General Relativity
Lecture 20

1 General relativity

General relativity is the classical (not quantum mechanical) theory of gravitation. As the gravitational interaction is a result of the structure of space-time, the mathematical form of general relativity underlies cosmology. Einstein developed general relativity in terms of a field equation. Remember that a field is a natural developmemnt required to explain special relativity, and the concept of "action-at-a-distance" cannot naturally be made to be relativistically invarient. Thus interactions are a result of modifications to space-time. The modifications are linear for weak interactions, but for strong interactions the modification of space-time results in non-linear behavior of the fields. This introduces self-energy terms which lead to singularities.

Special relativity is developed in the absence of gravity. General relativity assumes that the laws of physics are invarient in all free-falling (non-rotating) reference frames. This is the Einstein equivalence principle, $m_{inertia} = m_{gravitation}$. At small spatial scales, all reference frames in free-fall are equivalent, but in general curved space-time is assumed, and the metric (scale) tensor defines space-time geometry.

Mass and energy determine the structure of space-time, and the most appropriate form to represent this is the energy-momentum tensor. Thus, in Newtonian gravity, the source of a gravity field is the mass. In general relativity the source is the stress-energy tensor. The Einstein-Hilbert Lagrange density for the gravitational field in free-space is therefore;

$$\mathcal{L}_G = \frac{c^4}{16\pi G}(R + 2\overline{\Lambda})$$

Here R is the scalar curvature related to the Ricci tensor by;

$$R = R_{\alpha\beta}\, g^{\alpha\beta}$$

and $\overline{\Lambda}$ is a possible additional constant (cosmological constant) whose value can be determined by experiment. Stationary action using this Lagrangian gives the field equations for free space.

$$G_{\alpha\beta} = R_{\alpha\beta} - 1/2\, R g_{\alpha\beta} = \overline{\Lambda} g_{\alpha\beta}$$

The tensor $G_{\alpha\beta}$ is called the Einstein tensor. This tensor is equated to the energy-momentum tensor of the gravational field. The source of spatial curvature is then the

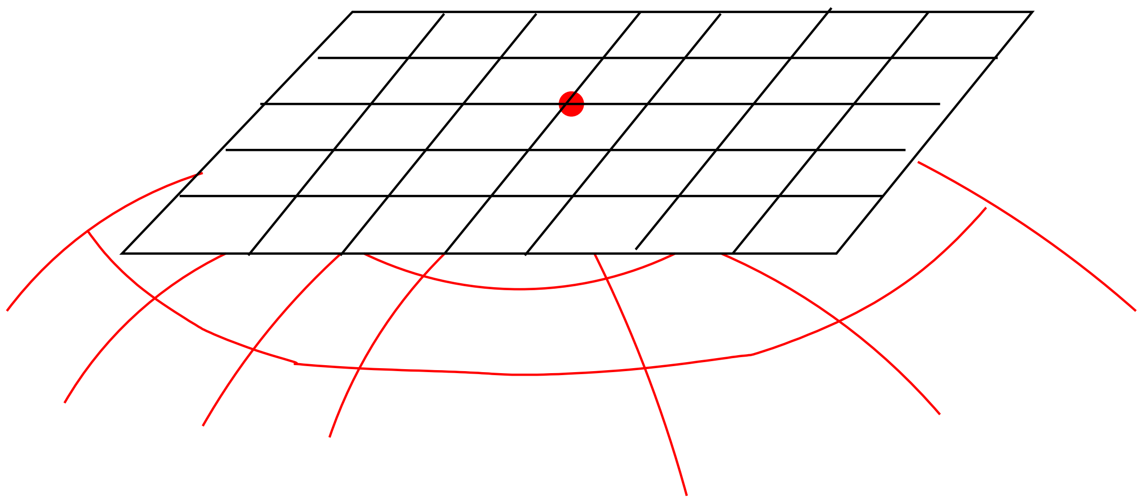

Figure 1: A tangential space time frame

energy-momentum tensor. All derivatives must be replaced by their counterparts in curved space-time - *ie* the covarient derivative. The covarient derivative is the change in a vector in curvalinear space as both components and unit vectors change upon displacement. The resulting equation has the form;

$$G_{\alpha\beta} = R_{\alpha\beta} - 1/2\, R g_{\alpha\beta} = \kappa T \alpha\beta$$

