Energy, Momentum, and Symmetries

1 Fields

The interaction of charges was described through the concept of a field. A field connects charge to a geometry of space-time. Thus charge modifies surrounding space such that this space affects another charge. While the abstraction of an interaction to include an intermediate step seems irrelevant to those approaching the subject from a Newtonian viewpoint, it is a way to include relativity in the interaction. The Newtonian concept of action-at-a-distance is not consistent with standard relativity theory.

Most think in terms of "force", and in particular, force acting directly between objects. You should now think in terms of fields, *i.e.* a modification to space-time geometry which then affects a particle positioned at that geometric point. In fact, it is not the fields themselves, but the potentials (which are more closely aligned with energy) which will be fundamental to the dynamics of interactions.

2 Complex notation

Note that a description of the EM interaction involves the time dependent Maxwell equations. The time dependence can be removed by essentially employing a Fourier transform. Remember in an earlier lecture, the Fourier transformation was introduced.

$$\mathcal{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, F(t)\, e^{-i\omega t}$$

$$F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, \mathcal{F}(\omega)\, e^{i\omega t}$$

Now assume that all functions have the form;

$$F(\vec{x}, t) \rightarrow F(\vec{x}) e^{i\omega t}$$

These solutions are valid for a particular frequency, ω. One can superimpose solutions of different frequencies through a weighted, inverse Fourier transform to get the time dependence of any function. Also note that this assumption results in the introduction of complex functions to describe measurable quantities which in fact must be real. As an example, look at Faraday's law.

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -i\omega \vec{B}(\vec{x})$$

In the above, the fields are complex which was introduced since we used the form, $e^{i\omega t}$, instead of a real harmonic form, $cos(\omega t)$. Thus to get a real result, we then choose to take the real part of the function, *ie* ;

$$Re[Ee^{i\omega t}] \,=\, E\,cos(\omega t)$$

For most calculations of energy and power, we need the time average, recalling that frequencies in most cases of interest are large, and that the instantaneous power is not as important as the time averaged value. Energy and power are obtained by squaring the fields or multiplying the field by a current. Therefore when multiplying the real component of two complex amplitudes;

$$Re[Ae^{i\omega t}] \,=\, 1/2[Ae^{-i\omega t} + A^{*}e^{i\omega t}]$$

$$Re[Be^{i\omega t}] \,=\, 1/2[Be^{-i\omega t} + B^{*}e^{i\omega t}]$$

$$\begin{aligned} &Re[A]Re[B] \,= \\ &\quad 1/4[ABe^{-i2\omega t} + A^{*}B + AB^{*} + A^{*}B^{*}e^{i\omega t}] \,= \\ &\quad 1/2\,Re[A^{*}B + ABe^{i2\omega t}] \end{aligned}$$

In the above, A^{*} is the complex conjugate of A. Applying a time average, the complex harmonic term, $e^{i2\omega t}$, vanishes. Thus;

$$\langle Re[A]Re[B]\rangle \,=\, (1/2)A^{*}(\vec{x})B(\vec{x})$$

Note A and B now depend only on spatial coordinates. (*ie* they are time independent)

3 Poynting vector

Recall that power input or loss of a charge is found from $P \,=\, \vec{F}\cdot\vec{V}$ where $\vec{F}$ is the force and $\vec{V}$ the velocity. Use the current density, $\vec{J} \,=\, \rho\vec{V}$ with ρ the charge density, to write the power which is expended in moving a charge by an electric field, $\vec{E}$.

$$P \,=\, \int\, d\tau\,\vec{E}\cdot\vec{J}$$

Insert from Ampere's Law the Maxwell equation;

$$\vec{J} \,=\, \vec{\nabla}\,\times\,\vec{H} - \frac{\partial\vec{D}}{\partial t}$$

$$P = \int d\tau \, [\vec{E} \cdot (\vec{\nabla} \times \vec{H} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t})]$$

Then use;

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{H})$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

To obtain;

$$P = -\oint (\vec{E} \times \vec{H}) \cdot d\vec{a} - \int d\tau \, [\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}]$$

The energy density is;

$$\mathcal{W} = (1/2)(\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H})$$

so that;

$$-P = \int d\tau \, (\frac{\partial \mathcal{W}}{\partial t}) + \oint (\vec{E} \times \vec{H}) \cdot d\vec{a}$$

