EM radiation

1 Review

Begin with a review of the potentials, fields, and Poynting vector for a point charge in accelerated motion. The retarded potential forms are given below. The source is evaluated at a time that is retarded by the propagation time of the EM wave. Causality has been applied.

$$V = \frac{1}{4\pi\epsilon} \int d^3x' \, \frac{\rho(\vec{x}', \, t' = t - R/c)}{R}$$

$$\vec{A} = \frac{\mu}{4\pi} \int d^3x' \, \frac{\vec{J}(\vec{x}', t' = t - R/c)}{R}$$

From these potential forms the fields were obtained. The electric field is given below. It is found to have two terms. The first term is a quasi-static field, and the second is a radiation field.

$$\vec{E} = \frac{q}{4\pi\epsilon}[\frac{\hat{n} - \vec{\beta}}{\gamma^2(1 - \vec{\beta}\cdot\hat{n})^3 R^2}]_r + \frac{q}{4\pi\epsilon c}[\frac{\hat{n} \times ([\hat{n} - \vec{\beta}] \times \dot{\vec{\beta}})}{(1 - \vec{\beta}\cdot\hat{n})^3 R}]_r$$

The radiation field in the above equation can be used to obtain the Poynting vector, and from the Poynting vector, the power flowing into solid angle, Ω.

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\epsilon_0 c}[\frac{|\hat{n} \times (\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}|^2}{(1 - \vec{\beta}\cdot\hat{n})^5}]_r$$

If the acceleration is in the direction of the velocity this power flow reduces to ($[\kappa = (1 - \vec{\beta}\cdot\hat{n})]$ with $\hat{n}$ the direction of observation and $\vec{\beta}$ the velocity);

$$\frac{dPower}{d\Omega} = \frac{\mu q^2 c}{16\pi^2} \frac{\dot{\beta}^2 \, sin^2(\theta)}{\kappa^5}$$

The time dependence of the radiated wave, $e^{i[kr - \omega t]}$, has been suppressed in the above expressions by assuming one works in frequency space using with the Fourier transform. Remembering that the time average Poynting vector for a wave is obtained from $(1/2)|\vec{E}|^2$, multiply the above expression by (1/2) to get the average power into the solid angle Ω, and drop the time dependent terms. Also apply the expression for the impedance of free space, $z_0 = \sqrt{\mu_0/\epsilon_0}$ in the above expression, and write the acceleration, $\dot{\vec{\beta}} = \vec{a}$. Finally set $\kappa = (1 - \vec{\beta}\cdot\hat{n}) = 1$ for non-relativistic motion.

$$\frac{dP}{d\Omega} = \frac{z_0 k^4 q^2 a^2 \, sin^2(\theta)}{32\pi^2\omega^4}$$

When integrated over the full solid angle ($\int \, d\Omega$);

$$P_T = (2\pi)\frac{z_0 k^4 q^2 a^2}{32\pi^2\omega^4} \int d\theta\, sin^3(\theta)$$

$$P_T = \frac{z_0 q^2 a^2 k^4}{12\pi\omega^4}$$

For an oscillating charge, the acceleration is measured by displacement of the charge from equilibrium.

$$x = x_0\, e^{i\omega t}$$

$$a = \frac{d^2x}{dt^2} = -\omega^2\, x$$

The term a^2 in the above equation is the average acceleration. Thus replace a^2 by $\omega^2 x_0^2$ and note that $(qx_0)^2$ is the electric dipole moment. The final result in terms of the electric dipole moment, p, becomes;

$$P_T = \frac{\mu_0\omega^4 p^2}{12\pi c}$$

The above equations are valid only when the particle is instantaneously at rest relative to the observer, *ie* the particle moves with non-relativistic velocity.

2 Radiated power for an accelerated dipole

The expression obtained for the radiated power above, requires one to obtain the electric dipole moment of an accelerated charge. Suppose an antenna with the geometry given in Figure 1. To proceed the current distribution in the antenna in the antenna is required. Assume that this distribution has the form;

$$I(z) = I_0(1 - \frac{2|z|}{d})\, e^{i\omega t}$$

This assumption, of course, is not exactly correct, but the radiation field is insensitive to the details of the current configuration. However, proceed by applying the equation of continuity. Note that the time dependence is assumed to be, $e^{-i\omega t}$

