Transformations Cont.

1 Relativistic Energy and Momentum

In order to develop expressions for the relativistic momentum and energy, require that momentum is conserved. Momentum conservation is assumed to be a general principle of mechanics, and non-relativistically, the time change of the momentum is a force. The integration of the force over a distance provides an energy. Thus for relativistic mechanics, define the momentum as a mass times the relativistic velocity. The mass is a function of the total speed, U, of the object, and will have a limiting value, m_0, equal to the non-relativistic mass when the object is at rest, $\mu(m_0, U)$. These assumptions allow a velocity transformation to satisfy the velocity transformation equations with a limiting rest mass as $U \to 0$.

Now set up an elastic collision between two indentical particles (same mass) in a geometry so that symmetry can be used to solve the resulting kinematic equations. As the collision is elastic, we require the incident and final momenta be the same (momentum conservation). First view the collision in a reference frame which describes the 2 incoming particles with equal but opposite velocities. This means that the total, incident momentum of the particles is 0. Then use symmetry to determine the velocities after the collision. Consider Fig. 1.

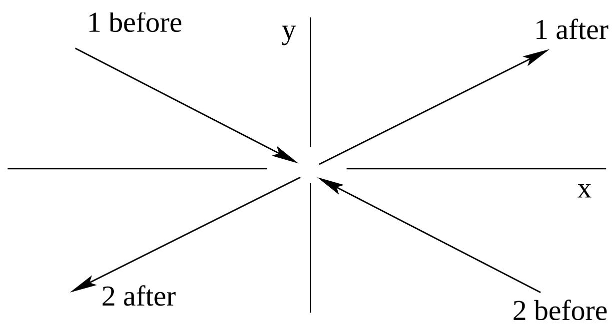

Figure 1: The collision of 2 particles of equal mass and equal velocities. Symmetry allows the velocities after the collision to be directly related to those before the collision.

Velocities before the collision;

$$U_{I1x} = -U_{I2x} = a$$

$$U_{I1y} = -U_{I2y} = b$$

$$U_{I1z} = -U_{I2z} = 0$$

Velocities after the collision;

$$U_{F1x} = -U_{F2x} = a$$

$$U_{F1y} = -U_{F2y} = -b$$

$$U_{F1z} = -U_{F2z} = 0$$

Note that momentum conservation confines the motion to the (x, y) plane, so drop terms in z and work in the (x, y) plane. Now make a transformation to a frame moving with a velocity a in the x direction, and apply the velocity addition equations to the velocities in the initial frame. This must be done carefully as the boost is not in the direction of the velocities in the initial frame. Thus before the collision;

$$U'_{I1x} = 0 \;\; U'_{I2x} = \frac{-2a}{1+(a/c)^2}$$

$$U'_{I1y} = \frac{b}{\sqrt{1-(a/c)^2}} \;\; U'_{I2y} = -\frac{\sqrt{1-(a/c)^2}}{1+(a/c)^2}\, b$$

$$U'_{I1} = \frac{b}{\sqrt{1-(a/c)^2}} \;\; U'_{I2z} = \frac{\sqrt{4a^2+b^2(1-(a/c)^2)}}{1+(a/c)^2}$$

The variable U'_{I1} is the total velocity of the of particle 1 in the new frame of refernce. After the collision the velocities have the forms;

$$U'_{F1x} = 0 \;\; U'_{F2x} = \frac{-2a}{1+(a/c)^2}$$

$$U'_{F1y} = \frac{-b}{\sqrt{1-(a/c)^2}} \;\; U_{F2y} = \frac{\sqrt{1-(a/c)^2}}{1+(a/c)^2}\, b$$

$$U_{F1} = \frac{b}{\sqrt{1(a/c)^2}} \;\; U_{F2} = \frac{\sqrt{4a^2+b^2(1-(a/c)^2)}}{1+(a/c)^2}$$

