# Examples

## 1 GPS System

The global positioning system was established in 1973, and has been updated almost yearly. The GPS calcualtes postion on the earth'ssurface by accurately measuring timing signals sent by GPS satelites. This information includes the time of the message, the precise orbital information, and general information on all GPS satelites. A receiver uses this information to determine transit time, and then computes the distances to each satelite. Given the satelite position, positions on hte earth's surface are determined by triangulation. Three satelites are sufficient, but because of clock error, four or more satelites are used, see Figure 1.

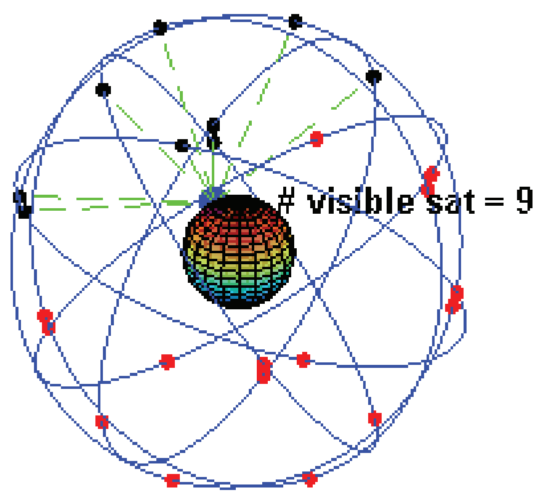



Figure 1: The figure shows the geometry of the GPS system

Because of the large velocity of light, $c$, clock error is amplified when calculating distances. One looks for intersections of spherical surfaces. Two surfaces intersect in a circular ring. The third surface intersects with this circle in no, one, or 2 points. One choses the point closest to the satelite in the case of 2 point intersection. The reciever has 4 unknowns, its position in 3D and a time shift, $b$. Thus

$$(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 = ([t_r + b - t_i]c)^2$$

$$i = 1, 2, 3, 4$$

The above equations are solved numerically. The atomic clocks on the satelites are synchronized to a universal time and is periodically corrected. In addition to correcting the clocks for relativistic effects ( even general relativity is required ), there are phase shifts, and propagation delays through the ionisphere.

# 2 Vector and Scalar Potentials

We used in the last lecture that;

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

giving $\vec{\nabla} \cdot \vec{B} = 0$ and from Faraday's law, $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$;

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

Now substitute these back into the other two Maxwell's equations.

$$\vec{\nabla} \cdot \vec{E} = -\nabla^2 V - \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{A} = \rho/\epsilon$$

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

$$\nabla^2 \vec{A} - \epsilon\mu \frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}(\vec{\nabla} \cdot \vec{A} + \epsilon\mu \frac{\partial V}{\partial t}) = -\mu \vec{J}$$

Thus we have reduced the number of Maxwell's equations from 4 to 2, but they are still coupled. At this point we assume that we can choose $\vec{\nabla} \cdot \vec{A} + \epsilon\mu \frac{\partial V}{\partial t} = 0$. This will be justified later. When this is done, separation of the above equations yields;

$$\nabla^2 V - \epsilon\mu \frac{\partial^2 V}{\partial t^2} = -\rho/\epsilon$$

$$\nabla^2 \vec{A} - \epsilon\mu \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}$$

The above equations are the inhomogeneous wave equation with sources $\rho$ and $\vec{J}$. We will see in the next section that these equations are consistent with the Lorentz transformation as outlined in the last lecture.

