Bremsstrahlung and the fields of moving charges - Lecture 16

1 Cherenkov Radiation

Cherenkov radiation occurs when a charged particle moves faster that EM radiation travels in that medium. Observe Figure 1. The observation point is closer to the present point than the retarded point. That is $V(t - t') > (c/n)(t - t')$. Here, n is the index of refraction of the medium. A wave front can be drawn as a line which begins at the present point and is tangent to the spherical wave front at the retarded point, as shown in the figure. Each spherical wave front from a point along the particle trajectory is tangent to this line. The angle of propagation of the wave front, θ, is obtained from the figure as;

$$cos(\theta) = \frac{ct/n}{vt} = \frac{1}{n\beta}$$

Previously the retarded point of the Lienard-Eiechert potentials was chosen because of causality. In the case of Cherenkov radiation, both retarded and advanced points are possible solutions. This can be seen in Figure 2. The positions of the observation and present points are related (c is used here but note that this is really c/n) by;

$$c(t - t') = |\vec{X} + \vec{V}(t - t')| = R$$

This equation is written for $\Delta t = (t - t')$;

$$c^2 \Delta t^2 = X^2 + V^2 \Delta t^2 + 2\vec{X} \cdot \vec{V} \Delta t.$$

This is a quadratic equation with solutions;

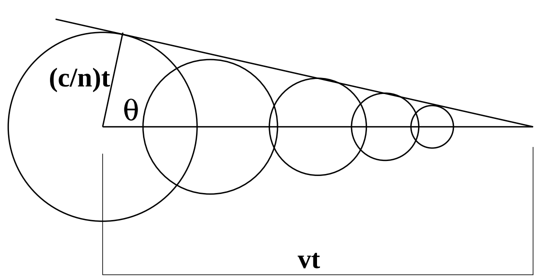

Figure 1: The geometry of wave fronts in Chernekov radiation

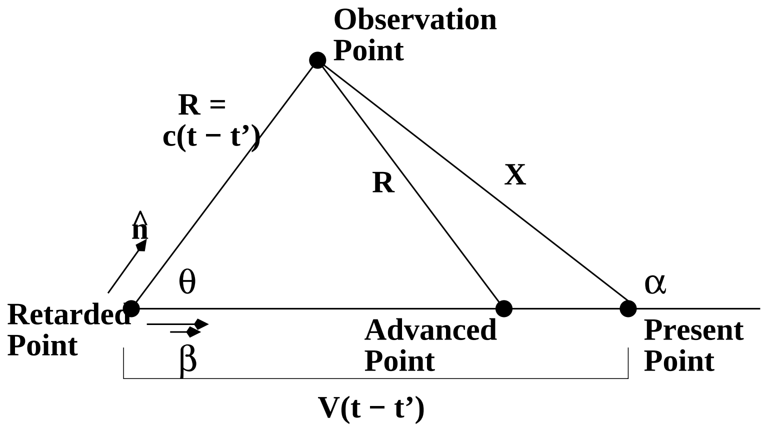

Figure 2: The geometry of Chernekov radiation showing the retarded, advanced, and observation points

$$\Delta t \;=\; \frac{-\vec{X}\cdot\vec{V} \pm \sqrt{(\vec{X}\cdot\vec{V})^2 - (V^2 - c_2)X^2}}{(V^2 - c^2)}$$

Then when $\Delta t > 0$ there is only one real positive root for $V < c$, which corresponds to the retarded point. For $V > C$ there is no real root for $\vec{X}\cdot\vec{V} > 0$ but when $\vec{X}\cdot\vec{V} < 0$ there are 2 solutions. This occurs for values of $\alpha > \pi/2$. From the discriminate;

$$cos(\alpha) \;=\; -[1 - (c/v)^2]^{1/2}$$

These solutions are the advanced and retarded points, so the Lienard-Wiechart potentials can be simply multiplied by 2 to include the advanced point as both retarded and advanved points are a distance X away from the observation point and contribute the same amplitude.