Now use the expression for the momentum $\vec{P} = \mu(m_0, U)\vec{U}$ and require that the mass, μ, reduce to m_0 as U goes to zero. Momentum conservation requires $P'_{x\,INC\,Total} = P'_{x\,Fnl\,Total}$. This means;

$$\mu(m_0, U'_{I1}) = \frac{1-(a/c)^2}{1+(a/c)^2}\, \mu(m_0, U'_{F2})$$

With U'_{I1} and U'_{F2} given above. Then let $b = 0$ so that;

$$\mu(m_0, 0) = \frac{1-(a/c)^2}{1+(a/c)^2}\, \mu(m_0, \frac{2a}{1+(a/c)^2})$$

Write $\mu(m_0, 0) = m_0$ and let $u = (\frac{2a}{1+(a/c)^2})$. This results in;

$$\mu(m_0, u) = \frac{m_0}{\sqrt{1-(u/c)^2}}$$

The relativistic momentum is identified as;

$$\vec{P}c = \gamma m_0 c^2 \vec{\beta}$$

Finally, the power is obtained from the force by the equation;

$$\frac{dE}{dt} = \vec{U} \cdot \frac{d\vec{P}}{dt}$$

In the above $\vec{U}$ is the mechanical velocity and $\vec{P}$ is the momentum. This equation can be rewritten as;

$$\frac{dE}{dt} = \frac{d}{dt} \frac{mc^2}{\sqrt{1-\beta^2}}$$

In the above $\beta = U/c$. This equation is integrated to obtain;

$$E = \gamma\, m_0 c^2 + E_0$$

Require this to reduce to the non-relativistic value defined by

$$T = E - m_0 c^2$$

where T is the kinetic energy. For $\beta \approx 0$ use a power expansion, $\gamma = 1 + \beta^2/2 + \cdots$;

$$T = (1/2) m_0 V^2$$

Thus we find that the integration constant, E_0, above vanishes and $E = \gamma m_0 c^2$

2 Transformation of a Force

As previously noted, force is no longer an invarient quantity. In fact it does not, in general, remain in the same direction upon a tranformation. From the above equations, the force is;

$$\vec{F} = \frac{d}{dt}(\gamma m_0 \vec{u})$$

Apply a Lorentz transformation describing a boost in the $\hat{z}$ direction.

$$F_z = \frac{d}{dt}(\gamma m_0 u_z) = \frac{m_0 \dot{u}_z}{\sqrt{1-(u_z/c)^2}} + \frac{m_0 u_z (u_z/c)\dot{u}_z}{(1-(u_z/c)^2)^{3/2}} = \frac{m_0 \dot{u}_z}{(1-(u_z/c)^2)^{3/2}}$$
$$F_x = \frac{d}{dt}(\gamma m_0 u_x) = \frac{m_0 \dot{u}_x}{\sqrt{1-(u_z/c)^2}} + \frac{m_0 u_x (u_z/c)\dot{u}_z}{(1-(u_z/c)^2)^{3/2}}$$

If we let $u_y = u_z = 0$ and $F_x = m_0 a'_x$ then;

$$a_z = a'_x/\gamma^3$$

and in a similar way;

$$a_x = a'_y/\gamma^2$$

Thus;

$$F_z = F'_z$$

$$F_x = F'_x/\gamma$$

3 Lorentz Transformation of Space/time in tensor notation

Now we put Lorentz transformations in more compact form, first observing that length and time are not invarient by themsleves, but are connected. The connection comes from the assumption that in all inertial coordinate frames;

$$x^2 + y^2 + z^2 = c^2 t^2$$

Put this transformation in tensor notation so that the components of a 4-D space vector are;

$$x'_0 = \gamma(x_0 - \beta x_1)$$

$$x'_1 = \gamma(x_1 - \beta x_0)$$

$$x'_2 = x_2$$

$$x'_3 = x_3$$

In this notation $x_0 = ct$ and we have assumed that the boost is in the x_1 direction. The above form is a tensor of rank 1 (a vector, A_i) in 4 dimensional space. Then look at the contraction of this vector with itself (a scalar product of the vectors).

$$A_i A^i \; = \; x_1^2 + x_2^2 + x_3^2 - c^2 t^2$$

To form a contraction consistent with tensor notation, use the metric of this space (called Minkowski space). Although as previously noted, covarient and contravarient forms are identical in a Cartesian system, write a subscript to label a covarient tensor and a superscript to label a contravarient one. The metric can be used to change the index of a tensor from covarient to contravarient (and from contravarient to covarient). Note that this metric in Cartesian coordinates is defined as;

$$g_{jk} \; = \; g^{jk} \; = \; \sum_i \frac{\partial x'^i}{\partial x_j} \frac{\partial x'^i}{\partial x_k}$$