The left side of the equation is the Einstein tensor which is the combination of the Ricci tensor $R_{\alpha\beta}$, the metric, $g_{\alpha\beta}$, and the scalar curvature, $R = R_{\alpha\beta}\, g^{\alpha\beta}$. The covarient divergence of the Einstein tensor vanishes, as does the energy momentum tensor, and both are symmetric. The Einstein tensor was chosen specifcally because was originally thought to be the only 4-component tensor depending on the metric which has the same transformational properties as the stress energy tensor. In addition, the general relativity equation should;

- Recover Newtonian gravity in the appropriate limit
- Satisfy the equivalence principle, $m_{inertia} = m_{gravity}$

2 Cosmological constant

In a flat-tangential space (Robertson-Walker metric with $k = 0$ see figure 1 and supplemental notes) there is a local free-falling frame in which the gravitational field is exactly cancelled by the intertial force. The cosmological constant is the vacuum energy tensor in the equation;

$$G_{\alpha\beta} = \frac{8\pi G}{c^4} T^{vac}_{\alpha\beta}$$

$$T^{vac}_{\alpha\beta} = \frac{\overline{\Lambda} c^4}{8\pi G} g_{\alpha\beta}$$

The general form of the stress-energy tensor for a fluid at rest is;

$$T_{\mu\nu} = P\, g_{\mu\nu} + (P + \rho c^2) u_\nu u_\mu / c^2$$

Here u is the 4-velocity. The Equation of State (EOS) for the fluid is then;

$$(P + \rho c^2)^{vac} = 0$$

$$P^{vac} = \frac{\overline{\Lambda} c^4}{8\pi G}$$

The cosmological constant exerts a positive P^{vac} for a positive value of $\overline{\Lambda}$. A postitive vacuum pressure counter-acts the gravitational attraction and increases the expansion of the universe. A negative pressure is required to explain recent experimental results in the CMW and the apparent acceleration in the expansion of the universe. This results in postulate that Dark Energy must exist; a topic which to be discussed later.

The cosmological constant has negative pressure equal to an energy density which presumably would be created as space expands. Simply put, energy is lost from a container if it does work on the container. A change in volume dV requires work to be done equal to $-pdV$. However, the amount of energy in the volume increases, but because the energy is equal to ρV where ρ is the density then $P = -\rho$ and P is negative.

3 A metric for low-density

The uniform baryonic mass in space is approximately 1 proton/m^3 or $10^{-29}\, g/cm^3$. Locally this is very small so assume a locally flat space. The metric is approximately Minkowskian;

$$g = \begin{pmatrix} \Lambda^2 & 0 & 0 & 0 \\ 0 & \Lambda^2 & 0 & 0 \\ 0 & 0 & \Lambda^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Now if space-time is expanding then $\Lambda \rightarrow \Lambda(t)$. Space stretches isotropically and locally it remains homogeneous. In a spherical frame the (Robertson-Walker metric) is;

$$ds^2 = \Lambda^2(t)[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 sin^2(\theta) d\phi^2] - c^2 dt^2$$