# 3 Invarience Examples

In the last lecture we noted that the 4-gradient operation $\partial_\alpha \to (\frac{\partial}{\partial x_0}, \vec{\nabla})$ forms a covarient 4-vector. Then using the metric defined in the last lecture, the contraction of this operator with itself is ;

$$\partial_\alpha \partial^\alpha = \partial_\alpha g_{\alpha\beta} \partial^\beta = \frac{\partial^2}{\partial x_0^2} - \nabla^2$$

This contraction yields a Lorentz invarient, the d'Alembertian, which is a Lorentz scalar. The d'Alembertian is defined using the operator, $\Box$, and applying normal complex algebra.

$$\Box = (\frac{\partial}{i\partial x_0}, \vec{\nabla})$$
$$\text{d'Alembertian} = \Box \cdot \Box = \Box^2$$

Thus the wave equation operator is a Lorentz scalar. Operation on a 3-vector function such as $\vec{A}$ must also produce also a 3-vector. From the above equations, this 3-vector is $\vec{J}$. You should now be able to show that the operaton on the scalar potential, $V$ (a form proportional to the time component of a 4 vector) must be the charge density. Contracting the 4-gradient operator with the 4-vector potential, $(V/c, \vec{A})$ gives;

$$\partial_\alpha A^\alpha = \frac{\partial V/c}{\partial x_0} + \vec{\nabla} \cdot \vec{A} = constant$$

The above is the equation which was set to zero in the last section in order to separate Maxwell's equations. We now see that this form must be a Lorentz constant. If it is zero in any frame is must be zero in all frames (the Lorentz condition). We will find that we can choose a frame in which the form vanishes so that it is zero in all frames. As another example, contract the 4-gradient with the current density 4-vector, $(c\rho, \vec{J})$. This gives;

$$\partial_\alpha j^\alpha = \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J}$$

As was noted previously, this is the equation of continuity, representing conservation of charge. In general, this operation is really a 4-divergence and a divergence operation which vanishes leads to a conservation equation. **Why??**

We found that the energy and momentum form a 4-vector $p^i = (E/c, \vec{p})$. Contract this 4-vector with itself.

$$p_i p^i = p^j g_{ji} p^i = E^2 - (pc)^2$$

This forms a Lorentz scalar which must be the same in all inertial frames. Choose the rest frame where $\vec{p} = 0$ in order to evaluate this constant. In this frame, $E = m_0 c^2$, the energy of the rest mass.

We also found the 4-velocity vector was defined by $U \to (\gamma c, \gamma \vec{u})$. Contraction of this 4-vector with itself gives;

$$U^i U_i = \gamma^2 c^2 - \gamma^2 u^2 = c^2$$

Thus

$$E^2 \; - \; (pc)^2 \; = \; (m_o c^2)^2$$

Finally consider the contraction of the field tensor with its dual. These tensors are given in the last lecture.

$$\mathcal{F}^{\alpha\beta} \; = \; \begin{array}{cc} \begin{array}{cccc} \phantom{E_x/c} & \beta & \rightarrow & \phantom{-E_z/c} \end{array} & \\ \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} & \begin{array}{c} \alpha \\ | \\ \mathrm{v} \end{array} \end{array}$$

$$\mathcal{G}_{\beta\alpha} \; = \; \begin{array}{cc} \begin{array}{cccc} \phantom{-B_x} & \alpha & \rightarrow & \phantom{-E_y/c} \end{array} & \\ \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z/c & -E_y/c \\ -B_y & -E_z/c & 0 & E_x/c \\ -B_z & E_y/c & -E_x/c & 0 \end{pmatrix} & \begin{array}{c} \beta \\ | \\ \mathrm{v} \end{array} \end{array}$$

$$\mathcal{F}^{\alpha\beta}\mathcal{G}_{\alpha\beta} \; = \; \mathcal{F}^{\alpha\beta} g_{\alpha\delta} g_{\beta\gamma} \mathcal{G}^{\delta\gamma}$$

The multiplication of these gives a Lorentz scalar;

$$\mathcal{F}^{\alpha\beta}\mathcal{G}_{\alpha\beta} \; = \; 4(\vec{E}/c) \cdot \vec{B}$$

# 4 Other Examples

## 4.1 Visual View of a moving rigid board

A rigid board as measured in its own rest frame has sides parallel to the z and the y axes. It moves with a uniform velocity, v, with respect to the z axis. What is the shape and orientation of the board with respect to an observer at the origin of the coordinate system, looking along the x axis, Figure 2?