Therefore, using the transformation properties of a vector;

$$A'^i \; = \; \sum_j \left(\frac{\partial x^i}{\partial x'^j}\right) A^j$$

The definition of the metric and the properties of the differentials show that;

$$A_i \; = \; \sum_j g_{ij} \, A^j$$

The above form for A^i are the components of a contravarient vector - a true spatial vector. The contraction of this vector with its covarient form gives the square of the length of this 4-vector.

$$A_i A^i \; = \; \sum_{ij} g_{ij} \, A^j A^i$$

From the above definition of the space/time metric, this contraction takes the form;

$$g_{ij} \; = \; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Sometimes a notation where the time coordinate of the first variable, x_0, is the interchanged with last, x_4, is used. This moves the first row in the above matrix to the last, and the signs of this metric are also usually reversed. Tensor notation allows a compact representation of a general coordinate transformation, and we can use this mathematical representation to calculates various Lorentz transformations.

Return to the Lorentz transformation which we now write as a matrix multiplying a vector. In the following, disregard the identification of a vector as covarient or contravarient since in a Cartesian frame they are the same.

$$A'_i = \Gamma_{ij} A_j$$

In the above equation, A_i are the vector components and Γ_{ij} are the elements of a Lorentz transformation. It is more familiar to use the mathematics of vectors and matricies, which we now apply.

$$\Gamma = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The Lorentz transformation looks like a rotation between the time and space coordinates as indeed we have preserved length in space/time by requiring that $A^i A_i$ is constant (summation convention). Remember that a vector is defined by its properties under a coordinate rotation. Thus for a rotation about the $\hat{z}$ axis, the rotation matrix has the form;

$$[R_{ij}] = \begin{pmatrix} cos(\theta) & sin(\theta) & 0 \\ -sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then a rotate about $\hat{z}$ to obtain the new coordinates (x', y') as shown in Figure 2. This rotation is defined by the transformation;

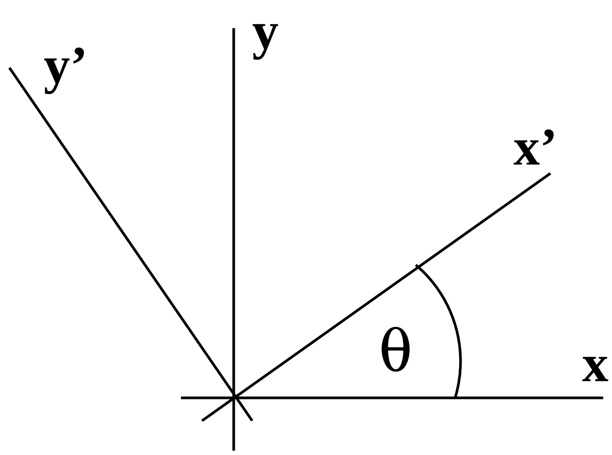

Figure 2: The rotational geometry for a counter-clockwise rotation about the $\hat{z}$ axis through an angle θ, producing the primed coordinate frame

$$x' = x\, cos(\theta) + y\, sin(\theta)$$

$$y' = -x\,sin(\theta) + y\,cos(\theta)$$

$$z' = z$$

We reproduced the result by vector/matrix multiplication;

$$\begin{pmatrix} x' \\ y' \\ x' \end{pmatrix} = \begin{pmatrix} cos(\theta) & sin(\theta) & 0 \\ -sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ x \end{pmatrix}$$

Now for a Lorentz transformation in Minkowski space, define a "rotation" α by;

$$\tanh(\alpha) = \beta$$

$$\cosh(\alpha) = \gamma$$

Then use $cosh^2(\alpha) - sinh^2(\alpha) = 1$ to obtain;

$$sinh(\alpha) = \beta\gamma$$

The Lorentz transformation looks like a "rotation" through an angle α, called "Rapidity", in a complex plane. In an arbitrary boost direction the Lorentz transformation is ;

$$x^{0\,\prime} = \gamma(x^0 - \vec{\beta} \cdot \vec{x})$$

$$\vec{x}' = \vec{x} + \frac{\gamma - 1}{\beta^2}(\vec{\beta} \cdot \vec{x})\vec{\beta} - \gamma\vec{\beta}x^0$$

In 3-D space and 1-D time, the components, x_i $i = 0, 1, 2, 3$, form a vector (4-vector) in 4-D Minkowski space, and transform between inertial frames by the matrix, Γ.