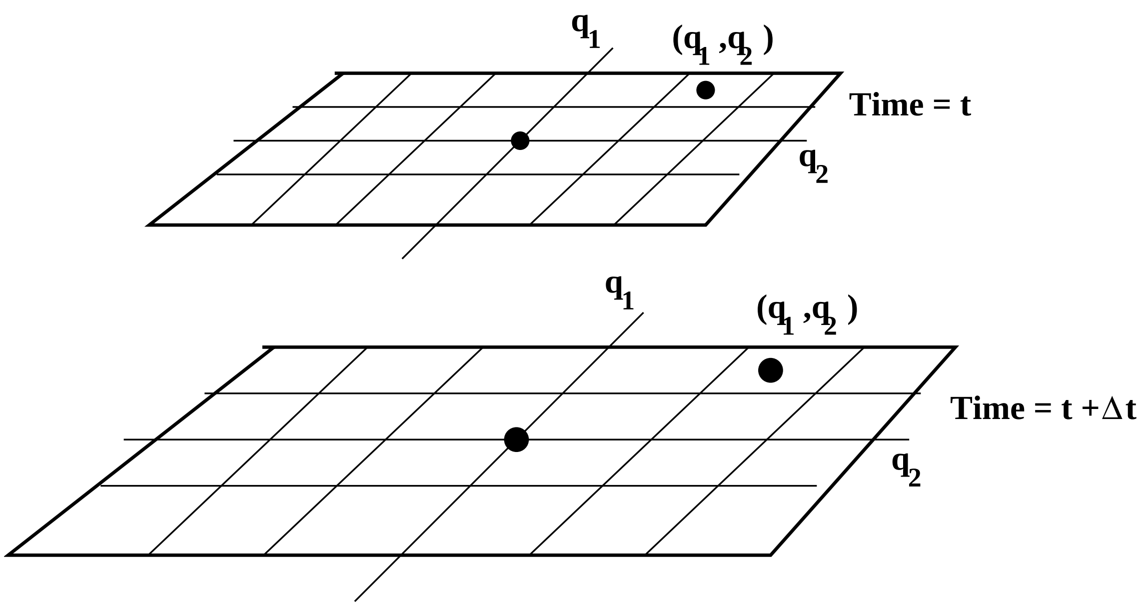

Figure 2: The expansion of space due to the metric

When $k = 0$ the space is flat. The anology is a rubber sheet which stretches isotropically, figure 2. Thus points on the sheet increase their relative distance by $\frac{d\Lambda}{dt}\,dt \;=\; \dot{\Lambda}\,dt$

4 Interpretation of the Doppler shift

In contrast to the usual explanation, galactic redshifts are then not caused by recessional velocities, but by the expansion of space. As illustrated in the figure above, the galaxies are relatively stationary with respect to local coordinates as space expands. This explains the isotropic behavior of space and the CMW. As space expands the wavelengths of the radiation increase, decreasing the frequencies. From the figure the separation of 2 events is given by the interval;

$$ds^2 \;=\; \Lambda^2(t)dq^i dq^i \;-\; c^2 dt^2$$

The sumation convention is assumed so that $i = 1, 2, 3$, and the ends of the interval are measured at the same time. Thus $dt = 0$.

$$ds \;=\; \Lambda^2(t)(d\vec{q})^2$$

As all time points are the same for a distance measurement;

$$l \,=\, \Lambda(t)(\vec{q}^{\,2})^{1/2}$$

Suppose a light signal is emitted at a point $(q_i, \cdots)$ and detected at the origin. The time for the light to travel a distance, l, is $\Delta t \,=\, l/c$. When the light reaches the origin, the spacial separtion has increased by Δl to a separation distance of;

$$l \,+\, \Delta l \,=\, \Lambda(t + \Delta t)(\vec{q}^{\,2})^{1/2}$$

The number of oscillations of the source during the time interval Δt is;

$$\Delta N \,=\, \nu \, \Delta t$$

and if the source does not move the wavelength for a stationary source is;

$$\lambda_0 \,=\, \frac{l}{\nu \Delta t} \,=\, \frac{\Lambda(t)(\vec{q}^{\,2})^{1/2}}{\nu \Delta t}$$

The actual wave length is the distance between the source and the number of oscillations;

$$\lambda \,=\, \frac{l + \Delta l}{\nu \Delta t} \,=\, \frac{\Lambda(t + \Delta t)(\vec{q}^{\,2})^{1/2}}{\nu \Delta t}$$

so that there is a wavelength shift given by

$$\frac{\lambda - \lambda_0}{\lambda_0} \,=\, \frac{\Lambda(t + \Delta t) - \Lambda(t)}{\Lambda(t)}$$

with $\Delta t \,=\, l/c$.