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t} = i\omega\rho$$

$$I = \int \vec{J}\cdot d\vec{A} = \int (\vec{\nabla}\cdot\vec{j})\, d\tau = -i\omega \int dz \int \rho\, dA$$

$$\frac{\partial I}{\partial z} = \frac{2I_0}{d} = i\omega\lambda$$

In the above, λ, is the charge per unit length. Then

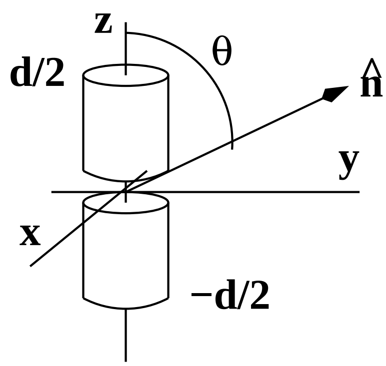

Figure 1: The geometry of a center-fed antenna. $\hat{n}$ gives the direction of observation of the radiated power

$$\lambda = \frac{2iI_0}{\omega d}$$

The dipole moment (time dependence suppressed) is

$$p = 2\int_0^{d/2} dz\, z(\frac{2iI_0}{\omega d}) = \frac{iI_0 d}{2\omega}$$

Then the radiated power is obtained from the dipole radiation equation previously developed.

$$P = \frac{I_0^2 (kd)^2}{48\pi}$$

The power input into the antenna is lost by driving the current, and goes into the radiation field. This appears as an effective resistance, R, in series with the antenna. Therefore on time average;

$$\frac{z_0 I_0^2 (kd)^2}{48\pi} = (1/2) I_0^2 R$$

In the above, R is called the radiation resistance. Let $k = 2\pi/\lambda$ for the wavelength, λ. Then

$$R = (\frac{\pi z_0}{6})(\frac{d}{\lambda})^2$$

As an example, suppose a long power transmission line (500 km) with frequency of 60 cycles/s

$$\lambda = \frac{3 \times 10^{10}}{60} = 5 \times 10^8 \text{ cm}$$

$$R_{rad} = 7.9\,\Omega$$

Compare this to the resistance of a copper wire. The resistivity of copper is $\rho = 1.7 \times 10^{-6}$

Ω-cm and $R = \frac{\rho L}{A}$. For a wire radius of 0.5 cm then $R = 108\ \Omega$.

3 Dipole radiation

Radiation due to absorption of an EM wave almost always results in dipole radiation and is used to describe scattering of the electromagnetic wave. Consider dipole radiation as an simple example of the more general case to be discussed in the following section. Begin with the retarded potentials. In the expressions below; $R = |\vec{r} - \vec{r}'|$;

$$V = \int d^3x' \, \frac{\rho(\vec{r}', t - R/c)}{R}$$

$$\vec{A} = \int d^3x' \, \frac{\vec{J}(\vec{r}', t - R/c)}{R}$$

where $R = |\vec{x} - \vec{x}'|$. Then let $r \gg r'$ so that $R \approx r - \vec{r} \cdot \hat{n}$ as seen in Figure 2. This assumes that the period of oscillation is much greater than the time for the EM wave to traverse the source. If this were not the case, then we would need to take into account phase shift differences in the radiation from different points in the source. In the figure $\hat{n}$ is in the direction of observation of the radiation. In the dipole approximation, neglect the factor of $\vec{r} \cdot \hat{n}$ in the denominator. However, this cannot be done in general because of phase differences between different points of the moving charges. In the case of the dipole, the retarded time is simply replaced by the present time. In the radiation zone $\vec{E}$ is perpendicular to $\vec{B}$ and $\vec{E}$, $\vec{B}$ are perpendicular to $\hat{n}$. Thus;

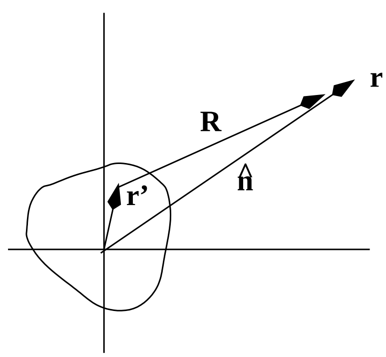

Figure 2: The geometry of dipole radiation

$$\vec{E} = \vec{B} \times \hat{n} = -\vec{\nabla} V - (1/c) \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Now $\vec{J} = \rho\vec{v}$, and write

$$\vec{A} \,=\, \frac{\mu}{4\pi R} \sum_i q_i\vec{v}_i \,=\, \frac{\mu}{4\pi R} \frac{d}{dt} \sum_i q_i\vec{r}_i$$