4 The energy/momentum 4 vector

The momentum and energy are written;

$$\vec{P}c = \vec{\beta}\gamma m_0 c^2$$

$$E = \gamma m_0 c^2$$

These are transformed using the velocity transformation equations between V_{0z} and $\vec{V}'$. Use the expression;

$$\gamma = \sqrt{\frac{1}{1-\beta^2}} = \gamma_0\gamma'(1 + V_0 V_z'/c^2)$$

This results in (note E is the energy NOT the electric field);

$$p_z c = \gamma(p_z' c - \beta E')$$

$$p_x c = p_x' c$$

$$p_y c = p_y' c$$

$$E = \gamma(E' - \beta p_z' c)$$

The above is the general form for a 4-vector transformation. Now contract this 4-vector with itself.

$$p^i p_i = g_{ij} p^i p^j = E^2 - (pc)^2$$

This forms a scalar value which is invariant under a Lorentz transformation. In the rest frame $p = 0$ and $E = m_0 c^2$. Thus;

$$E^2 = (pc)^2 + (m_0 c^2)^2$$

5 Velocity 4-Vector

Next consider the velocity transformations. These can be written as;

$$\gamma U_z = \gamma_0(\gamma' U_z' + \beta_0 \gamma' c)$$

$$\gamma U_x = \gamma' U_x'$$

$$\gamma U_y = \gamma' U_y'$$

$$\gamma c = \gamma_0(\gamma' c + \beta_0 \gamma' U_z')$$

This has the form of a 4-vector, so a velocity 4-vector can be defined by $(\gamma c, \gamma \vec{U})$

6 Current/density 4-Vector

Consider the equation of continuity;

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Suppose we a 4-D form $j = (c\rho, \vec{J})$ with the 4-gradient $\partial_i = \frac{\partial}{\partial x_i}$. Identify j as a 4-vector, and recover the equation of continuity by as the result of the contraction of these 4-D vectors. The result ia an invarient because the contraction forms a scalar quantity. To see this in tensor notation, recognize that the 4-vector gradient can be written as;

$$\vec{\nabla}_4 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_0}$$

Then consider the tensor contraction,$\vec{\nabla}_i j^i$, where we use the covarient 4-vector gradient operator and the contravarient 4-vector current/density. This contraction is;

$$\vec{\nabla}_i j^i = \sum_{ik} \frac{\partial}{\partial x_i} g^{ik} j_k$$

By insertion of the values for the current/density 4-vector, the equation of continuity is reproduced. Note here that the vector, $\vec{\nabla}_i$ is a covarient (not a contravarient) vector as opposed to a normal spatial vector.

7 Wave Eqn

Now also note that the 4-gradient contracted with itself gives the wave equation operator. Thus;

$$\vec{\nabla}_i \vec{\nabla}^i = \vec{\nabla}_i (g^{ij}) \vec{\nabla}_j = \frac{\partial^2}{\partial x_0} - \sum_i \frac{\partial}{\partial x_i}$$

This operator is a scalar invarient under a Lorentz transformation. Using Maxwell's equations, one obtains the wave equation operating on the Fields (either $\vec{E}$ or $\vec{B}$). However, consider here, the vector and scalar potentials.

8 4-Vector potential

Remember that;

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Then from Faraday's law;

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial\,(\vec{\nabla} \times \vec{A})}{\partial t}$$

Thus;

$$\vec{\nabla} \times (\vec{E} + \frac{\partial \vec{A}}{\partial t}) = 0$$

Because the curl of the gradient vanishes, write;

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

where V is a scalar potential.