Now look more carefully at this solution. The wave front represents a shock-front singularity. The general form of the angular distribution of the radiated power is written;

$$\frac{dP}{d\Omega} \;=\; (1/2\mu c)Re|\vec{E}^{*}\cdot\vec{E}|R^2$$

Develop the Fourier spectrum of the E field, by;

$$\mathcal{E} \;=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, e^{i\omega t}\, E(t)$$

$$E(t) \;=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, e^{-i\omega t}\, \mathcal{E}$$

The energy radiated is the time integral of the power. Therefore;

$$\frac{d\mathcal{W}}{d\Omega} = \frac{1}{2\pi\mu c} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \vec{E}^* \cdot \vec{E}\, e^{i(\omega'-\omega)t}$$

Interchange the time and frequency integrations and differentiate with respect to ω.

$$\frac{d\mathcal{W}}{d\Omega} = \frac{1}{2\mu c} |\mathcal{E}(\omega)|^2 R^2$$

The above is the energy radiated per unit solid angle. The equation has been multiplied by the differential area element, $R^2\, d\Omega$ and a factor of (1/2) is included so that always $\omega > 0$. The radiation field (Gaussian units) is;

$$E(t) = (q/c)[\frac{\hat{N} \times (\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}}{\kappa^3 R} = (e/c\kappa)\, \frac{d}{dt}[\hat{n} \times (\hat{n} \times \vec{\beta})]$$

The right side of the energy equation has the form $\mathcal{G}^*\mathcal{G}$.

$$\mathcal{G} = \frac{1}{(4\mu c)^{1/2}} \mathcal{E}(\omega) R = \frac{1}{(8\pi\mu c)^{1/2}} \int_{-\infty}^{\infty} dt\, e^{i\omega t}[\frac{\hat{n} \times (\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}}{\kappa^3}]$$

Change to present time, κt, and approximate the time in the phase component by $t' \approx t - \hat{n} \cdot \vec{r}/c$ for large r. Integrate by parts over the time, and then put this value for $\mathcal{G}$ back into the equation for the differential power above to obtain;

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{q^2\omega^2}{16\pi^2 c^3} | \int_{-\infty}^{\infty} dt[\hat{n} \times (\hat{n} \times \vec{\beta})]\, e^{i\omega(t - \hat{n}\cdot\vec{r}/c)}|^2$$

In a dielectric medium, the velocity of the EM wave is $c \to c/n = c/\sqrt{\epsilon}$ with n the index of refraction. Assume straight line motion so that $\vec{r} = \vec{V}t$.

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{\mu q^2 c\omega^2}{16\pi^2} |\hat{n} \times \vec{V}\, (\omega^2/2\pi) \int_{-\infty}^{\infty} dt\, e^{i\omega t(1 - \hat{n}\cdot\vec{r}/nc)}|^2$$

The integral is a δ function and results in the expression for the intensity of the Cherenkov radiation (Gaussian units);

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{q^2\beta^2}{nc} sin^2(\theta)|\delta(1 - (\beta/n)\, cos(\theta)|^2$$

The above δ function gives the Cherenkov angle as previously obtained. As an example, glass has an index of refraction of 1.5. The Cherenkov angle is then;

$$cos(\theta) = 1/(1.5\beta)$$

which gives a cut-off velocity of $\beta = 0.666$ An electron with this velocity has a momentum of 0.45 MeV.

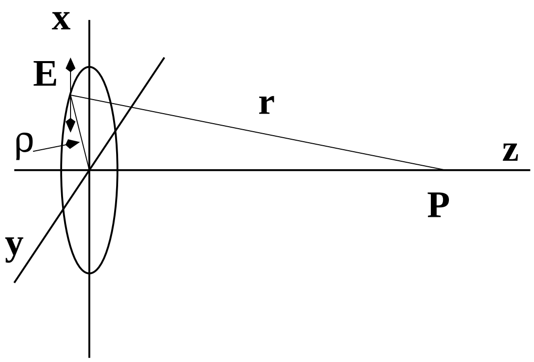

Figure 3: A schematic of a model for an incident plane wave scattered from a ring of atomic electrons in the $x, y)$ plane

2 Index of refraction

Choose a simple model which demonstrates that the index of refraction is due to an interference between the incident wave and the scattered wave. The scattered wave is due to radiation which occurs when the incident wave accelerates atomic electrons. Look at Figure 3. The atomic electrons in a ring of radius ρ in the (x, y) plane are caused to harmonically oscillate due to a plane wave incident along the z axis. Assume the electric vector of this wave is polarized in the x direction.

