Examples - Lecture 8

1 GPS System

The global positioning system, GPS, was established in 1973, and has been updated almost yearly. The GPS calculates postion on the earth's surface by accurately measuring timing signals sent by GPS satellites. The information which is sent, includes timing of the message, the precise orbital information of the satellite, and general information on all GPS satellites. The receiver uses this information to determine transit time of the signal, and then computes the distances to each satellite. Given the satellite positions, the position of the receiver on the earth's surface is then determined by triangulation. Three satellite signals would be sufficient, however because of clock uncertainties, four or more satellites are used; see Figure 1.

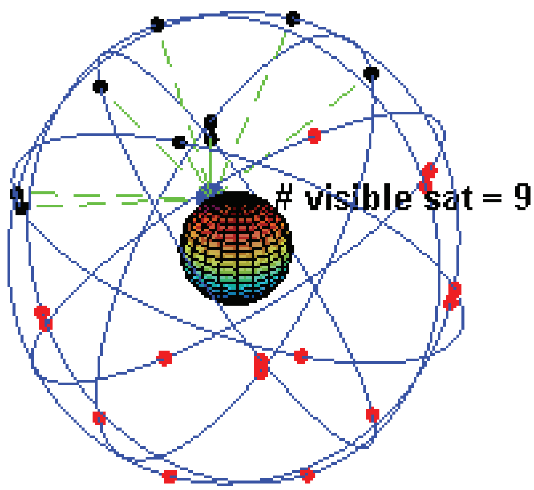

Figure 1: The figure shows the geometry of the GPS system

However, the velocity of light, c, is so large, that clock error is amplified when calculating distances, and also positions are obtained from the intersections of spherical surfaces as obtained from timing the signals. Two such surfaces intersect in a circular ring, and the third surface intersects this circle in 2 or fewer points, including zero (no intersection). The point closest to the satellite in the case of a 2 point intersection is chosen. The receiver has 4 unknowns (3 position in space and a time shift, b). Thus

$$(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 = ([t_r + b - t_i]c)^2$$

$$i = 1, 2, 3, 4$$

The above equations or the 4 observations are solved numerically. Atomic clocks on the satellites are synchronized to a universal time and are periodically corrected. In addition to correcting the clocks for relativistic effects (even general relativity is required), there are phase shifts, and propagation delays through the ionisphere.

2 Vector and Scalar Potentials

In the last lecture;

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

giving $\vec{\nabla} \cdot \vec{B} = 0$ and from Faraday's law, $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$;

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

Now substitute these back into the other two Maxwell's equations.

$$\vec{\nabla} \cdot \vec{E} = -\nabla^2 V - \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{A} = \rho/\epsilon$$

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

$$\nabla^2 \vec{A} - \epsilon\mu \frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}(\vec{\nabla} \cdot \vec{A} + \epsilon\mu \frac{\partial V}{\partial t}) = -\mu \vec{J}$$

Thus the number of Maxwell's equations is reduced from 4 to 2, but they are still coupled. At this point choose $\vec{\nabla} \cdot \vec{A} + \epsilon\mu \frac{\partial V}{\partial t} = 0$. This is the Lorenz condition and will be justified later. After applying this condition, separation of the above equations yields;

$$\nabla^2 V - \epsilon\mu \frac{\partial^2 V}{\partial t^2} = -\rho/\epsilon$$

$$\nabla^2 \vec{A} - \epsilon\mu \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}$$

These equations are the inhomogeneous wave equation with sources ρ and $\vec{J}$. It is shown in the next section that the equations are consistent with the Lorentz transformation as outlined in the last lecture.