For the moment, remember that the 3-vector potential, $\vec{A}$, can be written in terms of the current density 3-vector, $\vec{J} = \rho \vec{u}$, where $\vec{u}$ is the charge velocity and ρ the charge density.

$$\vec{A} = \frac{\mu}{4\pi} \int d\tau' \, \frac{\vec{J}(x', t')}{|\vec{x}' - \vec{x}|}$$

Now contract the 4-gradient operator with the above 3-vector and a yet unknown term which would represent the 4^{th} component of a 4-vector. The contraction of the current density, as described above, must result in the time dependence of the charge density. The Lorentz transformation of γ times the 3-velocity must give γc. Then note that the potential, V;

$$V = \frac{1}{4\pi\epsilon} \int d\tau' \, \frac{\rho(x', t')}{|\vec{x}' - \vec{x}|}$$

Obviously, V has the form of the scalar potential of electrostatics. This identification is shown in more detail later. For the moment the 4-potential is identified as $(V/c, \vec{A})$ with the elements of the 4-vector defined above.

Note The above operations used to identify the relativistic forms of various physical representations. Contraction of the covarient and contravarient representations of these forms using tensor algebra identifies the invarients and are used to write various equations in compact form.

9 Field Tensors

In the above section, the vector and scalar potentials form components of a 4-vector. The $\vec{E}$ and $\vec{B}$ fields are obtained from these potential forms, $(V/c, \vec{A})$, using for example, the equations;

$$E_x = -\frac{\partial A_x}{\partial t} - \frac{\partial V}{\partial x} = -c(\partial^0 A^1 - \partial^1 A^0)$$

$$B_x = -(\partial^2 A^3 - \partial^3 A^2)$$

By inspection it then appears that the fields are components of an antisymmetric tensor of rank 2. This tensor has the form;

$$\mathcal{F}^{\alpha\beta} = \begin{pmatrix} & \beta & \rightarrow & \\ 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \begin{matrix} \alpha \\ | \\ \mathrm{v} \end{matrix}$$

Therefore in covarient form Maxwell's equations are;

$$\partial_\alpha \mathcal{F}^{\alpha\beta} = \mu_0 j^\beta$$

This equation produces the 2 Maxwell equations;

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

The other 2 Maxwell equations are satisfied by requiring that the fields are obtained from the potentials. Now the field strength tensor is anti-symmetric so it also has a dual representation. This obtained by contraction the fiel tensor with the 4-D Levi-Civiti tensor;

$$\mathcal{G}^{\alpha\beta} = (1/2)\epsilon^{\alpha\beta\gamma\delta} \mathcal{F}_{\gamma\delta}$$

When require $\partial_\alpha \mathcal{G}^{\alpha\beta} = 0$, so that the following Maxwell equations are obtained.

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

The dual field strength tensor has the form;

$$\mathcal{G}^{\alpha\beta} = \begin{pmatrix} & \beta & \rightarrow & \\ 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix} \begin{matrix} \alpha \\ | \\ \text{v} \end{matrix}$$

Note that it is obtained by $E \rightarrow B$ and $B \rightarrow -E$. In 3-D the antisymmetric tensor of rank 2 has a dual which is a psedovector (axial vector). In the case of a rank 2 tensor in 4-D the dual is also a tensor of rank2.

10 Field Transformation

Before using covarient forms for the electromagnetic fields, develop their transformation properties by requiring Maxwell's equations to be invarient under a Lorentz transformation. Previously the wave equation for an electric or magnetic field was shown to be invarient. Below the 4 coupled Maxwell equations are forced to be invarient by a specific choice of the field transformations.

Charge transforms as a scalar and volume transforms as a length. Use Maxwell's eqns;

$$\nabla \bullet \mathbf{E} = \rho/\epsilon \qquad \nabla \bullet B = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times B = \mu \mathbf{J} + \mu\epsilon \frac{\partial \mathbf{E}}{\partial t}$$

The Lorentz transformation is chosen without loss of generality to have velocity along the x_1 axis.

$$x_0' = \gamma(x_0 - \beta x_1)$$

$$x_1' = \gamma(x_1 - \beta x_0)$$

$$x_2' = x_2$$

$$x_3' = x_3$$

These lead to the partials:

$$\frac{\partial}{\partial x} = \gamma \frac{\partial}{\partial x'} - (\beta/c) \frac{\partial}{\partial t'}$$

$$\frac{\partial}{\partial t} = \gamma \frac{\partial}{\partial t'} - v \frac{\partial}{\partial x'}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y'}$$
$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z'}$$