Every point is seen at the same time, $\tau$, at the position, $y_0$, on the y axis. The light is emitted at some earlier time and travels the distance, d.

$$d = \sqrt{x_0^2 + y_0^2 + z_0^2}$$

When $x_0 = y_0 = 0$ then the length of the board is $L/\gamma$ where $\gamma = [1 - \beta^2]^{-1/2}$. The time of observation of the corner is $\tau$, which is the proper time from that position. Thus the time

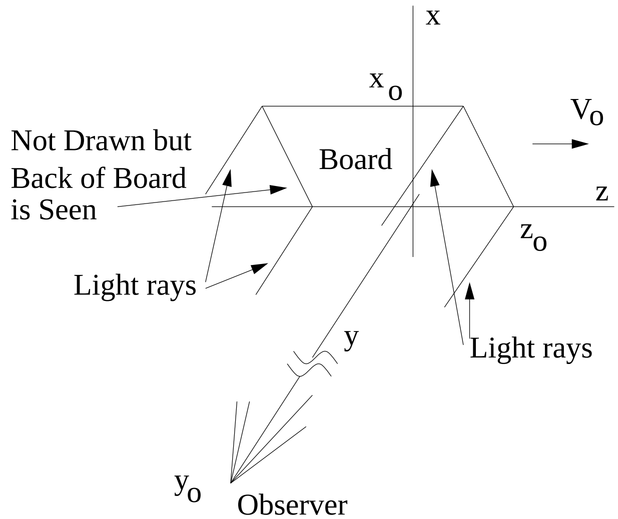



Figure 2: The figure shows the moving board in the (x,z) plane. Light rays come from each corner and arrive at the same time at the observer a long distance away

for a ray to travel from $(0, 0, z_0)$ to $(0, y_0, 0)$ is;

$$\tau_0 = d/c = [y_0^2 + z_0^2]^{1/2}/c$$

and during that same time a light ray travels from $(x_0, 0, z_0)$ to $(0, y_0, 0)$ ;

$$\tau_0 = [x_0^2 + y_0^2 + (z_0 - v\tau_0)^2]^{1/2}.$$

Solve these two equations for $\beta c\tau$

$$\delta z = \beta c\tau = z_0 - \sqrt{z_0^2 - x_0^2}.$$

Thus the corner at position $(x_0, 0, z_0)$ lags behind the corner $(0, 0, z_0)$ by a distance $\delta z$.

## 4.2 Field Transformation

Show that if the electric field vanishes in a given Lorentz frame then any charged particle obeying the Lorentz force law has constant speed (not velocity) in that frame.
The Lorentz force is:

$$mc\frac{du^\alpha}{d\tau} = qF^{\alpha\beta}u_\beta.$$

where u is the 4-velocity. The spatial and time components are;

$$\frac{d(\gamma mc\vec{v})}{dt} = q[\vec{E} + \vec{v}/c \times \vec{B}]. \text{ (Force)}$$

$$\frac{d(\gamma mc^2)}{dt} = \vec{F} \bullet \vec{v} = q\vec{E} \bullet \vec{v}. \text{ (power)}$$

when $\vec{E} = 0$ then;

$$\frac{d(\gamma mc^2)}{dt} = 0.$$

Thus $\gamma mc^2$ = constant.

