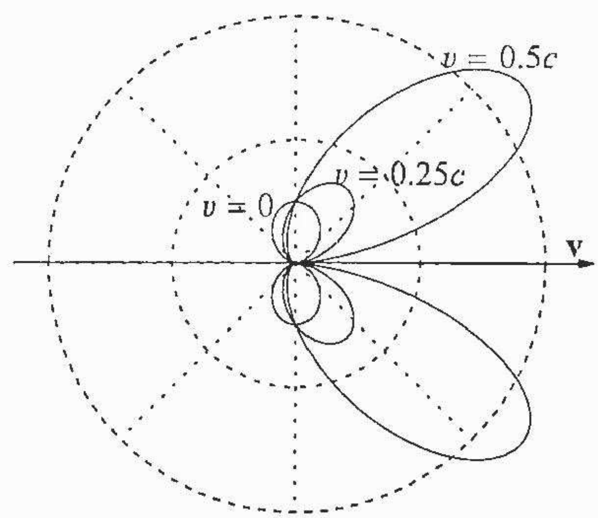
$$\hat{n}\times(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}} \;=\; \hat{n}\times\hat{n}\times\dot{\vec{\beta}}$$

$$|\hat{n}\times\hat{n}\times\dot{\vec{\beta}}|^2 \;=\; \dot{\beta}^2\, sin^2(\theta)$$

Here θ is the angle between the velocity (acceleration) and the direction of observation. In the MKS systen the angular distribution of the power is;

$$\frac{dPower}{d\Omega} \;=\; \frac{\mu q^2 c}{16\pi^2}\,\frac{\dot{\beta}^2\, sin^2(\theta)}{\kappa^5}$$

In the above, $\kappa \;=\; (1-\vec{\beta}\cdot\hat{n})$ with $\hat{n}$ the direction of observation. Note that when $\beta = 0$ $\kappa = 1$ and the power maximizes at $\theta = \pi/2$ with respect to the velocity. The maximum of the power as a function of angle approaches $\theta = 0$ as $\beta \rightarrow 1$. A contour plot of the power distribution is shown in Figure 3. Integration over the angles gives the total radiated power



Polar diagram of the energy loss angular distribution factor $\sin^2\theta/(1 - v\cos\theta/c)^5$ during bremsstrahlung for particle speeds $v = 0$, $v = 0.25c$, and $v = 0.5c$.

Figure 3: A contour plot of the power distribution as a function of the velocity of a linearly accelerated charge

(MKS);

$$Power \;=\; \frac{\mu q^2 c}{6\pi}\,\dot{\beta}^2\gamma^6$$

5.2 Perpendicular velocity and acceleration

If the motion is circular, choose the coordinate system shown in Figure 4. The motion of the charge is in the (x, z) plane. Evaluate;

$$|\hat{n} \times (\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}|^2 \;=\; \dot{\beta}^2[\kappa^2 + \frac{sin^2(\theta)\cos^2(\phi)}{\gamma^2}]$$

This gives the angular distribution of the power (MKS).

$$\frac{dP}{d\Omega} \;=\; \frac{\mu q^2 c}{16\pi^2}\,\frac{\dot{\beta}^2}{\kappa^3}[1 - \frac{sin^2(\theta)\cos^2(\phi)}{\gamma^2\kappa^2}]$$

Integrated over angle, the total power is (MKS);

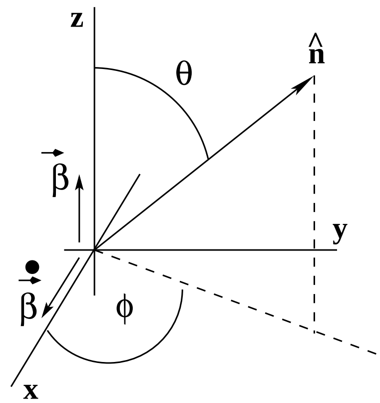


Figure 4: The geometry for an acceleration perpendicular to the velocity

$$P \;=\; \frac{\mu q^2 c}{6\pi}\,\dot{\beta}^2\gamma^4$$

5.3 Other examples

The non-relativistic radiation when the charge velocity $\beta \ll 1$ is obtained by setting $\gamma = 1$ in the above equations. This gives the Lamor equation for radiation.