3 Invarience Examples

In the last lecture it was noted that the 4-gradient operation $\partial_\alpha \rightarrow (\frac{\partial}{\partial x_0}, \vec{\nabla})$ forms a co-varient 4-vector. Then using the metric defined in the last lecture, the contraction of this operator with itself is ;

$$\partial_\alpha \partial^\alpha = \partial_\alpha g_{\alpha\beta} \partial^\beta = \frac{\partial^2}{\partial x_0^2} - \nabla^2$$

This contraction yields a Lorentz invarient, the d'Alembertian, which is a Lorentz scalar. The d'Alembertian is defined by the operator, $\Box$.

$$\Box = (\frac{\partial}{i\partial x_0}, \vec{\nabla})$$
$$\text{d'Alembertian} = \Box \cdot \Box = \Box^2$$

The wave equation operator is a Lorentz scalar. Operation on a 3-vector function such as $\vec{A}$ must also produce a 3-vector. From the above equations, this 3-vector is $\vec{J}$. You should now be able to show that the operation on the scalar potential, V (a form proportional to the time component of a 4 vector) must be the charge density. Contracting the 4-gradient operator with the 4-vector potential, $(V/c, \vec{A})$ gives;

$$\partial_\alpha A^\alpha = \frac{\partial V/c}{\partial x_0} + \vec{\nabla} \cdot \vec{A} = constant$$

The above [EH1]equation was set to zero in the last section in order to separate Maxwell's equations. Thus, this form must be a Lorentz constant, and if it is zero in any frame, it must be zero in all frames (the Lorentz condition). Therefore one can choose a frame in which the form vanishes and so it is zero in all frames. As another example, contract the 4-gradient with the current density 4-vector, $(c\rho, \vec{J})$. This gives;

$$\partial_\alpha j^\alpha = \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J}$$

As was noted previously, this is the equation of continuity, and describes conservation of charge. In general, this operation is really a 4-divergence and a divergence operation which vanishes leads to a conservation equation. **Why??**

The energy and momentum form a 4-vector $p^i = (E/c, \vec{p})$. Contract this 4-vector with itself.

$$p_i p^i = p^j g_{ji} p^i = E^2 - (pc)^2$$

This forms a Lorentz scalar which must be the same in all inertial frames. Choose the rest frame where $\vec{p} = 0$ in order to evaluate this constant. In this frame, $E = m_0 c^2$, the energy of the rest mass.

Also the 4-velocity vector was defined by $U \rightarrow (\gamma c, \gamma \vec{u})$. Contraction of this 4-vector with itself gives;

$$U^i U_i \;=\; \gamma^2 c^2 \;-\; \gamma^2 u^2 \;=\; c^2$$

Thus

$$E^2 \;-\; (pc)^2 \;=\; (m_o c^2)^2$$

Finally, consider the contraction of the field tensor with its dual. These tensors were found in the last lecture.

$$\mathcal{F}^{\alpha\beta} \;=\; \begin{array}{cc} \begin{array}{cccc} & \beta & \rightarrow & \end{array} & \\ \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} & \begin{array}{c} \alpha \\ | \\ \mathrm{v} \end{array} \end{array}$$

$$\mathcal{G}_{\beta\alpha} \;=\; \begin{array}{cc} \begin{array}{cccc} & \alpha & \rightarrow & \end{array} & \\ \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z/c & -E_y/c \\ -B_y & -E_z/c & 0 & E_x/c \\ -B_z & E_y/c & -E_x/c & 0 \end{pmatrix} & \begin{array}{c} \beta \\ | \\ \mathrm{v} \end{array} \end{array}$$

$$\mathcal{F}^{\alpha\beta}\mathcal{G}_{\alpha\beta} \;=\; \mathcal{F}^{\alpha\beta} g_{\alpha\delta} g_{\beta\gamma} \mathcal{G}^{\delta\gamma}$$

The multiplication of these gives a Lorentz scalar;

$$\mathcal{F}^{\alpha\beta}\mathcal{G}_{\alpha\beta} \;=\; 4(\vec{E}/c) \cdot \vec{B}$$

4 Other Examples

4.1 Visual View of a moving rigid board

A rigid board as measured in its own rest frame has sides parallel to the z and the y axes. It moves with a uniform velocity, v, with respect to the z axis. What is the shape and orientation of the board with respect to an observer at the origin of the coordinate system, looking along the x axis, Figure 2?

Every point is seen at the same time, τ, at the position, y_0, on the y axis. The light is emitted at some earlier time and travels the distance, d.