## 4.3 Another field transformation

You are given that $\vec{E} \cdot \vec{B}$ is a Lorentz constant. This can be shown by observing that the contraction $\mathcal{F}^{\alpha\beta}\mathcal{G}_{\alpha\beta}$ is a scalar, where $\mathcal{F}^{\alpha\beta}$ is the field strength tensor and $\mathcal{G}_{\alpha\beta}$ is its dual. Now suppose $\vec{E}' = 0$ in a reference frame and then boost this in the 1 direction to another frame. In that frame;

$$E_1 = E_1' = 0$$

$$E_2 = \gamma(E_2' + \beta B_3') = \gamma\beta B_3'$$

$$E_3 = \gamma(E_3' - \beta B_2') = -\gamma\beta B_2'$$

$$B_1 = B_1' = 0$$

$$B_2 = \gamma(B_2' - \beta E_3')$$

$$B_3 = \gamma(B_3' + \beta E_2')$$

Any $\vec{B}'$ that generates the $\vec{E}$ and $\vec{B}$ above satisfies the requirement that $\vec{E} \cdot \vec{B} = 0$. However, for an explicit solution choose $B_2' = 0$ so that;

$$E_1 = 0$$

$$E_2 = \gamma\beta B_3'$$

$$E_3 = 0$$

Then;

$$B_1 = B_1' = 0$$

$$B_2 = \gamma(B_2' - \beta E_3') = 0$$

$$B_3 = \gamma(B_3' + \beta E_2') = \gamma B_3'$$

Then $E_3 = 0$, $B_3 \neq 0$ but $\vec{E} \cdot \vec{B} = 0$

## 4.4 Energy and Momentum Conservation

We have found that in any frame a system of particles has (units where $c = 1$ so that momentum and mass are measured in energy units);

$$[\sum_i E_i]^2 - \sum_i [(\vec{p}_i)]^2 = M^2 = E_{CM}^2 \text{ constant}$$

For 2 particles with one at rest;

$$E_{CM} = [m_1^2 + m_2^2 + 2\, E_1 m_2]^{1/2}$$

Also from this we can find the velocity of the CM system with respect to this reference frame, see Figure 3;

$$\vec{\beta}_{CM} = \frac{\sum_i \vec{p}_i}{E_{cm}}$$

$$\gamma_{cm} = \frac{\sum_i E_i}{E_{cm}}$$

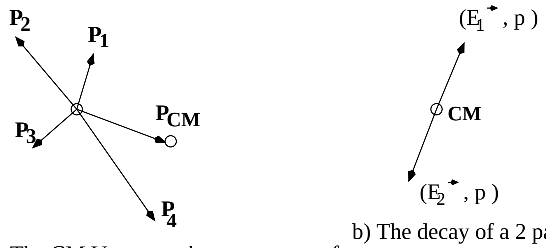
The energy of a particle in the CM frame, assuming 2-particle decay, Figure 3;

$$E_{1\,CM} = \frac{M^2 + (m_1^2 - m_2^2)}{2M}$$

$$M = E_{1\,CM} + E_{2\,CM}$$

Find the maximum energy that can be transferred in an electron-electron collision using realtivistic kinematics.

Energy and momentum conservation give;




a) The CM Vector as the vector sum of the momentum vectors of a system

b) The decay of a 2 particl

Figure 3: Figure 2a shows the CM vector as the sum of the individual momentum vectors of a many body system. Figure 2b shows the decay of a 2 particle system in the CM

$$E_1 + E_2 = E'_1 + E'_2 = E_0$$

$$\vec{p_1} + \vec{p_2} = \vec{p'_1} + \vec{p'_2} = \vec{p_0}$$

Remove the dependence on $(\vec{p'_2}, E'_2)$

$$E'^2_2 - p'^2_2 = m^2 = (E_0^2 - p_0^2) + m^2 - 2E_0 E'_1 - 2\vec{p_0} \bullet \vec{p'_1}$$

Let $a = \frac{m_0^2}{2E_0}; b = \frac{\vec{p_0}}{E_0} = \vec{\beta_{cm}}$

$$[1 - b^2 cos^2(\theta_1)]p'^2_1 + 2ab\, cos(\theta_1)\, p'_1 - (a^2 - m^2) = 0$$

The solution is;

$$p'_1 = \frac{-ab\, cos(\theta_1) \pm \sqrt{a^2 - m^2[1 - b^2\, cos^2(\theta_1)]}}{[1 - b^2 cos^2(\theta_1)]}$$