Substitute into Maxwell's equations. The form is invarient if;

$$E'_x = E_x \qquad B'_x = B_x$$

$$E'_y/c = \gamma(E_y/c - \beta B_z) \qquad B'_y = \gamma(B_y + \beta E_z/c)$$

$$E'_z/c = \gamma(E_z/c + \beta B_y) \qquad B'_z = \gamma(B_z - \beta E_y/c)$$

and ;

$$\rho' = \gamma(\rho - (\beta/c)\bullet\mathbf{J})$$

$$\mathbf{J}' = \gamma(\mathbf{J} - c\beta\rho)$$

In a more compact development consider the matrix form of the Lorentz transformation.

$$\vec{x}' = \Gamma\vec{x}$$

$$\vec{x}^{\dagger} = \vec{x}^{\dagger}\Gamma^{\dagger}$$

Because we expect

$$\vec{x}'^{\dagger}\mathcal{F}'^{\alpha\beta}\,\vec{x}' = \vec{x}^{\dagger}\mathcal{F}^{\alpha\beta}\,\vec{x}$$

Substitute using the Lorentz transformation matrix and its transpose to obtain;

$$F'^{\alpha\beta} = \Gamma F^{\alpha\beta}\Gamma^{\dagger}$$

Multiply the above matricies to obtain a new field tensor and compare the components. This gives the transformation properties of the fields. For a boost in the 1 direction

$$E'_1 = E_1\,; \;\; B'_1 = B_1$$

$$E'_2/c = \gamma(E_2/c - \beta B_3)\,; \;\; B'_2 = \gamma(B_2 + \beta E_3/c)$$

$$E'_3/c = \gamma(E_3/c + \beta B_2)\,; \;\; B'_3 = \gamma(B_3 - \beta E_2/c)$$

This is the same transformation found earlier when we required Maxwell's equations to remain invarient under a Lorentz transformation. If there is **ONLY** a magnetic field in the unprimed frame, then;

$$\vec{E}'/c = -\vec{\beta} \times \vec{B}'$$

11 Field of a point charge

Apply the transformations to transform the field of a point charge at rest ($\vec{E}' = \frac{q}{r^2}\hat{r}$) to a frame moving with constant velocity (Gaussian units). Restrict the $\vec{E}$ to a plane and apply a boost in the 1 direction about which the $\vec{E}$ field is symmetric. This is shown in Figure 3. Apply the Lorentz transformation to obtain the following; (assume that $t = 0$);

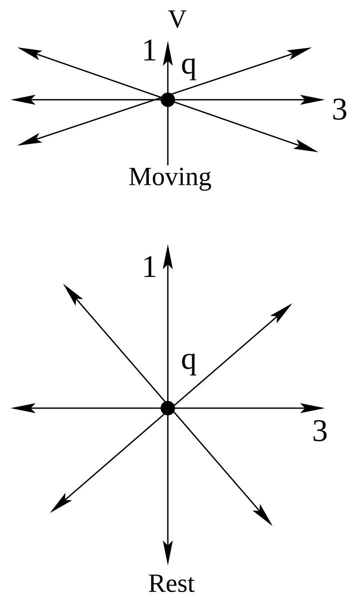

Figure 3: The field of a point charge in the (1,3) plane at rest and in a frame boosted in the 1 direction

$$E_1 = E_1' = \frac{1}{4\pi\epsilon}\frac{qx_1'}{[x_3'^2 + x_1'^2]^{3/2}} = \frac{1}{4\pi\epsilon}\frac{q\gamma x_1}{[x_3^2 + \gamma^2 x_1^2]^{3/2}}$$

$$E_3 = \gamma E_3' = \frac{1}{4\pi\epsilon}\frac{q\gamma x_3'}{[x_3'^2 + x_1'^2]^{3/2}} = \frac{1}{4\pi\epsilon}\frac{q\gamma x_3}{[x_3^2 + \gamma^2 x_1^2]^{3/2}}$$

$$B_2 = \frac{1}{4\pi\epsilon}\frac{q\gamma\beta x_3'}{c[x_3'^2 + \gamma^2 x_1'^2]^{3/2}} = \beta\, E_3/c$$

Put this into spherical coordinates in the unprimed frame.