The above equation is interpreted as showing that the power expended by a moving charge using fields, $P = \int d\tau \, (\vec{E} \cdot \vec{J})$, is obtained by decreasing the energy in a volume of the field and the outward flow of energy through a surface surrounding this volume. Then define the Poynting vector ;

$$\vec{S} = \vec{E} \times \vec{H}$$

This is the energy flow through the surface of the volume. For a simple example, consider Figure 1 which represents the current flowing through a cylindrical resistor, R, of length L. The current, I, produces a magnetic field at the surface of the resistor, radius a. It has a value obtained from Ampere's law;

$$\oint \vec{B} \cdot \vec{dl} = B(2\pi a) = \mu \, I$$

The $\vec{B}$ field points along the cylindrical surface in a direction given by the right hand rule. The $\vec{E}$ field acts in the axial direction having a value of V/L. The power loss from the circuit equations is VI. Now look at the Poynting vector;

$$\vec{P} = (1/\mu)\vec{E} \times \vec{B} = -(1/\mu)(V/L)(\frac{\mu I}{2\pi a}) \, \hat{r} = -\frac{VI}{2\pi a L} \, \hat{r}$$

The Poynting vector P gives the power flowing into the resistor per unit surface area from

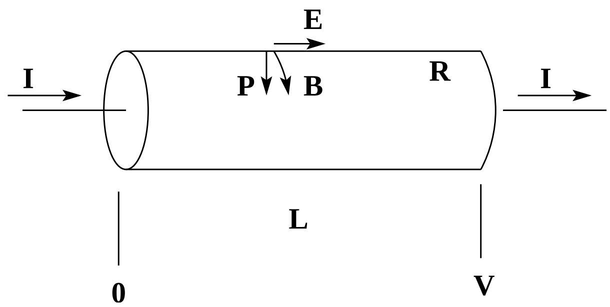

Figure 1: Evaluation geometry for the Poynting vector at the surface of a resistor

the fields. This connects the functions of these circuit components with operations on the fields, and field energies.

Recall that a vanishing divergence equation can be related to a conservation principle. Define a 4-vector ($S^\alpha = \mathcal{W}, \vec{S}$), as a relativistic Poynting 4-vector. If $\vec{J} \cdot \vec{E} = 0$, a covarient conservation equation is obtained;

$$\frac{\partial S^\alpha}{\partial x^\alpha} = 0$$

4 Momentum flow

In a similar way to the power flow, use Maxwell's equations and the Lorentz force on an element of charge, dq,to develop the time change of momentum. The differential Lorentz force is written;

$$d\vec{F} = d\,q[\vec{E} + (\vec{v} \times \vec{B})]$$

Upon writing this for a volume charge density;

$$\frac{d\vec{P}_M}{dt} = \int [\rho\vec{E} + \vec{J} \times \vec{B}]\, d^3x$$

In the above expression, $\vec{P}_M$ is the mechanical momentum due to the moving charge. Substitute

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon \text{ and } \vec{\nabla} \times \vec{H} = \vec{J} + (1/c^2\mu)\,\frac{\partial\vec{E}}{\partial t}$$

and use the remaining Maxwell equations to write the equation;

$$\frac{d\vec{P}_M}{dt} + \frac{d}{dt}\int [\epsilon\,\vec{E}\times\vec{B}]\,d^3x \;=$$
$$\epsilon\int [\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\vec{\nabla}\,\times\,\vec{E}) +$$
$$c^2\vec{B}(\vec{\nabla}\cdot\vec{B}) - c^2(\vec{B}\times(\vec{\nabla}\,\times\,\vec{B}))]\,d^3x$$

Now note that $\vec{P}_{field}$ is identified by the momentum per unit volume in the field;

$$\vec{P}_{field} \;=\; (1/c^2)\int (\vec{E}\times\vec{H})\,d^3x$$

Described in words, the above equation expresses that the momentum per unit volume equals energy flowing through the surface area divided by its velocity, c and the volume of the momentum, $(ct)Area$, (Energy/c)/[(ct)Area]. The energy flow per unit time (power) through an area is given by the Poynting vector, $\vec{S} \;=\; \vec{E}\times\vec{H}$. Note that this power flows with a velocity, c, and the Energy/c equals the momentum of the electromagnetic wave. Divide the remaining term into its vector components. In the 1 direction we find;

$$[\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\vec{\nabla}\,\times\,\vec{E})]_1 \;=\; \sum_i \frac{\partial}{\partial x^i}\,[E_1E_i \,-\, (1/2)\vec{E}\cdot\vec{E}\delta_{i1}]$$