For Maximum energy transfer $\theta_1 = \pi$ and in the ultra-relativistic case $(m/a)^2 \approx 0$

$$p'_1 \approx \frac{a(1+b)}{(1-b)(1+b)} = \frac{a}{1-b} \approx a$$

The energy transfer is;

$$\omega = E_1 - E'_1 \approx E_1 - \frac{E_0^2 - p_0^2}{2E_0} = E_1 \left[1 - \frac{E_2}{E_0}(1 - cos(\theta_{12}))\right]$$

# 5 Mandelstam Variables

Cross sections measured at relativisitc energies are sometimes written in terms of the relativistic Maldestam invarients defined by ;

$$s = (p_1 + p_2)^2 = (m_1^2 + 2E_1E_2 - 2\vec{p}_1 \cdot \vec{p}_2 + m_2^2)$$

$$t = (p_1 - p_3)^2$$

$$u = (p_1 - p_4)^2$$

In the above equations $p_i$ is the energy/momentum 4-vector $(E, pc)$ with $c = 1$. Thus $p^2 = E^2 - |\vec{P}|^2 = m^2$. The interaction diagram of the particles is given in Figure 4. In an interpretation that particles 1 and 2 are incident and 3 and 4 are reaction products, $s$ is the relativistic energy in the CM system, $t$ represennts the 4-momentum transfer in the reaction, and $u$ is a crossing channel which does not have an obvious interpretation. The parameters are not independent.

$$s + t + u = 2[\mu_{in-CM}^2 + \mu_{out_CM}^2]$$

In the above $\mu$ represents the CM mass (Energy) for the in and out chanels respectively.

# 6 Kinematics

As observed in the last section, the scattering momentum in a collision is obtained from the solution of a quadratic equation, which can be solved if the scattering angle is known. This solution can have 2 real solutions for selected values of incident masses and incident momenta. This is most easily seen in the figure 5 from the perspective of the Center of Momentum. In non-relativisitc kinematics one often discusses a collision in terms of the Center of Mass. However this is ill defined in relativity, and the use of the Center of Momentum, CM, is most appropriate. In the CM frame, the sum of the incident momenta vanishes, and since momentum conserved, the momentum will also vanish after the collision. Then a boost to the momentum in the CM frame is illustrated by the vector additions shown in the figure. However be careful here. The additions actually must be done using relativistic relations.

When the Lorentz transformation is applied to transform the momentum and energy between two Lorentz frames, one obtains in polar coordinates;

$$p\, cos(\theta) = \gamma(p'\, cos(\theta') + \beta E')$$

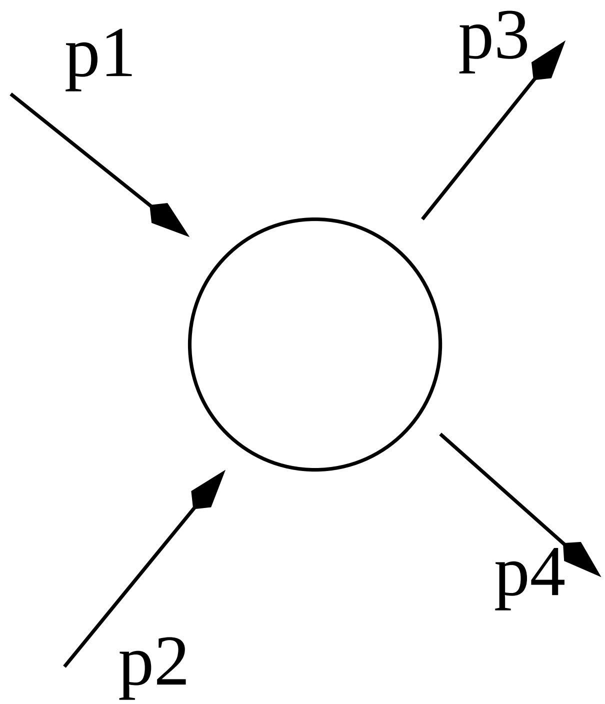



Figure 4: A diagram showing the incident and reaction channels in a collision at relativistivc energ terms of the Maldestam variables.