In the above equation $q\vec{r}$ is the electric dipole moment p, and $\ddot{p}$ is the second time deravitive of the dipole moment (an acceleration term). The fields in the radiation zone, where E, B are perpendicular to each other and perpendicular to the energy flow, become;

$$\vec{B} \,=\, \frac{\mu}{4\pi c} (\frac{\ddot{p} \times \hat{n}}{r})$$

$$\vec{E} \,=\, \frac{\mu}{4\pi} (\frac{(\ddot{p} \times \hat{n}) \times \hat{n}}{r})$$

The radiation intensity is given by the Poynting vector. For the dipole case multiply by the differential area $R^2 d\Omega$ and obtain the radiated power as;

$$\frac{dP}{d\Omega} \,=\, \frac{\mu\ddot{p}^2}{16\pi^2 c} \, sin^2(\theta)$$

In this expression θ is the angle of observation with respect to the dipole moment. A similar development gives the power and fields for magnetic dipole radiation.

4 General look at radiation

In general all EM fields are created by a system of charges and currents. Assume that one can work in frequency space, so look for solutions for each frequency, ω, in the spectrum. Therefore the time dependence is;

$$\rho(\vec{r}, t) \,\rightarrow\, \rho(\vec{r})e^{-i\omega t}$$

$$\vec{A}(\vec{r}, t) \,\rightarrow\, \vec{A}(\vec{r})e^{-i\omega t}$$

Then the vector potential is;

$$\vec{A}(\vec{r}, t) \,=\, \frac{\mu}{4\pi} \int dx'^3 \int dt' \, \frac{\vec{J}(\vec{r}', t')}{|\vec{r}' - \vec{r}|} \, \delta(t' - \frac{|\vec{r}' - \vec{r}|}{c} - t)$$

Causality is preserved by the δ function. The equation for the charge density can be obtained from the equation of continuity.

$$\rho \,=\, (1/i\omega)(\vec{\nabla} \cdot \vec{J})$$

When a harmonic form is assumed, $e^{i\omega t'}$, the time dependence can be removed after integration over t' and replacing $\vec{J}(\vec{r'})e^{ik[(\vec{r'}-\vec{r})]}\, e^{i\omega t}$ by $\vec{J}\, e^{ik[(\vec{r'}-\vec{r})]}$ so that the vector potential becomes the static equation;

$$\vec{A}(\vec{r}) \;=\; \frac{\mu}{4\pi} \int dx'^3 \, \frac{\vec{J}\, e^{ik[(\vec{r'}-\vec{r})]}}{|\vec{r'} - \vec{r}|}$$

In the abobe, $k = \omega/c$. The The fields are constructed from the potentials. Suppose the source dimensions are d, and invesitgate solutions for $\lambda \gg d$ and $\lambda \ll d$. The integral over the source is confined to integration over $\vec{r}'$. There are several distance scales.

1. Near Field $d \ll \lambda > r$
2. Far Field $d \ll \lambda \ll r$
3. Intermediate field $d \ll \lambda \approx r$

Here one always assumes that at least $d \approx r' \ll \lambda$. If this is not the case then there are phase problems to address. With this assumption, the exponential in the integrand can be written;

$$e^{ikr[1-2\vec{r}\cdot\vec{r'}/r^2+r'^2/r^2]^{1/2}} \;\approx\; e^{ir[1-(r'/r)\,cos(\theta)]}$$

Expand this in a power series of $(r'/r$ and neglect for the moment the denominator.. The m^{th} term of the vector potential is ;

$$\vec{A}_m \;=\; \frac{\mu}{4\pi}(e^{ikr}/r)\,\frac{(ik)^m}{m!}\,[1 + \frac{a_1}{ikr} + \cdots + \frac{a_m}{(ikr)^m}] \int d\tau' \, \vec{J}\,(r'\,cos(\theta))^m$$

The coefficients in this series, a_j are integers. For the near field $kr \ll 1$;

$$A_m \rightarrow \frac{(-1)^m \mu a_m}{4\pi m!}(1/r^{m+1}) \int d\tau' \, \vec{J}(r'\,cos(\theta))^m$$