$$P \;=\; \frac{\mu q^2 c}{6\pi}\dot{\beta}^2$$

Other examples of the electric field generated by an accelerating charge are shown in Figure 5. Note the concentration of the transverse components of the field at varions positions. Remember the transverse field is a radiation component while the longitudinal field is a static one. The transverse components correlate with the value of acceleration. Suppose the charge moves harmonically along the z axis of a Cartesian reference frame.

$$z(t') \;=\; a\,cos(\omega t')$$

$$\dot{z}(t') \;=\; -a\omega\,sin(\omega t')$$

$$\ddot{z}(t') \;=\; -a\omega^2\,cos(\omega t')$$

Evaluate the radiative power as follows;

$$|\hat{n} \times (\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}|^2 \;=\; (\hat{n} \cdot \dot{\vec{\beta}})^2(\hat{n} - \vec{\beta})^2 + [\hat{n} \cdot (\hat{n} - \vec{\beta})]^2\dot{\beta}^2 - 2(\hat{n} - \vec{\beta}) \cdot \dot{\vec{\beta}}(\hat{n} \cdot \dot{\vec{\beta}})[\hat{n} \cdot (\hat{n} - \vec{\beta})]$$

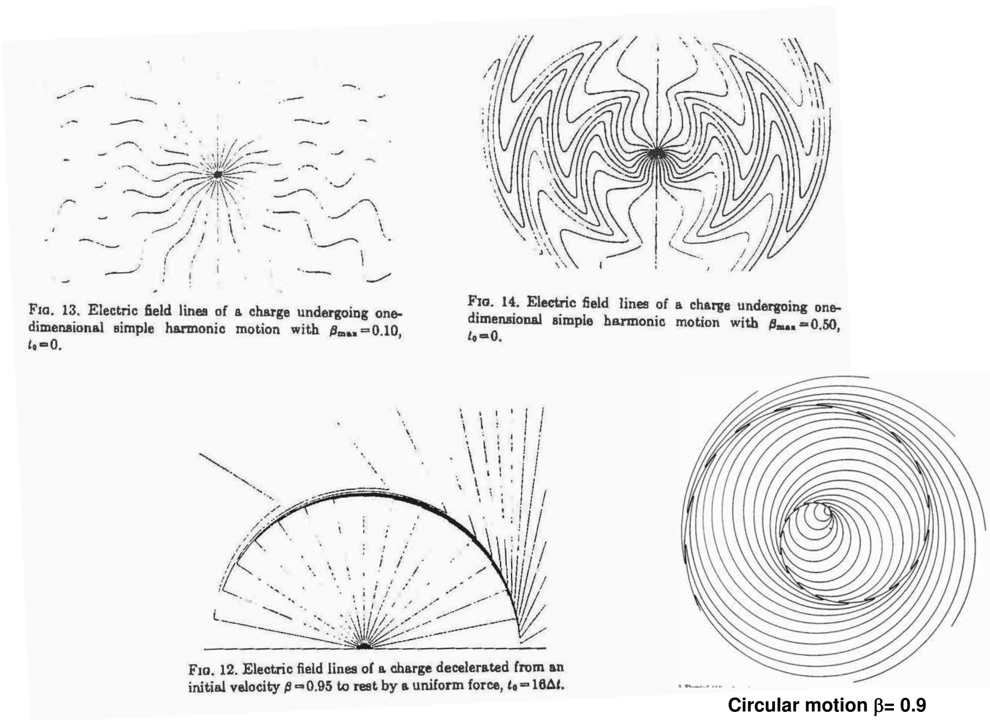


Figure 5: Examples of the electric field for an accelerated charge

$$(\hat{n} - \vec{\beta})^2 \;=\; 1 + \beta^2 - 2\hat{n}\cdot\vec{\beta}$$

Collect terms;

$$|\hat{n} \times (\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}|^2 \;=\; \dot{\beta}^2\, sin^2(\theta)$$

$$\frac{dP}{d\Omega} \;=\; \frac{\mu q^2 c}{16\pi^2}\,\frac{\dot{\beta}^2\, sin^2(\theta)}{\kappa^5}$$

Substitute for $\dot{\beta}$

$$\frac{dP}{d\Omega} \;=\; \frac{\mu q^2}{16\pi^2 c}\,\frac{a^2\omega^4\, sin^2(\theta)}{\kappa^5}$$

Note that $qa = p$ is the electric dipole moment. One usually finds electric dipole radiation written as;

$$\frac{dP}{d\Omega} \;=\; \frac{\mu}{16\pi^2 c}\, p^2\omega^4\, sin^2(\theta)$$

The factor, κ^5, is a relativistic effect which corrects the classical value for high frequencies. In summary, a charged particle radiates when it is accelerated, and this radiation is a relativistic effect.