5 Solutions for the Robertson-Walker metric for $k = 0$

Define;

$$h(t) \,=\, \frac{1}{\Lambda} \frac{d\Lambda}{dt} \,=\, \dot{\Lambda}/\Lambda$$

$$\dot{h} \,=\, \ddot{\Lambda}/\Lambda \,-\, (\dot{\Lambda}/\Lambda)^2$$

$$\ddot{\Lambda}/\Lambda \,=\, \dot{h} \,+\, h$$

The Einstein field equations for the Robertson-Walker metric with $k = 0$ (flat space) are;

$$-3(\frac{dh}{dt} \,+\, h^2) \,=\, (\kappa/2)\rho c^4$$

$$(\frac{dh}{dt} + 3h^2) = (\kappa/2)\rho c^4$$

with $\kappa = \frac{8\pi G}{c^4}$. The solution is

$$h(t) = \frac{2}{3(t - t_0)}$$

Here t_0 is a constant of integration. Then $t - t_0$ is the age of the universe. Note the singularity when $t = t_0$ Substitution gives

$$h(t) = [\frac{8\pi G\rho(t)}{3}]^{1/2}$$

and ;

$$t - t_0 = \frac{2}{3h(t)} = [\frac{1}{6\pi G\rho(t)}]^{1/2}$$

Where $\rho(t)$ is the density of the universe at the beginning of time. Since

$$h(t) = \frac{d}{dt} ln(\Lambda)$$

then

$$\Lambda^2 = \gamma^{2/3}(t - t_0)$$

where γ is a constant of integration. Space-time stretches so expand $\Lambda(t)$ about the present time point t_p.

$$\Lambda(t) = \Lambda(t_p) + (t - t_p)\Lambda'(t_p + (1/2)(t - t_p)^2\Lambda''(t_p) \cdots$$

$$\Lambda(t) = \Lambda(t_p)[1 + (t - t_p)H_0 + (1/2)(t - t_p)^2 qH_0^2 \cdots]$$

This relates the Hubble constant to the age of the universe;

$$H_0 = [\frac{1}{\Lambda}\frac{d\Lambda}{dt}]_{tp} = h(t_p) = \frac{2}{3(t_p - t_0)}$$

There is a deceleration parameter q given by ;

$$-qH_0^2 = -(2/9)\frac{1}{(t_p - t_0)^2} = -1/2$$

The Robinson-Walker metric is similar to the experimental determination of the geometry of space-time at the present epoch. Some values of the parameters are;

$$\rho(t_p) = 2 \times 10^{-29}\, g/cm^3$$
$$h(t_p) = 3.34 \times 10^{-18}\, s$$
$$t_p - t_0 = 0.632 \times 10^{10}\, yr$$
$$q_0 = 1/2$$

Finally $\nu = h(t)\, l$, and for the present time;

$$H_0 l/c = \frac{\lambda - \lambda_0}{\lambda}$$

Which defines the Hubble constant

6 Cosmic Sound Waves

Sound waves are pressure waves that propagate longitudinally. At a time 380,000 years after the Big Bang, the temperature falls below 3000°K and electrons began to combine with protons to form atomic hydrogen. The number of electrons falls by 10^4 and the universe is 1000 times smaller and more than 10^9 times more dense than at present. At this time after creation, the photon mean free path becomes larger than the dimensions of the universe, and it lasted about 50,000 years. The CMW represents observation of the last scattered photons from the plasma just before this transition.