$$\vec{E} = \frac{q(1-\beta^2)}{r^2[1-\beta^2\, sin^2(\theta)]^{3/2}}\hat{r}$$

Now note that the field is radial and compressed perpendicular to the direction of motion. The paradox here is that at all points in space, the $\vec{E}$ field points to the apparent position of the charge. Given causality, the question is how does an observer far away from the charge know where the charge will be? The answer is that the observed field only points to the position where the charge is expected to be. If the charge is accelerated in some way, and the information of this acceleration does not reach the observer during the time of observation due to the maximum propagation velocity, c, the observer would believe that the charge contiuned moving in the same direction at a constant velocity. However, if the field lines were curved in some way, radiation radiation would result. But a radiating charge loses energy, and in a frame where the charge is at rest no energy is lost. Remember with respect to the laws of physivs, all intertial frames of reference are equivalent.

12 Wire carring a current

Suppose a long straight wire carring a current, I. The curent produces a magnetic field, but there is no electric field as there is no net charge density on the wire. Suppose a charge, q is placed at rest near the wire. There is no force on the charge as its velocity is zero and there is no E field. However suppose we transform into a moving frame so that the charge has a constant velocity. In this frame there is a magnetic $\vec{V} \times \vec{B}$ force on the charge so that it moves either toward or away from the wire depending on the directions of the field and the velocity. This presents a paradox, and its resolution is instructive. Look at the current on the wire. As a model, assume that the electrons (negative charges) are moving to produce the current and the atoms of positive charge do not move, but they do cancel the charge of the negative electrons. Now transform to a frame so that the positive charges move. In this frame there will be a change to the positive charge density, $\lambda^+ = \delta q/\delta L$, because the elemental length δL is contracted to $\delta L/\gamma$. The negative charge density also changes, but because of the non-linear addition of velocities in the velocity transformations, it does not change by the same amount. The negative charge density in the moving frame is $\lambda^- = \gamma[1-\beta_0\beta]\lambda$, β_0 is the initial velocity of the electrons, and β the velocity of the moving frame with respect to the rest frame. Therefore the net charge density is;

$$\lambda_{net} = \lambda^+ - \lambda^- = \gamma\beta_0\beta$$

Since there is now a net charge density on the wire there will be an electric field. The force due to the electric field will cancel the force due to the magnetic field, so the charge does not move toward (or away from) the wire. Of course a relativistic transformation couples electric and magnetic fields as was developed previously.

13 Proper time

Suppose there are 2 space time points located at $(ct_1, \vec{x}_2)$ and $(ct_2, \vec{x}_2)$. The time for a light ray to travel between these points is;

$$d\tau^2 = (t_2 - t_1)^2 - (1/c)^2(\vec{x}_2 - \vec{x}_1)^2$$

If the two events have the same space point, $d\tau$ would be the time recorded on a clock at rest in that frame of reference. Whenever two events can just be connected by a light ray that leaves one point and arrives at the other, then such events are simultaneous and $d\tau = 0$. If the value of $d\tau$ is real then the points are "time-like" and events at these points can be observed. If $d\tau$ is imaginary the events would be "space-like" and could not be observed. Write the differential;

$$d\tau^2 = dt^2 - (1/c)^2(\frac{d\vec{x}}{dt})^2 \, dt^2 = dt^2(1 - \beta^2) = (dt/\gamma)^2$$

Note here that;

$$U^\alpha = \frac{dx^\alpha}{d\tau} = \gamma \frac{dx^\alpha}{dt} = \gamma V^\alpha$$

which is the relativistic velocity.

14 World Line

To understand the world line, consider a projection of a line of events onto 1 dimension of space and the 1 dimension of time. There is a line in this space representing the sequence of events in time as a function of a 1-D position in space. This is shown in Figure 4. The lines at $x = \pm ct$ represent the light cone. In a 2-D space plus 1-D time this is a geometric cone with axis along the time direction. An event which occurs at $t = 0$ (the present) lies at the apex of the light cone in the past and will be connected to positions within the light cone in the future. If any event at some point in space/time lies outside the light cone, it cannot pass information to the present point until the light cone moves sufficiently to encompass this point. Another way to observe this is to use the constancy of the velocity of light. If the proper time is imaginary, the event is "space like" and lies outside the light cone. If the proper time is real, the event is "time like" and lies within the light cone. If the proper time is zero, it lies on the light cone.