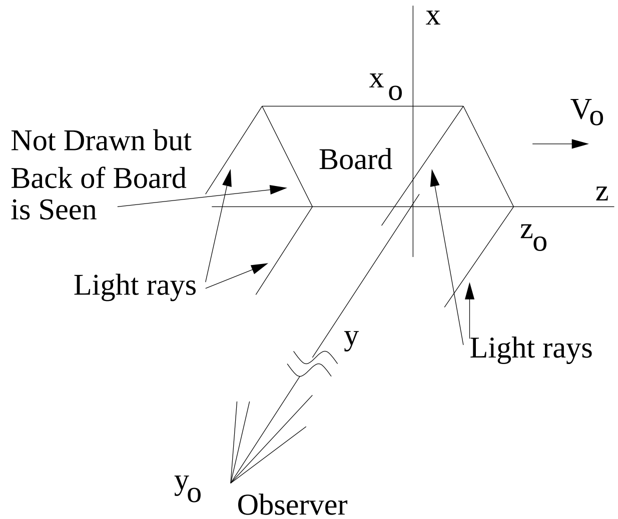

Figure 2: The figure shows the moving board in the (x,z) plane. Light rays come from each corner and arrive at the same time at the observer a long distance away

$$d = \sqrt{x_0^2 + y_0^2 + z_0^2}$$

When $x_0 = y_0 = 0$ then the length of the board is L/γ where $\gamma = [1 - \beta^2]^{-1/2}$. The time of observation of the corner is τ, which is the proper time from that position. Thus the time for a ray to travel from $(0, 0, z_0)$ to $(0, y_0, 0)$ is;

$$\tau_0 = d/c = [y_0^2 + z_0^2]^{1/2}/c$$

and during that same time a light ray travels from $(x_0, 0, z_0)$ to $(0, y_0, 0)$;

$$\tau_0 = [x_0^2 + y_0^2 + (z_0 - v\tau_0)^2]^{1/2}.$$

Solve these two equations for $\beta c\tau$

$$\delta z = \beta c\tau = z_0 - \sqrt{z_0^2 - x_0^2}.$$

Thus the corner at position $(x_0, 0, z_0)$ lags behind the corner $(0, 0, z_0)$ by a distance δz.

4.2 Field Transformation

If electric field vanishes in a given Lorentz frame, then any charged particle obeying the Lorentz force law has constant speed (not velocity) in that frame.

The Lorentz force is:

$$mc\frac{du^\alpha}{d\tau} = qF^{\alpha\beta}u_\beta.$$

where u is the 4-velocity. The spatial and time components are;

$$\frac{d(\gamma mc\vec{v})}{dt} = q[\vec{E} + \vec{v}/c \times \vec{B}]. \text{ (Force)}$$

$$\frac{d(\gamma mc^2)}{dt} = \vec{F} \bullet \vec{v} = q\vec{E} \bullet \vec{v}. \text{ (power)}$$

when $\vec{E} = 0$ then;

$$\frac{d(\gamma mc^2)}{dt} = 0.$$

Thus γmc^2 = constant.

4.3 Another field transformation

The expression, $\vec{E} \cdot \vec{B}$, is a constant Lorentz invarient. This can be shown by observing that the contraction $\mathcal{F}^{\alpha\beta}\mathcal{G}_{\alpha\beta}$ is a scalar, where $\mathcal{F}^{\alpha\beta}$ is the field strength tensor and $\mathcal{G}_{\alpha\beta}$ is its dual. Now suppose $\vec{E}' = 0$ in a reference frame and then boost this in the 1 direction to another frame. In that frame;

$$E_1 = E'_1 = 0$$

$$E_2 = \gamma(E'_2 + \beta B'_3) = \gamma\beta B'_3$$

$$E_3 = \gamma(E'_3 - \beta B'_2) = -\gamma\beta B'_2$$

$$B_1 = B'_1 = 0$$

$$B_2 = \gamma(B'_2 - \beta E'_3)$$

$$B_3 = \gamma(B'_3 + \beta E'_2)$$

Any $\vec{B}'$ that generates the $\vec{E}$ and $\vec{B}$ above satisfies the requirement that $\vec{E} \cdot \vec{B} = 0$. However, for an explicit solution choose $B'_2 = 0$ so that;

$$E_1 = 0$$

$$E_2 = \gamma\beta B_3'$$

$$E_3 = 0$$

Then;

$$B_1 = B_1' = 0$$

$$B_2 = \gamma(B_2' - \beta E_3') = 0$$

$$B_3 = \gamma(B_3' + \beta E_2') = \gamma B_3'$$

Then $E_3 = 0$, $B_3 \neq 0$ but $\vec{E} \cdot \vec{B} = 0$

4.4 Energy and Momentum Conservation

In any frame, a system of particles takes the following form (units where $c = 1$ so that momentum and mass are measured in energy units).