The electrons are bound to atoms. Assume the binding is approximated by a linear force, resulting in the non-relativistic harmonic force equation;

$$m\frac{d^2x}{dt^2} + kx = qE\, e^{i\omega t}$$

The 1^{st} term is the intertial force, ma, the 2^{nd} term is the restoring force keeping the electron bound to the nucleus, and the 3^{rd} term is the driving force due to the EM field.
The solution is ;

$$x = x_0 E_{i\omega t} = \frac{qE/m}{\omega_0^2 - \omega^2}\, e^{i\omega t}$$

The time derivatives give the velocity and acceleration which are placed in the non-relativistic expression for the radiation field.

$$E = (e/4\pi\epsilon c)[\hat{n} \times \hat{n} \times \dot{\vec{\beta}}]\,[e^{i\omega t}/r]$$

Now multiply by the number of electrons per unit area in the plane, N, and integrate over the surface of the plane $2\pi \int \rho \, d\rho$. Change the variable to an integration over the distance from the electron to the observation point on the z axis, $r^2 = \rho^2 + z^2$ and approximate the value of t' as $t' \approx t - r/c$. This results in the expression;

$$E_{total} = \frac{qN}{4\pi\epsilon c} \int \rho \, d\rho \int d\phi$$

$$E_{total} = \frac{2\pi q N \omega^2 x_0}{4\pi\epsilon c} e^{i\omega t} \int_z^\infty r dr \, e^{-i\omega r/c}/r$$

$$E_{total} \approx \frac{2\pi i q N \omega^2 x_0}{4\pi\epsilon c} e^{i\omega(t-r/c)}$$

To this result add the incident wave $e^{i\omega(t-z/c)}$. Now write the wave at the point z as due to a wave having the form,

$$E = E_0 \, e^{i\omega(t - (n-1)\Delta z/c - z/c)}$$

The incident wave propagation from the plane is just $e^{i\omega(t-z/c)}$ the additional term $(n-1)\Delta z/c$ is the time delay experienced by the wave when passing through the dielectric material having index n and thickness, Δz. For small values of Δz expand this term to write the field at the observation point as;

$$E \approx E_0 e^{i\omega(t-z/c)}[1 - \frac{i\omega(n-1)\Delta z}{c}]$$

Divide through by the thickness, δz, and identify this result with the previous calculation of the scattered field;

$$(n-1) = \frac{2\pi q^2 (N/\Delta z)}{4\pi\epsilon m(\omega_0^2 - \omega^2)}$$

The term $N/\Delta z$ is the number density of the electrons.

3 Radiation and coherence

Now an accelerated charge radiates energy. Suppose 2 charges are in symmetric positions on a loop and spin around the z axis as shown in Figure 4. This results in radiation because the charges are accelerated. Now suppose the loop is filled with a continuium charge distribution which rotates. Contrary to intuition, there is no radiation. This occurs because of coherence of the radiation from each of the elemental charge distributions cancels the radiation field. This is seen as follows. The charge distribution of the two elements as shown in the figure can be written;

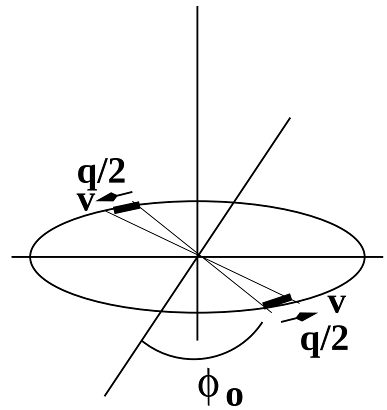

Figure 4: Radiation from sets of charge rotating in a circle

$$\rho \;=\; (q/2)\frac{\delta(r-a)}{a^2}\,\delta(\theta-\pi/2)[\delta(\phi+\phi_0)+\delta(\phi-\phi_0)]$$