Before decoupling, the plasma was a strongly coupled system of electromagnetic radiation and charged particles (electrons and protons) figure 3. It was similar to a dense plasma fluid which could sustain plasma sound waves (electron/field oscillations) of high velocity ($\sqrt{1/3}$c), as the radiation pressure and energy density were >> than that of the baryons. The photon density was 10^{12} photons/cm^3 compared to a baryon density of 10^3 baryons/cm^3.

The temperature of the CMW is presently 2.73 deg K, and it is isotropic after subtraction of the local anisotropies and motion of the earth. There are varitions in the temperature of the order 30×10^6 deg K due to differencies in the densities of the universe at the location where the photons last scattered 4.

These variations were initially due to the random fluctuations in the Big Bang which were smoothed by inflation. The remaining asymmetries then grew more pronounced as gravitation attracted matter to regions more dense from regions less dense regions. This produced gallactic superclusters and voids.

Before decoupling of the electromagnetic radiation, plasma oscillation created waves identified by their pressure changes. These pressure ridges formed scattering centers from which the CMB scattered 5. The height of the energy peak in the CMB is sensitive to the baryon and Dark Matter densities at the time of last scattering and the position of the peaks

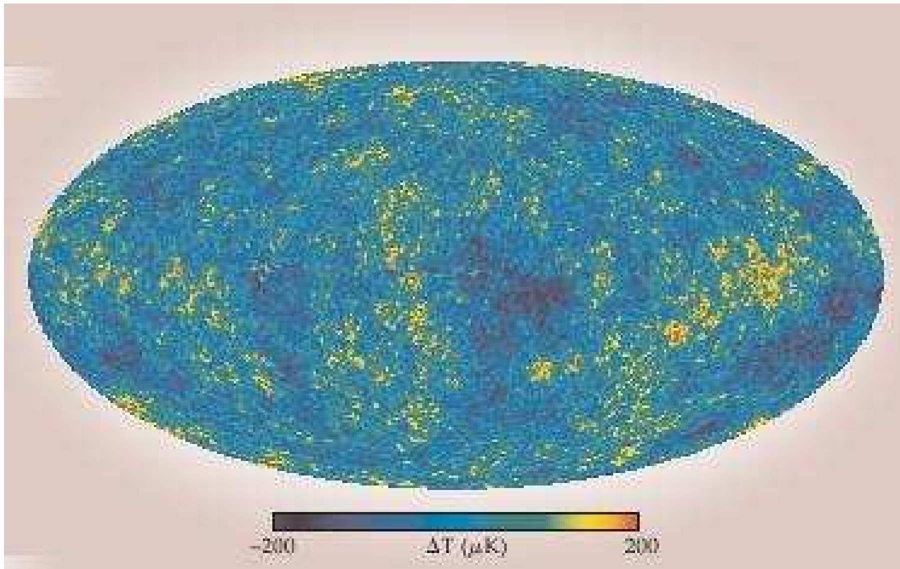

Figure 3: The measured anisotropies of the CMW

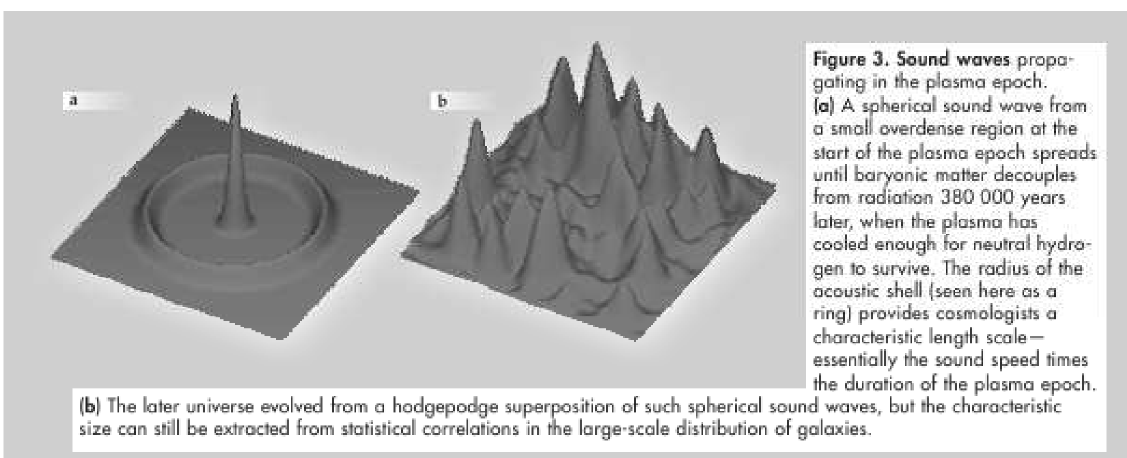

Figure 4: The surface of the pressure wave from which the CMW last scattered

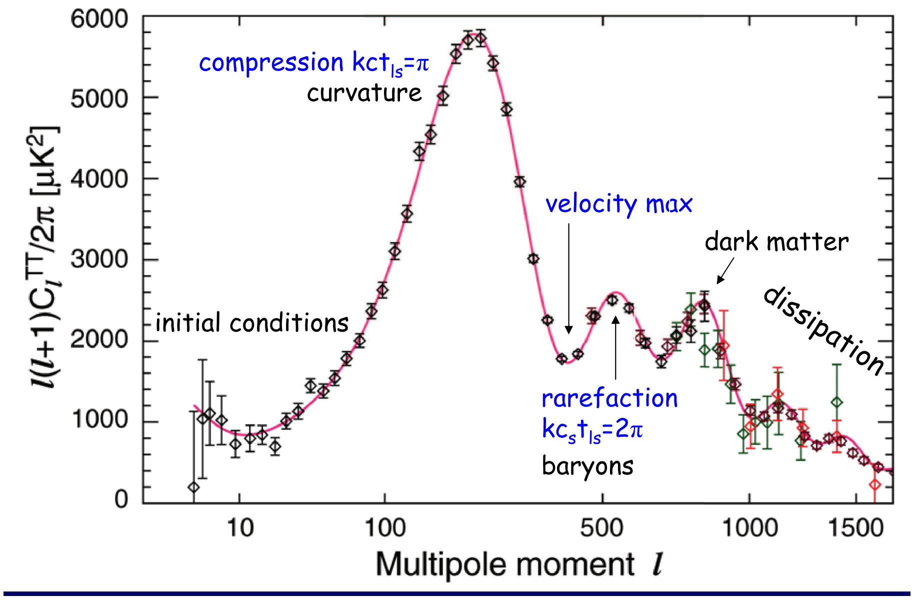

Figure 5: CMW energy spectrum showing the acoustic peaks