Since $e^{ikr} \approx 1$, the leading term is independent of k. Thus this term has a time dependence of $e^{-i\omega t}$ but no traveling wave develops. In the far field case $kr \gg 1$ so that;

$$\vec{A} \rightarrow \frac{\mu(-ik)^m\, e^{ikr}}{4\pi r m!} \int \vec{J}(r'\,cos(\theta))^m$$

using this expression, the vector potential becomes;

$$\vec{A}(\vec{r}) \,=\, \frac{\mu e^{ikr}}{4\pi r} \int d\tau' \, \frac{\vec{J}(r')\, e^{-ikr' \, cos(\theta)}}{1 - (r'/r)\, cos(\theta)}$$

This expression is valid for $(r'/r) \leq d/r \ll 1$. Here one must be careful to evaluate the phase where terms of order $(r'/r)^2$ have been neglected. However so long as $2\pi r'/\lambda \ll 1$, expand the exponential and the denominator in powers of (r'/r).

$$\frac{e^{-ikr' \, cos(\theta)}}{1 - (r'/r)\, cos(\theta)} \,=\, 1 + (1/r - ik) r' \, cos(\theta) \,+ \cdots$$

In this region the fields fall like $1/r$ and are transverse to the radial vector r. The potential is dominated by the lowest, non-zero term in the expansion. Note that this is just the multipole expansion previously developed for a static charge distribution. Thus the radiation problem for a specified, harmonic frequency, is determined by the lowest, non-zero multipole moment of the potential. Note that the development requires a harmonic time dependence. If this is not the case, then the source forms must be broken into Fourier components by a Fourier transformation.

5 Electric dipole

In the last section using $m = 0$, the vector potential will be;

$$\vec{A} \approx \frac{\mu e^{ikr}}{4\pi r} \int d\tau' \, \vec{J}(\vec{r}')$$

If this does not vanish, then an integration by parts gives;

$$\int d\tau' \, \vec{J} \,=\, -\int \, d\tau' \, r' (\vec{\nabla} \cdot \vec{J}) \,=\, -i\omega \int \, d\tau' \, \vec{x}' \rho(\vec{r}')$$

A harmonic time dependence is assumed and used in the equation of continuity. Note that $\int \, d\tau \, \vec{x} \rho$ is the dipole moment of the charge distribution. Thus;

$$\vec{A} \,=\, \frac{-i\mu}{4\pi r} \, e^{ikr} \, \omega \vec{p}$$

Here choose $\vec{p} \,=\, p_0 e^{-i\omega t} \hat{z}$. The magnetic field is then;

$$\vec{B} \,=\, \vec{\nabla} \,\times\, \vec{A}$$

The vector potential points along the direction of observation, so to have $\vec{p}$ point along the $\hat{z}$ axis, project $\hat{z}$ onto $\hat{r}$.

$$\vec{A} \,=\, \frac{-i\mu\omega}{4\pi r} \, e^{ikr} \, p[cos(\theta)\, \hat{r} \,-\, sin(\theta)\, \hat{\theta}]$$

The magnetic field is obtained from $\vec{\nabla} \times \vec{A}$ in spherical coordinates.

$$\vec{\nabla} \times \vec{A} = [-\frac{k\mu\omega}{4\pi}(e^{ikr}/r)p\,sin(\theta) + \\ \frac{i\mu\omega}{4\pi}(e^{ikr}/r^2)p\,sin(\theta)]\hat{\phi}$$

Drop the last term as it has the functional form, $1/r^2$, to obtain;

$$\vec{B} = -\frac{k\mu\omega p_0}{4\pi}p\,sin(\theta)(e^{ikr}/r)\,\hat{\phi}$$

The electric field is;

$$\vec{E} = \frac{i}{\omega\mu\epsilon}\vec{\nabla} \times \vec{B}$$

$$\vec{E} = -\frac{\mu\omega^2 p}{4\pi\mu\epsilon c^2}\,sin(\theta)\,(e^{ikr}/r)\,\hat{\theta}$$

The time averaged Poynting vector is;

$$\langle S\rangle = (1/2)\vec{E} \times \vec{H} = \frac{\mu p^2\omega^4}{32\pi^2 cr^2}\,sin^2(\theta)\hat{r}$$

The total power radiated is obtained by integration over the solid angle.