$$\sum_i (E_i)^2 - \sum_i (\vec{p}_i)^2 = M^2 = E_{CM}^2 \text{ constant}$$

For 2 particles with one at rest;

$$E_{CM} = [m_1^2 + m_2^2 + 2\, E_1 m_2]^{1/2}$$

Also from this, the velocity of the CM system with respect to this reference frame is, see Figure 3;

$$\vec{\beta}_{CM} = \frac{\sum_i \vec{p}_i}{E_{cm}}$$

$$\gamma_{cm} = \frac{\sum_i E_i}{E_{cm}}$$

The energy of a particle in the CM frame, assuming 2-particle decay, Figure 3;

$$E_{1\,CM} = \frac{M^2 + (m_1^2 - m_2^2)}{2M}$$

$$M = E_{1\,CM} + E_{2\,CM}$$

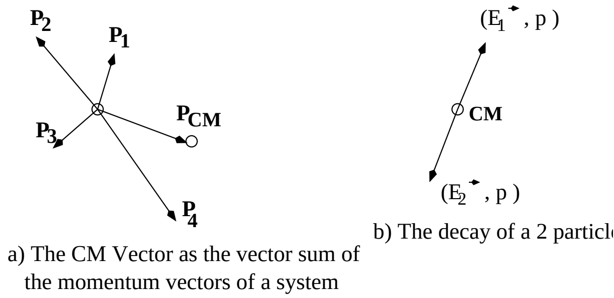

Figure 3: Figure 2a shows the CM vector as the sum of the individual momentum vectors of a many body system. Figure 2b shows the decay of a 2-particle system in the CM

Find the maximum energy which can be transferred in an electron-electron collision using realtivistic kinematics.

Energy and momentum conservation give;

$$E_1 + E_2 = E_1' + E_2' = E_0$$

$$\vec{p_1} + \vec{p_2} = \vec{p_1'} + \vec{p_2'} = \vec{p_0}$$

Remove the dependence on $(\vec{p_2'}, E_2')$

$$E_2'^2 - p_2'^2 = m^2 = (E_0^2 - p_0^2) + m^2 - 2E_0E_1' - 2\vec{p_0} \bullet \vec{p_1'}$$

Let $a = \frac{m_0^2}{2E_0}; b = \frac{\vec{p_0}}{E_0} = \vec{\beta_{cm}}$

$$[1 - b^2 cos^2(\theta_1)]p_1'^2 + 2ab\, cos(\theta_1)\, p_1' - (a^2 - m^2) = 0$$

The solution is;

$$p_1' = \frac{-ab\, cos(\theta_1) \pm \sqrt{a^2 - m^2[1 - b^2\, cos^2(\theta_1)]}}{[1 - b^2 cos^2(\theta_1)]}$$

For Maximum energy transfer $\theta_1 = \pi$ and in the ultra-relativistic case $(m/a)^2 \approx 0$

$$p_1' \approx \frac{a(1+b)}{(1-b)(1+b)} = \frac{a}{1-b} \approx a$$

The energy transfer is;

$$\omega = E_1 - E_1' \approx E_1 - \frac{E_0^2 - p_0^2}{2E_0} = E_1 \left[1 - \frac{E_2}{E_0}(1 - cos(\theta_{12}))\right]$$

5 Mandelstam Variables

Cross sections measured at relativistic energies are sometimes written in terms of the relativistic Maldestam invariants defined by ;

$$s = (p_1 + p_2)^2 = (m_1^2 + 2E_1E_2 - 2\vec{p}_1 \cdot \vec{p}_2 + m_2^2)$$

$$t = (p_1 - p_3)^2$$

$$u = (p_1 - p_4)^2$$

In the above equations p_i is the energy/momentum 4-vector (E, pc) with $c = 1$. Thus $p^2 = E^2 - |\vec{P}|^2 = m^2$. The interaction diagram of the particles is given in Figure 4. In the assignment where particles 1 and 2 are incident and 3 and 4 are reaction products, s is the relativistic energy in the CM system, t represents the 4-momentum transfer in the reaction, and u is a crossing channel which does not have an obvious interpretation. The parameters are not independent.

$$s + t + u = 2[\mu_{in-CM}^2 + \mu_{out_CM}^2]$$

In the above, μ represents the CM mass (Energy) for the in and out chanels respectively.