6 Appendix - Green function for a scalar wave

Now look for a solution, λ, for one of the 4 coordinates represented by the 4-potential $(V, \vec{A})$. To find a solution we choose to develop an appropriate Green function, G. The equations to be solved are;

$$\nabla^2\lambda \;-\; \mu_0\epsilon_0\frac{\partial^2\lambda}{\partial t^2} \;=\; -4\pi s(x,t)$$

$$\nabla^2 G \;-\; \mu_0\epsilon_0\frac{\partial^2 G}{\partial t^2} \;=\; -4\pi\delta(\vec{r} - \vec{r}')\delta(t - t')$$

The Green function is a solution to the wave ewuation for a point source. The solution for a point souce is more easily found, as for example it previously was used with symmetry and homogeneity. Here to find this solution, use any complete set of functions appropriate to the geometry of the problem. Then to proceed, multiply the first eqyation by G, the second equation by λ, and subtract one from the other.

$$\vec{\nabla}\cdot[\lambda\vec{\nabla}G \;-\; G\vec{\nabla}\lambda] \;-\; [(1/c^2\frac{\partial}{\partial t}[\lambda\frac{\partial G}{\partial t} - G\frac{\partial\lambda}{\partial t}] \;=$$
$$-4\pi\delta(\vec{r} - \vec{r}')\delta(t - t')\lambda \;+ 4\pi\, S\, G$$

Integrate over space and time.

$$\int_{-\infty}^{t+\epsilon} dt' \oint [\lambda \vec{\nabla} G \, - \, G \vec{\nabla} \lambda] \, - \, (1/c^2) \int \, dx'^3 [\lambda \frac{\partial G}{\partial t} - G \frac{\partial \lambda}{\partial t}]_{-\infty}^{t+\epsilon} \, =$$
$$-4\pi\lambda + 4\pi \int \, dt \int \, dx'^3 \, s \, G$$

The boundary conditions are Cauchy on an open boundary in 4-d space. Choose G and $\vec{\nabla} G$ to vanish on the boundaries. (*ie* Cauchy boundary conditions give a unique solution to the wave equation) Apply causality to set the values at $t' = -\infty$ and $t' = t + \epsilon$ to zero. The result is

$$\lambda \, = \int \, dt \int \, dx'^3 \, s \, G$$

The above equation shows that the solution is just the product of the point solution times the source strength at that point integrated over the volume. Now find a representation for the Green function, G.

$$\nabla^2 G \, - \, \mu_0 \epsilon_0 \frac{\partial^2 G}{\partial t^2} \, = \, -4\pi \delta(\vec{r} - \vec{r'}) \delta(t - t')$$

As indicated earlier, choose G and $\vec{\nabla} G$ to vanish on the boundary, and apply causality. Expand the δ function and the Green's function, G, in a Fourier transform. For compactness in notation below, let $\tau = t - t'$ and $\vec{R} = \vec{r} - \vec{r'}$.

$$f(t) \, = \, \frac{1}{\sqrt{2\pi}} \int \, d\omega \, \tilde{\mathrm{f}}(\omega) \, e^{i\omega t}$$
$$\tilde{\mathrm{f}}(\omega) \, = \, \frac{1}{\sqrt{2\pi}} \int \, dt \, f(t) \, e^{-i\omega t}$$

Then by substitution;

$$f(t) \, = \, \frac{1}{2\pi} \int \, dt' \, f(t') \int \, d\omega \, e^{i\omega(t-t')}$$

To conclude;

$$2\pi \delta(t - t') \, = \int \, d\omega \, e^{i\omega(t-t')}$$

In a similar way, obtain the expansion for δR. This results in;

$$\delta(\vec{r} - \vec{r'}) \delta(t - t') \, = \, \frac{1}{(2\pi)^4} \int \, d\vec{k} \int \, d\omega \, e^{i\vec{k} \cdot (\vec{r} - \vec{r'})} \, e^{-i\omega(t-t')}$$