$$\langle P\rangle = \frac{\mu p^2\omega^4}{12\pi c^2}$$

6 Magnetic dipole radiation

Return to the general expansion developed earlier and take the 2^{nd} term in the series for $\vec{A}$. This gives;

$$\vec{A} = \frac{\mu}{4\pi}(e^{ikr}/r)\,(-ik/1!)\int d\tau'\,\vec{J}(r')(r'\,cos(\theta))$$

Write $r'\,cos(\theta) = \hat{n}\cdot\vec{r}'$ for $\hat{n}$ in the direction of the radiation. Then expand as follows.

$$(\hat{n}\cdot\vec{r}') = (1/2)[\vec{r}' \times \vec{J} \times \hat{n}] + (1/2)[(\vec{J}\cdot\hat{n})\vec{r}' + (\hat{n}\cdot\vec{r}')\vec{J}]$$

The first term is anti-symmetric in (r', J) and the 2^{nd} is symmetric. The first term produces magnetic dipole radiation and the second electric quadrupole radiation. Now $\vec{M} = (1/2)(\vec{r}' \times \vec{J})$ where M is the magnetization (magnetic moment per unit volume). Thus the integral of the anti-symmetric term of $(\hat{n}\cdot\vec{J}) = (\hat{n} \times \vec{r}' \times \vec{J})$ over the volume is $\hat{n} \times \vec{m}$ where

$\vec{m}$ is the magnetic dipole moment. The vector potential becomes;

$$\vec{A} = \frac{ik\mu}{4\pi}(\hat{n} \times \vec{m})(e^{ikr}/r)$$

$$\vec{m} = (1/2) \int d\tau' \, (\vec{r}' \times \vec{J}(\vec{r}'))$$

The fields can be obtained from A.

$$\vec{B} = \frac{\mu}{4\pi}[k^2(\hat{n} \times \vec{m}) \times \hat{n}(e^{ikr}/r) + [3\hat{n}(\hat{n} \cdot \vec{m} - \vec{m}](1/r^3 - ik/r^2))e^{ikr}$$

The first term is the radiation field. Then

$$\vec{E} = -\frac{z_0 k^2(\hat{n} \times \vec{M})}{4\pi}(e^{ikr}/r)$$

The time averaged Poynting vector is;

$$\langle S \rangle = \frac{\mu m^2 \omega^4}{32\pi^2 c^3}(sin^2(\theta)/r^2)\hat{r}$$

7 Quadrupole radiation

First note that the parity symmetry ($\vec{r} \rightarrow -\vec{r}$) of the electric dipole ($l = 1$) is odd and in fact all higher odd electric multipoles will be odd, while all magnetic multipoles will be even. Thus the magnetic dipole ($l = 1$) has even parity. This is the reason the expression for the symmetry of vector potential was used so that appropriate multipoles could be combined. The ratio of magnetic to electric multipoles is approximately β for the same order of multipolarity. Then use the symmetric term in the expression as discussed in the last section. One writes the equation;

$$(\hat{n} \cdot \vec{r}') = -(1/2)[\vec{r}' \times \vec{J} \times \hat{n}] + (1/2)[(\vec{J} \cdot \hat{n})\vec{r}' + (\hat{n} \cdot \vec{r}')\vec{J}]$$

Look at the second term;

$$\int d\tau' \, [(\hat{n} \cdot \vec{r}')\vec{J} + (\hat{n} \cdot \vec{J})\vec{r}'] = -i\omega \int d\tau' \, \vec{r}'(\hat{n} \cdot \vec{r}')\rho(\vec{x}')$$

In the above the continuity equation has been used. This is an electric quadrupole source. Substitute into the expression for the vector potential to obtain;

$$\vec{A} = \frac{\mu}{8\pi}(e^{ikr}/r)(k\omega) \int d\tau' \, \vec{r}'(\hat{n} \cdot \vec{r}')\rho(\vec{x}')$$

The magnetic field is;

$$\vec{B} = ik\hat{n} \times \vec{A} = -\frac{ik^2\omega\mu}{8\pi} \int d\tau' \, \vec{r}'(\hat{n} \cdot \vec{r}')\rho(\vec{x}')$$

Now write a quadrupole tensor in the form;

$$Q_{ij} = \int d\tau \, [3x_i x_j - r^2\delta_{ij}]\rho$$

$$\hat{n} \times \int d\tau \, \vec{r}'(\hat{n} \cdot \vec{r}')\rho = (1/2)\hat{n} \times \vec{Q}$$

$$\vec{Q} = \sum Q_{ij}\hat{n}_i$$

The magnetic field becomes;

$$B = \frac{ik^2\mu\omega}{24\pi}(e^{ikr}/r) \, \hat{n} \times \vec{Q}$$

The radiated power per solid angle;

$$\frac{dP}{d\Omega} = \frac{c^2 z_0}{1152\pi^2} k^6 |(\hat{n} \times \vec{Q}) \times \hat{n}|^2$$

8 A half-wave, center-fed linear antenna

Remember that the current must vanish at the ends of the antenna wire. Assume that the wire is thin so that radial currents can be neglected, and we know that the actual radiation field is rather insensitive to the current distribution.