6 Kinematics

As observed in the last section, the momentum of a scattered particle due to a collision is obtained from the solution of a quadratic equation, which can be solved if the scattering angle is known. This solution can have 2 real solutions for selected values of incident masses and incident momenta. This is most easily seen in the figure 5 from the perspective of the Center of Momentum, (CoM). In non-relativisitc kinematics one often discusses a collision in terms of the Center of Mass (CM). However, CM is ill defined in relativity, and the use of the CoM, is most appropriate. In the CoM frame, the sum of the incident momenta vanish, and since momentum conserved, the momentum also vanishs after the collision. A boost to the momentum in the CoM frame is illustrated by the vector additions shown in the figure. However be careful here. The additions actually must be done using relativistic relations.

When the Lorentz transformation is applied to transform the momentum and energy between two Lorentz frames, one obtains in polar coordinates;

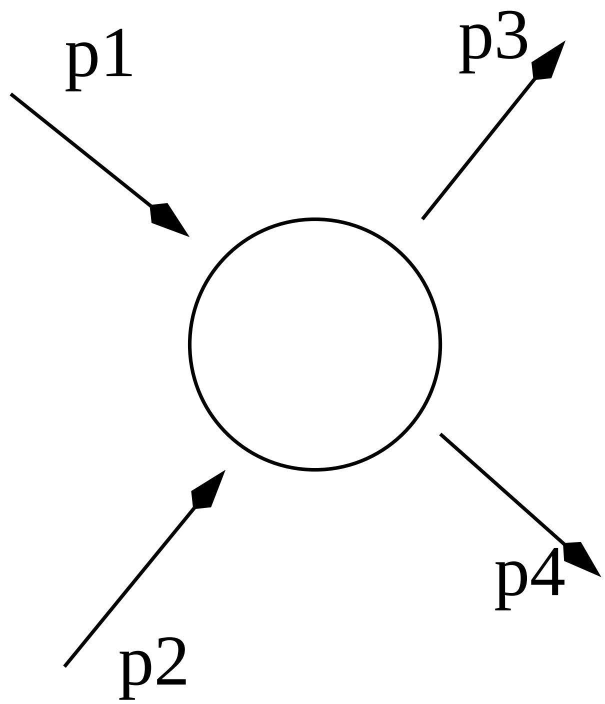

Figure 4: A diagram showing the incident and reaction channels in a collision at relativistivc energ terms of the Maldestam variables.

$$p\,cos(\theta) \;=\; \gamma(p'\,cos(\theta') + \beta E')$$

$$p\,sin(\theta) \;=\; p'\,sin(\theta')$$

$$E \;=\; \gamma(E' \;+\; \beta p'\,cos(\theta'))$$

$$\phi \;=\; \phi'$$

In general,

$$p_1 \;=\; p\,cos(\theta)$$

$$p_2 \;=\; p\,sin(\theta)\,cos(\phi)$$

$$p_3 \;=\; p\,sin(\theta)\,sin(\phi)$$

with β, γ the CM velocity and γ factor.

The conservation of energy and momentum for a collision $A + B \;\rightarrow\; C + D$, assuming B is