Rewrite the charge per unit length so that additional symmetric units of charge add;

$$\lambda \;=\; \frac{q}{(n+1)\,\pi a}\delta(\phi-\pi k/N-\phi_0) \quad k=0,1,2,\cdots,N-1$$

All of the charge elements rotate with the frequency ω. The charge density is then written;

$$\rho_k \;=\; \frac{q}{(n+1)\pi}\frac{\delta(r-a)}{a^2}\,\delta(\theta-\pi/2)\,\delta(\phi-\pi k/N-\omega t)$$

Now work out the Fourier components of the distribution.

$$\rho_k \;=\; \sum A_n\, cos(n[\phi-\omega t])$$

The coefficients A_n are found using othogonality of the Fourier functions, so that;

$$\rho_k \;=\; \frac{q}{(n+1)2\pi}\frac{\delta(r-a)}{a^2}\,\delta(\theta-\pi/2)\,\sum_n\, cos(2\pi kn/N)\,cos(n[\phi-\omega t])$$

Then;

$$\rho \;=\; \sum_{k=0}^{N-1}\,\rho_k$$

The sum over k provides the following

$$\sum_{k=0}^{N-1}\, cos(2\pi nk/N) \;=\; cos(n\pi[1-1/N])\,\frac{sin(n\pi)}{\sin(n\pi/N)}$$

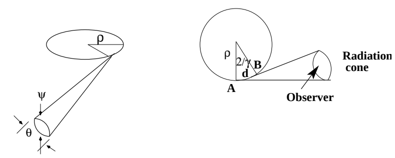

Figure 5: Synchrotron radiation beams from a circular accelerator

This is zero unless $n = N$ or 0. Thus the radiation has multipolarity N. But if $N \to \infty$ there will be no radiation. In effect the radiation from each charge element contributes coherently, cancelling the radiated power.

4 Synchrotron radiation

Synchrotron radiation occurs when a particle is accelerated perpendicular to its velocity as in a circular orbit. It occurs naturally for electrons moving around the magentic field lines in the upper atmosphere of the earth. It is generated in electron accelerators to produce EM radiation mainly used to study the structure of materials and biological samples. Suppose a relativistic particle in a circular orbit as shown in Figure 5. The radius of the orbit is ρ, and assume a pulse of beam which radiates. Note that if the beam is continuious then, as previously, no radiation occurs.
The arc distance $A\, to\, B$ is $d = 2\rho/\gamma$. The time for the particle to travel this distance is $t = d/v$. The first photon leaves the initial pulse from A and travels a distance t/c during this time interval. Thus the distance between the 1^{st} and last photon in the emission cone is ΔL;

$$\Delta L = \frac{2\rho}{\gamma}[1/\beta - 1] \approx \rho/\gamma^3$$

The time for the EM pulse to travel this distance is $\rho/(\gamma^3 c)$ which determines the frequency cut-off for the radiation; $\omega_c = \gamma^3 c/\rho$. Obtain the radiated power from the expression for the power per solid angle per frequency obtained in the Cherenkov radiation section;

$$\frac{d^I}{d\omega d\Omega} = \frac{q^2\omega^2}{16\pi^2 c^3}|\int_{-\infty}^{\infty} dt[\hat{n} \times (\hat{n} \times \vec{\beta})]\, e^{i\omega(t - \hat{n}\cdot\vec{r}/c)}|^2$$

Choose a coordinate system as shown in Figure 6 and substitute for the velocity and the distance from the retarded point to the present point, r

$$\hat{n} \times (\hat{n} \times \vec{\beta}) = \beta\hat{n} \times (\hat{n} \times [\hat{\epsilon}_{\parallel}\, sin(vt/\rho) + \hat{x}\, cos(vt/\rho)])$$

$$\hat{n} \times (\hat{n} \times \vec{\beta}) = \beta(-sin(vt/\rho)\,\hat{\epsilon}_{\parallel} + cos(vt/\rho)\, sin(\theta)\,\hat{\epsilon}_{\perp})$$