sets the acoustic length scale determining the distance to the surface of last scattering, figure 6.

Emergence of the baryon acoustic peak

The successive panels in the figure show a calculation of an acoustic wave spreading from a pointlike primordial density perturbation, and its subsequent effect on various forms of matter and radiation. Each panel is labeled by its time since the end of inflation and the corresponding redshift z. Each shows mass profiles of overdensity versus radial distance from the initial perturbation for the various components of the cosmos: nonbaryonic cold dark matter (black), ordinary baryonic matter made up of nuclei and electrons (blue), the cosmic photon field (red), and the neutrino background (green).

All the overdensities begin as tiny perturbations on a nearly homogeneous universe. Radii are given in a comoving coordinate system that is expanding along with the universe. Two observers at rest with respect to the cosmic microwave background remain at constant comoving separation.

(a) The simulation begins with a primordial perturbation in which a small patch of space is slightly denser than the rest of the universe. The perturbation is adiabatic, so that all species of particles have the same initial perturbation. Because there is a net overdensity in this small region, a net gravitational force attracts matter from distant regions toward the origin.

(b) At early times, the universe is ionized and the nuclei and electrons are tightly coupled to the photon field. The initial overdensity of the photons creates a radiation overpressure that travels away from the origin as a sound wave in the baryon–photon plasma fluid. Meanwhile, the cold dark matter, slow and impervious to photons, stays near the origin while the relativistic neutrinos stream freely away.