$$p\,sin(\theta) \;=\; p'\,sin(\theta')$$

$$E \;=\; \gamma(E' \,+\, \beta p'\,cos(\theta'))$$

$$\phi \;=\; \phi'$$

In general,

$$p_1 \;=\; p\,cos(\theta)$$

$$p_2 \;=\; p\,sin(\theta)\,cos(\phi)$$

$$p_3 \;=\; p\,sin(\theta)\,sin(\phi)$$

with $\beta$, $\gamma$ the CM velocity and $\gamma$ factor.

The conservation of energy and momentum for a collision $A + B \,\rightarrow\, C + D$, assuming $B$ is initally at rest using the mass in energy units $mc^2 \rightarrow M$ is;

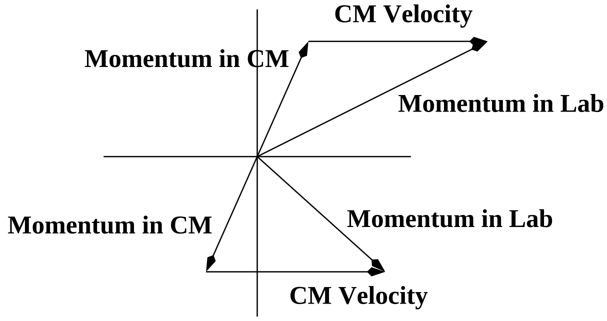



Figure 5: The figure shows how the addition of the CM momentum is added to the momenta in the CM system to obtain the momentum in the laboratory system

$$E_D = (E_A + M_B) - E_c$$

$$\vec{p}_D = \vec{p}_A - \vec{p}_C$$

Use $E^2 = (Pc)^2 + (m_0 c^2)^2$ and solve for the momentum of particle $C$

$$p_C = \frac{\alpha\beta \pm \sqrt{\alpha^2 - m_c^2(1-\beta^2)}}{1-\beta^2}$$

$$\beta = p_A\, cos(\theta)/(E_A + M_B)$$

$$\alpha = (M_A^2 + M_B^2 + M_C^2 - M_D^2 + 2E_A E_B)/2(E_A + M_B)$$

# 7 Compton Scattering

X-Ray scattering from atomic electrons is called Compton scattering. Compton scattering has the largest scattering cross section for low energy photons interacting with materials. It was an important experimental verification of the quantum theory of electromagnetic radiation. We now derive the equations for scattering of a photon of energy, $E$, from a free electron at rest, using both energy and momentum conservation. This is the process, $\gamma + e \rightarrow \gamma' + e'$ .

Conservation of Energy

$$E_\gamma + m_e = E'_\gamma + E'_e$$

Conservation of Momentum

$$\vec{p_\gamma} = \vec{p'_\gamma} + \vec{p'_e}$$

Remove the dependence on the scattered electron from the above equations. Therefore;

$$p_e'^2 = p_\gamma^2 + p_\gamma'^2 - 2p_\gamma p'_\gamma \, cos(\theta)$$

$$E_e'^2 = [E_\gamma + m_e]^2 - 2E'_\gamma[E_\gamma + m_e] + E_\gamma'^2$$

The photon has no mass so, $E_\gamma = p_\gamma$. Note that in the above equations we have taken $c = 1$ so that all units are in energy. Use $E_e^2 - p_e^2 = m_e^2$ and combine the above equations into;

$$2p_\gamma - 2p'_\gamma p_\gamma + 2p'_\gamma p_\gamma \, cos(\theta) - 2p'_\gamma m_e = 0$$