The right side of the above equation is the divergence of rank 2 tensors. Collecting all components for the $\vec{E}$ as well as the $\vec{B}$ field, this tensor has the form

$$\mathcal{T}_{\alpha\beta} \;=\; \epsilon[E_\alpha E_\beta + c^2B_\alpha B_\beta - (1/2)(\vec{E}\cdot\vec{E} + c^2\vec{B}\cdot\vec{B})]$$

The tensor, $\mathcal{T}$, is called the Maxwell stress tensor. In integral form of the above equation is;

$$\frac{d}{dt}[\vec{P}_M \,+\, \vec{P}_{Field}]_\alpha \;=\; \oint \mathcal{T}_{\alpha\beta}\,da_\beta$$

The integral over $d\,a_\beta$ is an integral over the area surrounding the volume containing the field momentum. Now identify $\mathcal{T}_{\alpha\alpha}$ as the pressure $(force/area)$ on the surface α . This has units of energy density (Force/area). The tensor element $\mathcal{T}_{\alpha\beta}$ when $\alpha\neq\beta$, is the momentum density times the velocity, or a shearing force in the area element α. In differential form the above equation is;

$$\vec{\nabla}\cdot\mathcal{T} \,+\, (\rho\vec{E} + \vec{J}\times\vec{B}) \;=\; -\frac{\partial\vec{G}}{\partial t}$$

In this equation, $\vec{G} \;=\; (1/c^2)\vec{E}\times\vec{H}$ is the momentum density, and the equation represents momentum conservation. Thus the momentum density through a surface plus mechanical momentum density of the charge currents equals the time change of the momentum density in the enclosed volume.

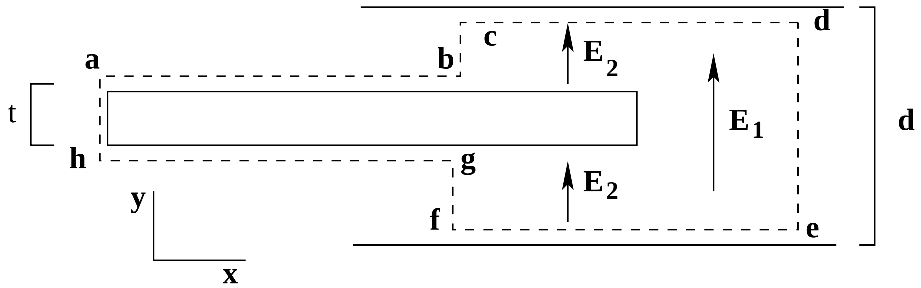

Figure 2: The surfaces on which we will evaluate the Maxwell stress tensor.

5 Examples

Look at the z component of the force applied by the fields in Cartesian coordinates.

$$\mathcal{T}_{zz} = (\epsilon/2)[E_z^2 - E_y^2 - E_x^2] + (\epsilon c^2/2)[B_z^2 - B_y^2 - B_x^2]$$

$$\mathcal{T}zy = \epsilon(E_z E_y + c^2 B_z B_y)$$

$$\mathcal{T}zx = \epsilon(E_z E_x + c^2 B_z B_x)$$

As an example, calculate the force on a conducting plate inserted in a capacitor. Simple analysis shows that the field of the capacitor includes charge on the conducting plate, and this causes a force on the plate pulling it into the capacitor. Consider Figure 2. Ignore dimensions perpendicular to the plane of the drawing and assume that the fields within the capacitor are uniform and perpendicular to the planar surfaces, *ie* the field of an infintite parallel plate capacitor.

The dashed line indicates a surface (include dimensions in and out of the drawing) over which the stress tensor is evaluated. There is only an E field to consider. Integration over ha is essentially zero as there is little field at this point. Integration over ab and gh cancel as does integration over cd and ef. Then integration over bc and fg are equal and only field components of E_y are non-zero.

$$E_1 = V/d$$

$$E_2 = V/(d-t)$$

Let the width of the conducting plate be w. Use the outward normal as the direction of the area vectors. The field tensor is then;

$$\mathcal{T} = \epsilon \begin{pmatrix} -(1/2)E_y^2 & 0 & 0 \\ 0 & (1/2)E_y^2 & 0 \\ 0 & 0 & -(1/2)E_y^2 \end{pmatrix}$$