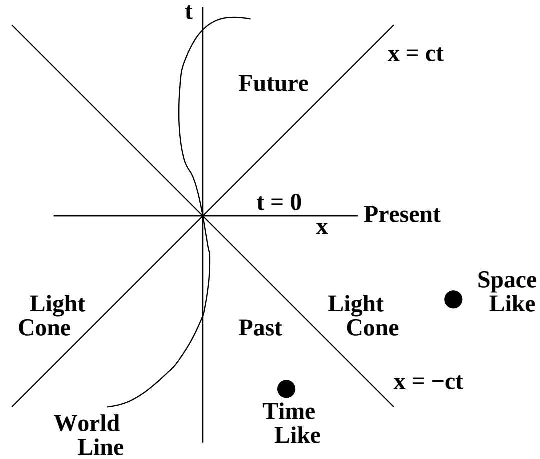

Figure 4: The light cone and world lines, showing “space like” and “time like” events

15 Lorentz force

As previously written, the Lorentz force is;

$$\vec{F} = \frac{d\vec{p}}{dt} = q(\vec{E} + c\vec{\beta} \times \vec{B})$$

Now consider the following contraction;

$$F^{\alpha\delta}\, g_{\delta\beta}\, U^{\beta} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} \gamma c \\ \gamma V_x \\ \gamma V_y \\ \gamma V_z \end{pmatrix}$$

The matrix is obtained by matrix multiplication of the Field tensor $\mathcal{F}^{\alpha\delta}$ by the metric $g_{\delta\beta}$. The above equation results in

$$\mathcal{F}^{\alpha\delta}\, g_{\delta\beta}\, U^{\beta} = \begin{pmatrix} \gamma(E_xV_x/c + E_yV_y/c + E_zV_z/c) \\ \gamma(E_x/c + B_zV_y - B_yV_z) \\ \gamma(E_y/c - B_zV_x - B_xV_z) \\ \gamma(E_z/c - B_yV_x - B_xV_y) \end{pmatrix}$$

Note that $d\tau = dt/\gamma$ with $d\tau$ the proper time. We make the connection to the non-covarient equation by remembering that the power equals $\vec{F} \cdot \vec{V} = q\vec{E} \cdot \vec{V}$. The covarient form of the Lorentz force is then written;

$$(m_0c)\frac{dU^{\alpha}}{d\tau} = q\mathcal{F}^{\alpha\beta}U_{\beta}$$

16 Ehernfest Paradox

Ehernfest in 1909 proposed a problem which seemed to contradict the postulates of relativity. In its simplest form,the problem compares the circumference, C, of a rigid disk with its radius, R, as the disk acquires an angular velocity, ω. Applying the Fitzgerald-Lorentz contraction, the circumference contracts, while the radius which is perpendicular to the velocity remains constant. Thus the value of C/2R is not equal to the constant π but depends on the value of ω. In fact, this result was still proposed in a recent paper.

However, in the first place there is the issue that the rotating frame is not an inertial frame. Comparison of lengths require simultaneous comparison of the ends of rods and times on clocks in diferent inertial frames. While one can devise a way to make a measurment of a proper length in any frame which is instantaneously at rest, the two systems, one at rest and one undergoing an acceleration, are not symmetric. This was pointed out in the analysis of the Twin Paradox discussed earlier. Thus comparison of lengths and times using recriprocal inversions of transformations is not correct.
Resolution of this paradox is not completely resolved. However, its solution is tied to the assumption that the disk is a rigid body which acquires an acceleration. The concept of a rigid body is not consistent with the postulate that information cannot propagate faster than c, so elastic constants can only respond to forces between sections of a body within in a finite time interval. An analysis of a rotating ring, taking into account the limiting velocity, c, results in a length expansion of the ring circumference of $[1 + b^2]^{1/2}$ to first order in β. In addition the radius changes by the same factor. Thus a ring consistent with the relativity postulate of limiting velocity, would expand by the above factor. but the radius also expands. Finally, while this analysis does not resolve the problem of a rotating disk, it does help to show that a complete analysis of the response of the disk to its acceleration is needed. The paradox, as formulated is inconsistent with relativistic mechanics.