Represent G by the Fourier transform;

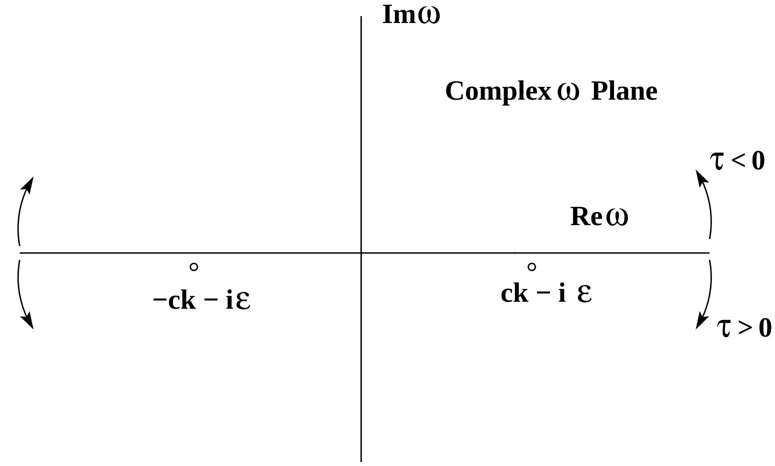


Figure 6: Diagram for integration of the fourier coefficient in the complex ω plane

$$G = \int d\omega \int d\vec{k}\, e^{i\vec{k}\cdot\vec{R}}\, e^{-i\omega\tau}\, g(\vec{k},\omega)$$

By Fourier transformation of the wave equation with boundary conditions, an algebraic equation results with solution;

$$g(\vec{k},\omega) = 1/(4\pi)^3 \frac{1}{k^2 - \omega^2/c^2}$$

Note g has singularities at $\pm kc$. The inverse Fourier transform must be evaluated in the complex ω plane. Since G must satisfy causality, $G = 0$ when $t < t'$ ($\tau < 0$), which represents an outgoing wave when $\tau > 0$. The integral to be evaluated is;

$$G = 1/(4\pi)^3 \int dk^3 \int d\omega \frac{e^{i\vec{k}\cdot\vec{R}}\, e^{-i\omega\tau}}{k^2 - (\omega + i\epsilon)^2/c^2}$$

The Cauchy integral theorem states that if $f(x)$ is analytic;

$$\oint dx \frac{f(x)}{x - x_0} = 2\pi\, i \text{ [Sum of Residues]}$$

The infinitesmal displacement, $-i\epsilon$, moves the poles below the $Re\,\omega$ axis in order to comply with causality. Integration proceeds as shown in the Figure 6. For $\tau < 0$ we close the integral loop in the upper half plane. Within this loop there are no poles so $G = 0$ consistent with causality. When $\tau > 0$ close the loop in the lower half plane. The calculus of residues gives;

$$G = (c/2\pi^2) \int dk^3\, e^{i\vec{k}\cdot\vec{R}} \frac{sin(ck\tau)}{k}$$

Integrate over angles and rewrite the result

$$G = (2c/\pi R) \int_0^\infty k^2\, dk \, \frac{sin(kR)\, sin(ck\tau)}{k^2}$$

$$G = (2c/\pi R) \int_{-\infty}^\infty dk\, [e^{ick(\tau - R/c)} - e^{ick(\tau + R/c)}]$$

$$G = \delta(\tau - R/c)$$

as only $\tau > 0$ is allowed. Putting this back into the equation for λ

$$\lambda = \int dx^3\, s(\vec{R}, t - R/c)/R$$

This is the retarded potential form, which indicates that the effective potential at the field point is due to the potential at the earlier time of the source point. The scalar and vector potentials are obtained by integration over the volume at the retarded value of the time;

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon} \int dx'^3\, \frac{\rho(\vec{x}', t - (1/c)|\vec{r} - \vec{r}'|)}{|\vec{r} - \vec{r}'|}$$

$$\vec{A}(\vec{r}, t) = \frac{\mu}{4\pi} \int dx'^3\, \frac{\vec{J}(\vec{x}', t - (1/c)|\vec{r} - \vec{r}'|)}{|\vec{r} - \vec{r}'|}$$