The force on the plate is;

$$F_x = \int \mathcal{T}_{xx} dA_x$$

$$F_x = (\epsilon/2)\frac{V^2}{(d-t)^2}\, w(d-t) - (\epsilon/2)\frac{V^2}{d^2}\, wd$$

$$F_x = \frac{\epsilon V^2 wt}{2d(d-t)}$$

In this simple case the same result can be obtained from the field energy. Suppose the plate extends into the capacitor a distance x. Use the energy density in the fields to obtain;

$$W = (1/2)[\epsilon E^2 + (1/\mu)B^2] \times Volume$$

$$W = (1/2)\epsilon[\frac{V^2}{d^2}\, d(L-x)w + \frac{V^2}{(d-t)^2}\,(d-t)xw]$$

$$F_x = -\frac{dW}{dx} = (1/2)\epsilon[-\frac{V^2}{d}\, w + \frac{V^2}{(d-t)}\, w]$$

Which results in the same answer.

6 Another example 1

Now consider another, more complicated example. Find the radiation pressure on a spherical, metalic particle of radius, a. The particle is assumed to be perfectly conducting. First develop an expresion for the radiation pressure using an approximation. The Poynting vector divided by c^2 is the momentum density in the EM fields ($Momentum - c/(area - time - c^2)$. The force is the time change of the momentum density, $\vec{G}$, so that (assume a plane EM wave in the z direction);

$$\frac{dP_z}{dt} = F_z = 2c \int \vec{G}_z dA_z$$

In the above dA_z is the surface area of the spherical particle struck by the wave. The factor of 2 comes from the change of momentum between the incoming to outgoing wave. The integral extends over the lower hemisphere and we want the time average. For a plane wave in MKS units $B_0 = E_0/c$.

$$F_z = 2/c \int r^2\, d\,cos(\theta)\, d\phi\, cos(\theta)\, ((1/2)Re[\vec{E} \times \vec{H}^*]_z)$$

Then for the plane wave, $H_0 = B_0/\mu = E_0/c\mu = \epsilon\, c\, E_0$. Substitution and integration yields;

$$F_z \;=\; \pi a^2 \epsilon E_0^2$$

However, to really work this problem correctly one must use the stress tensor. This means the fields surrounding the spherical particle must be obtained. Therefore a solution to the problem of a plane wave scattered from a conducting sphere is required. Assume here that these fields are known, *i.e.* both the incident and the scattered wave at the spherical surface of the particle. The fields are to be expressed in spherical coordinates. The force on the particle is obtained from;

$$F_z \;=\; (1/2)[\int\, da_Z\, |\mathcal{T}_{zz}| \;+\; \int\, dA_x\, |\mathcal{T}_{zx}| \;+\; \int\, dA_y\, |\mathcal{T}_{zy}|]$$

The factor of (1/2) comes from the time average of the harmonic wave fields, and the time average stress tensor components $|\mathcal{T}_{\alpha\beta}|$ is used. Put this in spherical coordinates and average over the azmuthal angle. Note that these fields are the total field (incident plus scattered) at the surface $d\vec{\sigma}$. To obtain these field components, look at Figure 2. The direction cosines of the axes in spherical coordinates are;

$$l = sin(\theta)\, cos(\phi)$$

$$m = sin(\theta)\, sin(\phi)$$

$$n = cos(\theta)$$

and $l\, d\sigma \;=\; d\sigma_x$, etc. Then;

$$E_x \;=\; E_r\, sin(\theta)\, cos(\phi) \;+\; E_\theta\, cos(\theta)\, cos(\phi) \;-\; E_\phi\, sin(\phi)$$

$$E_y \;=\; E_r\, sin(\theta)\, sin(\phi) \;+\; E_\theta\, cos(\theta)\, sin(\phi) \;+\; E_\phi\, cos(\phi)$$

$$E_z \;=\; E_r\, cos(\theta) \;-\; E_\theta\, sin(\theta)$$

There is substantial algebra in order to collect terms when substituting the fields into the stress tensor, and averaging over the azmuthal angle. Here we will only need $\mathcal{T}_{zz}$ to continue using the approximation originally used. Thus, let $ct = cos(\theta)$ and $st = sin(\theta)$

$$\mathcal{T}_{zz} \;=\; (\epsilon/2)[E_r^2(ct^2 - st^2) + E_\theta^2(st^2 - ct^2) - E_\phi^2 - 4E_r E_\theta\, ct\, st]$$

Ignore the scattered wave and choose a plane wave incident along the z axis. For this case $\mathcal{T}_{zy} = \mathcal{T}_{zx} = 0$.