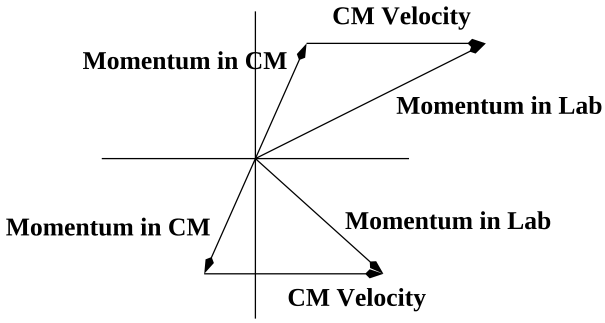

Figure 5: The figure shows how the addition of the CM momentum is added to the momenta in the CM system to obtain the momentum in the laboratory system

initially at rest using the mass in energy units $mc^2 \rightarrow M$ is;

$$E_D \,=\, (E_A \,+\, M_B) \,-\, E_c$$

$$\vec{p}_D \,=\, \vec{p}_A \,-\, \vec{p}_C$$

Use $E^2 \,=\, (Pc)^2 \,+\, (m_0 c^2)^2$ and solve for the momentum of particle C

$$p_C \,=\, \frac{\alpha\beta \,\pm\, \sqrt{\alpha^2 \,-\, m_c^2(1-\beta^2)}}{1-\beta^2}$$

$$\beta \,=\, p_A\, cos(\theta)/(E_A + M_B)$$

$$\alpha \,=\, (M_A^2 + M_B^2 + M_C^2 - M_D^2 + 2E_A E_B)/2(E_A + M_B)$$

7 Compton Scattering

Photon (Gamma-Ray) scattering from atomic electrons is called Compton scattering, and Compton scattering has the largest cross section for low energy photon interactions with materials. It was an important experimental verification of the quantum theory of electromagnetic radiation. The equations for scattering of a photon of energy, E, from a free electron at rest, using both energy and momentum conservation are obtained below. This is the process, $\gamma \,+\, e \,\rightarrow\, \gamma' \,+\, e'$.

Conservation of Energy

$$E_\gamma \,+\, m_e \,=\, E'_\gamma \,+\, E'_e$$

Conservation of Momentum

$$\vec{p}_\gamma = \vec{p}'_\gamma + \vec{p}'_e$$

Remove the dependence on the scattered electron from the above equations. Therefore;

$$p_e'^2 = p_\gamma^2 + p_\gamma'^2 - 2p_\gamma p'_\gamma\, cos(\theta)$$

$$E_e'^2 = [E_\gamma + m_e]^2 - 2E'_\gamma[E_\gamma + m_e] + E_\gamma'^2$$

The photon has no mass so, $E_\gamma = p_\gamma$. Note that in the above equations, $c = 1$ so that all units are in energy. Use $E_e^2 - p_e^2 = m_e^2$ and combine the above equations into;

$$2p_\gamma - 2p'_\gamma p_\gamma + 2p'_\gamma p_\gamma\, cos(\theta) - 2p'_\gamma m_e = 0$$

$$\frac{2m_e}{p'_\gamma} - \frac{2m_e}{p_\gamma} = 2[1 - cos(\theta)]$$

Because the photon is quantized, $\lambda = h/p$, where λ is the wavelength, h is Planck's constant, and p is the photon momentum. Substitution gives the Compton equation;

$$\lambda' - \lambda = \frac{h}{2m_e}[1 - cos(\theta)]$$

8 The Proca Equation

Previously it was found that the 4-vector potential of the electromagnetic field is $(V/c, \vec{A})$. From the 4-vector energy-momentum vector, a Field energy-momentum relation could be written;

$$(V/c)^2 - |\vec{A}|^2 = (m_0c^2)^2$$

by comparing to the particle energy momentum relation;

$$E^2 - |\vec{p}c|^2 = (m_0c^2)^2$$

The above represents the connection between energy and momentum of the electromagnetic field. The Lorentz invariant expression for the above equation can be written;

$$(\partial_\alpha\partial^\alpha + \mu^2)A^\lambda = j^\lambda$$

Solutions to this equation satisfy the homogeneous equations ;

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

The mass has units of inverse length since the unit of the scalar potental is energy per charge per length. This is observed in the solution for the scalar potential (time component of the above equation). When the charge is at rest, the solution takes the form;

$$V = (Q/r)e^{-\mu r}$$

Thus the exponential decrease in the potential as a function of distance depends on the field (in this case the photon) mass. However, if the mass of the electromagnetic field Vanishes, the potential decreases as r^{-1} as observed. Because the mass of the field vanishes, Gauss's law is valid. Or perhaps because Gausss law is observed to be correct, the photon must be massless.