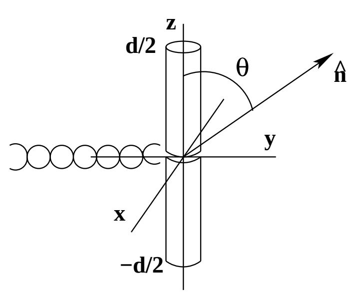

Figure 3: The linear center fed antenna. We neglect the radial width of the antenna wire

In the dipole approximation the radiation field is;

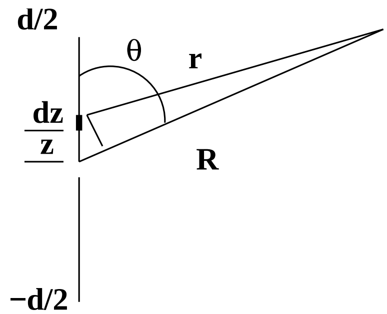

Figure 4: The geometry to properly include the phase factor for the antenna field

$$\vec{E} = -\frac{\mu\omega^2 p}{4\pi} sin(\theta)(e^{ikr}/r)\hat{\theta}$$

In the above equation, $\vec{p}$ is the electric dipole moment. Now suppose the charge moves in an antenna wire such that $I = I_0 \, cos(\frac{\pi}{d}z) \, e^{i\omega t}$. Recall in a previous section, electric dipole radiation was developed using a current distribution, $I(z,t) = I_0[1 - 2z/d]e^{i\omega t}$. Both assumptions assume a a current distribution which vanishes at the ends of the antenna wire and is harnonic in time. The later assumption is a solution of the boundary value problem for current in a thin wire antenna, and is a better approximation than that originally used. Then;

$$q = q_0 \, cos(\tfrac{\pi}{d}z) \; q_0 = I_0/\omega$$

The differential dipole moment is $dp = q \, dz$. Then since $d \approx \lambda$ so the phase of each oscillating charge element must into taken into account. Do this by summing the amplitudes of small antenna elements and properly including their phases.

$$dE = -\frac{\mu\omega^2}{4\pi} sin(\theta) \, (e^{ikr/r})(q \, dz)$$

$$dE = -\frac{\mu\omega^2}{4\pi} sin(\theta) \, (e^{ikr/r}) \, cos(\tfrac{\pi}{d}) \, z \, dz$$

Consider Figure 4, where the phase term for the factor e^{ikr} is approximated.

$$R = r + z \, cos(\theta)$$

Substitution gives;

$$dE = -\frac{\mu q_0 \omega^2}{4\pi} sin(\theta) \, (e^{ikR/R}) \, e^{ikz \, cos(\theta)} \, cos(\tfrac{\pi}{d}) \, z \, dz$$

Integration gives;

$$E = \frac{\mu q_0 \omega^2\, sin(\theta)}{4\pi}(e^{ikR}/R)[\frac{2cos(\pi/2\, cos(\theta))}{cos^2(\theta) - 1}]$$
$$E = \frac{\mu q_0 \omega^2}{2\pi}(e^{ikR}/R)[\frac{cos(\pi/2\, cos(\theta))}{sin(\theta)}]$$

The Poynting vector is;

$$\langle S \rangle = \frac{(2q_0 \mu \omega^2\, sin(\theta))^2}{2\mu c (4\pi)^2 R^2}(2\pi/d)^2[(\frac{cos(kd/2\, cos(\theta))}{k^2\, cos^2(\theta) - (\pi/2)^2}]^2$$

The radiation resistance is obtained from Power $= I^2 R$

$$R = \text{Power}/(\frac{dq}{dt})^2 = P/\omega^2 q_0$$

The radiation resistance can be evaluated numerically in this case by integrating over the solid angle for the power. The result is;

$$R = 73.1\ \Omega$$