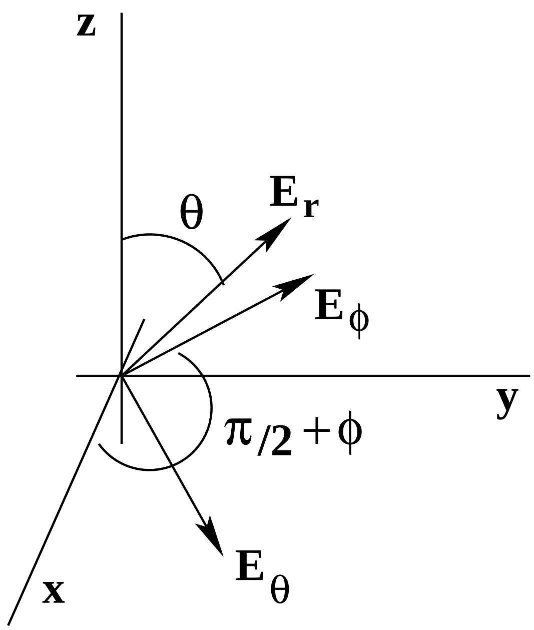

Figure 3: The surfaces on which we will evaluate the Maxwell stress tensor.

$$\mathcal{T}_{zz} = -(\epsilon/2)E_x^2 - (1/2\mu)B_y^2$$

Integration proceeds easily to obtain the previous result.

7 Another example 2

As another example, find the force on the Nothern hemisphere exerted by the Sothern hemisphere of a uniformly charged sphere with density $\rho = \frac{Q}{4\pi R^3/3}$. Here Q is the total charge and R is the radius. Use the stress tensor, $Tij = \epsilon(E_i E_j - 1/2\,\delta_{ij}E^2)$ where the B field component of the tensor is set to zero.Since spherical coordinates match the geometry, the field is written as;

$$\vec{E}(r) = E_x\hat{x} + E_y\hat{y} + E_z\hat{z} = \kappa(Q/R^2)\hat{r}$$

$$\vec{E}(r) = \kappa(Q/R^2)[sin(\theta)\,cos(\phi)\,\hat{x} + sin(\theta)\,sin(\phi)\,\hat{y} + cos(\theta)\,\hat{z}]$$

The net force must be in the $\hat{z}$ direction and is divided into 2 parts, 1) the hemispherical cap and 2) the disk along the equator. For both, one only needs;

For part 1

$$(T \cdot d\vec{a})_z|_{r=R} = T_{zx}da_x \; + \; T_{zy}da_y \; + T_{zz}da_z$$

Then $d\vec{a} \; = \; R^2\, sin(\theta)\, d\theta\, d\phi\, \hat{r}$

$$da_x \; = \; (R^2\, sin(\theta)\, d\theta\, d\phi)(sin(\theta)\, cos(\phi)\, \hat{x}$$

$$da_y \; = \; (R^2\, sin(\theta)\, d\theta\, d\phi)(sin(\theta)\, sin(\phi)\, \hat{y}$$

$$da_x \; = \; (R^2\, sin(\theta)\, d\theta\, d\phi)(cos(\theta)\, \hat{z}$$

Substitute these into the contraction of the stress tensor with the area vector.

$$(T \cdot d\vec{a})_z|_{r=R} = \epsilon\, (\frac{Q}{4\pi\epsilon R})^2 [sin^3(\theta)\, cos(\theta) cos^2(\phi)\, d\theta\, d\phi +$$

$$sin^3(\theta)\, cos(\theta) sin^2(\phi)\, d\theta\, d\phi +$$

$$(1/2)\, (cos^2(\theta) - sin^2(\theta))\, sin(\theta)\, cos(\theta)\, d\theta\, d\phi]$$

Collect terms and integrate over the top hemisphere. The result is;

$$F1_z \; = \; \kappa \frac{Q^2}{8R^2}$$

For part 2

Use the outward normal for the outward disk. This $d\vec{a} \; = \; da_z\, \hat{z}$. The field on this surface is;

$$\vec{E}_{disk} \; = \; \kappa(Q/R^3)\vec{r}|_{\theta=\pi/2}$$

$$\vec{E}_{disk} \; = \; \kappa(Q/R^3) r [cos(\phi)\hat{x} \; + \; sin(\phi)\hat{y}]$$

The force evaluated by the integral of $T_{zz}da_x$ over the disk with;

$$T_{zz} \; = \; 1/2[\epsilon(E_z^2 - E_y^2 - E_x^2)]$$

$$F2_z \; = \; \kappa \frac{Q^2}{16R^2}$$

The total force is the sum of the two components.