(c) The sound wave continues outward in the baryon–photon plasma, traveling at $c/\sqrt{3}$. The cold-dark-matter perturbation has also been spreading because the outgoing sphere of photons and neutrinos has decreased the gravitational pull toward the origin, causing the converging velocity flow of the initial perturbation to pool at finite radius rather than at the origin. That's also why the neutrinos don't appear as a sharp spike in radius.

(d) By $z = 1081$, the universe has cooled enough that the electrons and nuclei combine to form neutral atoms. Photons decouple from the baryonic matter, and most of them have never again scattered off an electron. The sound speed plummets.

(e) With the photons completely decoupled, the shell of baryons that had been propagating with the sound wave grinds to a halt in the comoving coordinates at a radius of 480 Mly (dashed gray line).

(f) In the absence of significant radiation pressure, gravitational instability takes over the distribution of baryons. Their overdensity in the shell attracts dark matter from its more homogeneous reservoir, while the dark-matter overdensity near the origin attracts baryons back toward where they started. Because there's much less baryonic than dark matter, the density contrast between the acoustic peak and its surroundings drops.

(g) By redshift 10, around the time when galaxies are beginning to form, the dark matter and baryons are almost in lockstep. We are left with an overdensity centered at the origin and on a shell at 480 Mly due to the imprint of the sound wave. Both overdense regions are more likely to form galaxies than are less dense neighboring regions. That is, one expects to see a small excess of galaxy pairs separated by about 480 Mly. And indeed that predicted excess has now been found in large redshift surveys of galaxies (see figure 4).

Of course the universe has many such patches, some overdense, some underdense, all overlapping. So don't expect to see bull's-eye patterns in the data. But the characteristic comoving length can be found statistically, and it can serve as a cosmological yardstick. (Figure adapted from ref. 11.)

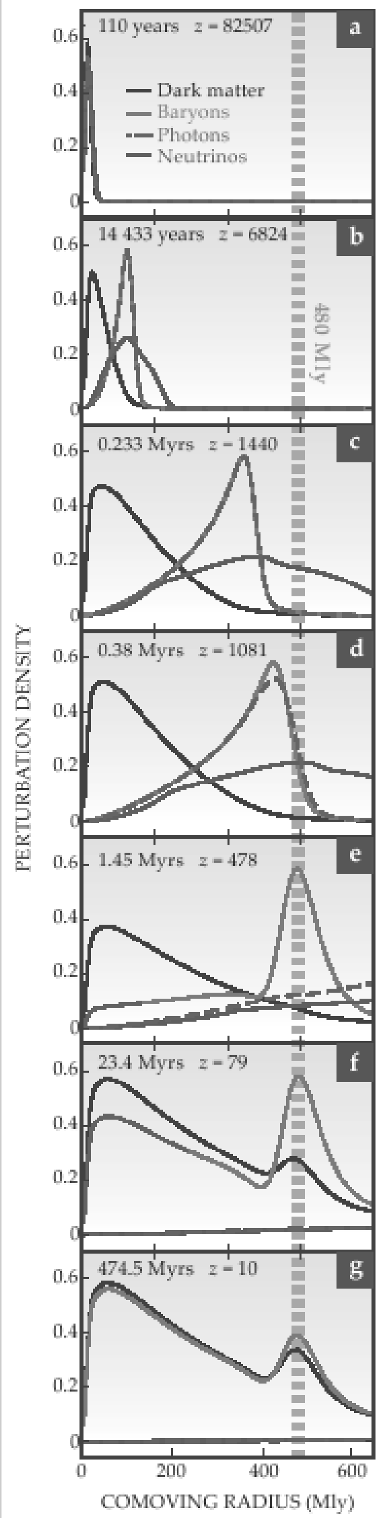

Figure 6: The sensitivity of the CMW spectrum to the last scattering time