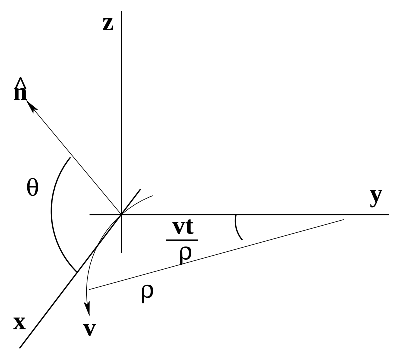

Figure 6: The geometry used for synchrotron radiation of an ultra-relativistic particle

Also;

$$\hat{n} \cdot \vec{r} \,=\, (\rho/c) sin(vt/\rho)\, cos(\theta)$$

Integration proceeds in the ultra-relativistic approximation. The result is

$$\frac{d^2 I}{\omega\, d\Omega} \;=\; \frac{27\, e^2 c}{32\pi^3 c\rho} \frac{\lambda_c}{\lambda} \gamma^8 [1 + (\gamma\psi)^2]^2 [K^2_{2/3}(\eta) \,+\, \frac{(\gamma\psi)^2}{1 + (\gamma\psi)^2} K^2_{1/3}(\eta)]$$

In this equation

$$\lambda_c = \frac{4\pi\rho\gamma^{-3}}{3}$$

$$\eta = \frac{\lambda_c [1 + (\gamma\psi)^2]^{3/2}}{2\lambda}$$

$K^j_{i/2}$ are modified Bessel functions of the 2^{nd} kind.

5 Virtual photon

A charged particle of momentum $\vec{p}_I$ and energy E_I cannot decay into a photon and a charged particle having the same mass. Such a process is shown in Figure **??**. Conserve both energy and momentum, and write;

$$\vec{p}_I \,=\, \vec{p}_F \,+\, \vec{p}_\gamma$$

$$E_I \,=\, E_F \,+\, E_\gamma$$

Then combining the equations

$$M_\gamma^2 \,=\, E_\gamma^2 - p_\gamma^2 \,=\, 2[M^2 + p_I p_F\, cos(\theta) \,-\, E_I E_F]$$

In the above M is the mass of the particle and θ is shown in the figure. Now $M_\gamma^2 < 0$ so that M_γ is imaginary. However, in some cases it is useful to think that this photon has some of the the properties of a real photon. Indeed in QM such a virtual particle can live within the uncertainty relation $\Delta E\,\Delta t \,\approx\, \hbar$. For later notation we let this photon have energy $\omega = E_I - E_F$ and momentum $\vec{q} \,=\, \vec{P}_I - \vec{p}_F$. Then

$$M_\gamma^2 = \omega^2 - q^2 \,<\, 0$$

6 Summary

At low frequencies the spectrum of radiated quanta is;

$$\frac{dI}{d\omega} \,=\, \frac{q^2}{4\pi^2 cv}\,[ln(\frac{1.123\gamma v}{wb_{min}}) - (v^2/2c^2)]$$

Then use

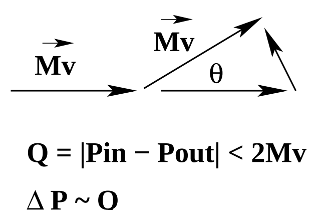

Figure 7: The geometry for the momentum transfer

$$\Delta\rho\,\Delta p \,\approx \hbar \text{ with } \Delta\rho \,=\, b$$

$$b_{min} \,=\, \hbar/2Mv \,=\, \hbar/Q_{max}$$

$$\beta \approx = \, 1$$

Re-write the above equation in the form;

$$\frac{dI}{d\omega} \,=\, \frac{q^2}{4\pi^2 cv}\,[ln(\frac{1.123\gamma mc^2}{\hbar w}) - (v^2/2c^2)]$$

The Thompson Cross Section

$$\frac{d\sigma}{d\Omega} = (\frac{q^2}{(4\pi)^2 \epsilon m c^2})^2[1/2(1 + cos^2(\theta))]$$