$$\frac{2m_e}{p'_\gamma} - \frac{2m_e}{p_\gamma} = 2[1 - cos(\theta)]$$

Because the photon is quantized, $\lambda = h/p$, where $\lambda$ is the wavelength, $h$ is Planck's constant, and $p$ is the photon momentum. Substitution gives the Compton equation;

$$\lambda' - \lambda = \frac{h}{2m_e}[1 - cos(\theta)]$$

# 8 The Proca Equation

Previously we found that the 4-vector potential of the electromagnetic field is $(V/c, \vec{A})$. From the 4-vector energy-momentum vector we anticipate that a Field energy-momentum relation could be written;

$$(V/c)^2 - |\vec{A}|^2 = (m_0 c^2)^2$$

in anaology to the particle energy momentum relation;

$$E^2 - |\vec{p}c|^2 = (m_0 c^2)^2$$

The above would represent an energy momentum relation for the electromagnetic field. In analogy to an expression for a particle of mass, $\mu$, in units of length, we have an differential equation that takes the form;

$$(\partial_\alpha \partial^\alpha + \mu^2)A^\lambda = j^\lambda$$

Solutions to this equation satisfy the homogeneous equations ;

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

The mass has units of inverse length since the unit of the scalar potential is energy per charge per length. We see this whne observing the solution for the scalar potential (time component of the above equation). When the charge is at rest, the solution takes the form;

$$V = (Q/r)e^{-\mu r}$$

Thus the potential decreases exponentially with distance depending on the field mass. Thus only if the mass of the electromagnetic field vanishes will the potential decrease inversely as r. Because the mass of the field vanishes, Gauss's law is valid.

# 9 Rapidity

Rapidity is a measure of relativistic velocities. Note that for low velocities speeds essentially add, but at high velocity relativistic addition is non-linear. However, using rapidity as a measure of speed, rapidities add for parallel velocities. The rapidity, $\alpha$, of an object is defined by;

$$\alpha = artanh(\beta) \text{ with } \beta = v/c$$

Write a Lorentz boost in the $x_1$ direction in terms of a rotation in Minkowski space.

$$\begin{pmatrix} x'_0 \\ x'_1 \end{pmatrix} = \begin{pmatrix} \cosh(\alpha) & -\sinh(\alpha) \\ -\sinh(\alpha) & \cosh(\alpha) \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

Now choose two successive transformations defined by $\alpha_1$ and $\alpha_2$ in the same direction. Thus apply the lorentz transformations as;

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \cosh(\alpha_2) & -\sinh(\alpha_2) \\ -\sinh(\alpha_2) & \cosh(\alpha_2) \end{pmatrix} \begin{pmatrix} \cosh(\alpha_1) & -\sinh(\alpha_1) \\ -\sinh(\alpha_1) & \cosh(\alpha_1) \end{pmatrix}$$

Use the exponential forms;

$$\cosh(\alpha) = [e^{\alpha} + e^{-\alpha}]/2$$

$$\sinh(\alpha) = [e^{\alpha} - e^{-\alpha}]/2$$

to show that;

$$\left(\begin{array}{cc} A & B \\ C & D \end{array}\right) = \left(\begin{array}{cc} \cosh(\alpha_2+\alpha_1) & -\sinh(\alpha_2+\alpha_1) \\ -\sinh(\alpha_2+\alpha_1) & \cosh(\alpha_2+\alpha_1) \end{array}\right).$$

Therefore rapidities add for boosts in the same direction. Note that we previously identified $\cosh(\alpha) \,=\, \gamma \,=\, \dfrac{1}{\sqrt{1-\beta^2}}$

Rapidity is used in relativistic collisions of particles, particularly heavy ion collisions. The energy and momentum of such a particle in units with $c = 1$ is ;

$$E \,=\, m\cosh(\alpha)$$

$$|\vec{p}| \,=\, m\sinh(\alpha)$$

The rapidity is then;

$$\alpha \,=\, \text{artanh}(\frac{|\vec{p}|}{E}) \,=\, ln[\frac{E+p_1}{E-p_1}]$$