8 Magnetic pressure in a solenoid

Previously an expression for the magnetic field in the interior of a long solenoid using Ampere's law was developed. If there are N turns of wire per unit length each carring a current, I, the magnetic field inside the solenoid is constant, directed along the solenoidal axis, and equal to $\vec{B} = \mu_0 NI\,\hat{z}$. The stress tensor is diagonal and has the form;

$$\mathcal{T}_{zz} = \frac{1}{2\mu_0} B_z^2$$

$$\mathcal{T}_{yy} = -\frac{1}{2\mu_0} B_z^2$$

$$\mathcal{T}_{xx} = -\frac{1}{2\mu_0} B_z^2$$

There is an outward, radial presure on the windings of the solenoid. To find the outward force on the upper hemisphere with length, L, evaluate the equation,

$$F_y = \int \mathcal{T}_{yy} d\,A_y$$

Then $d\,\vec{A}_y = R\,d\phi\,dz\,\hat{r} \cdot \vec{y} = R\,d\phi\,dz\,sin(\phi)$

$$F_y = R\,L \int_0^\pi \mathcal{T}_{yy}\,sin(\phi) = \frac{2RLB_z^2}{2\mu_0} = \mu_0 RL(NI)^2$$

This problem can also be worked using the Lorentz force $\vec{F} = q\vec{V} \times \vec{B}$, see figure 4. Use $qV = NI\,dl$ and for a length L of the solenoid, the force on the upper hemisphere is ;

$$F_y = NI\,RL\,(B^2/2) \int_0^\pi sin(\phi)\,d\phi = \mu_0 RL(NI)^2$$

The factor of 1/2 in the expression for the force can be obtained either by recognizing that 1/2 of the current causes a B field which interacts with the other half of the current, or that the B force decreases from the interior value to zero across the solenoid windings. The resulting answer is then the same when calculated by the stress tensor. Such a force force can be substantial in a high field high current superconducting solenoid and cause the windings to move. Movement can quench the superconductor, so that the high current would then pass through a resistive load. In such a case, the energy in the field is then released as heat resulting in a run-away condition that could destroy the magnet.

With the exception of the example of the EM wave interacting with the metalic sphere, the other examples previously discusssed, are simple and do not show the power of the stress tensor when in finding the EM force.

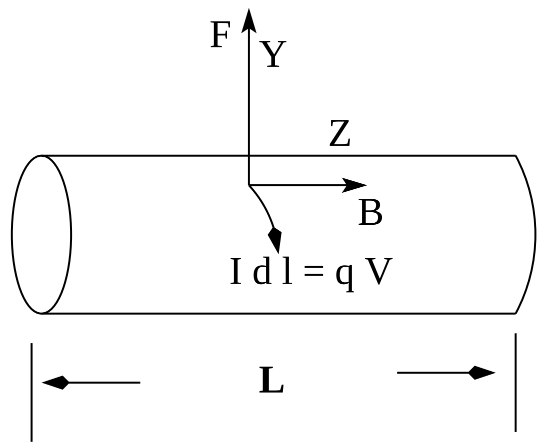

Figure 4: A figure used to find the outward pressure on the windings of a long solenoid

9 Momentum conservation

As another example consider a long coaxial cable of length, L, with inner radius a and outer radius b. There is a charge per unit length of λ on the conductors and they carry a current I in opposite directions. The fields between the conductors are;

$$\vec{E} = \frac{1}{2\pi\epsilon}\frac{\lambda}{\rho}\hat{\rho}; \;\; \vec{B} = \frac{\mu}{2\pi}\frac{I}{\rho}\hat{\phi}$$

The power flow down the cable is;

$$\text{Power} = \int \vec{S}\cdot d\vec{A} = (1/\mu)\int (\vec{E}\times\vec{B})\cdot\hat{z}\, dA_z$$

$$\text{Power} = (1/\mu)\frac{\lambda\mu I}{4\pi^2\epsilon}\int d\phi \int \frac{\rho}{\rho^2}\,d\rho =$$

$$I[\frac{\lambda}{2\pi\epsilon}\, ln(b/a)] = IV$$

The momentum in the field is ;

$$\vec{P}_{field} = (1/c^2)\int \vec{S}\, d^3x = \frac{\mu\lambda IL}{2\pi}\, ln(a/b)\,\hat{z}$$

Nothing appears to move, at least the CM of the system is constant, so what does the field momentum represent? Look at the current loop in Figure 5. Charges on the left are accelerated toward the top and charges on the right decelerated toward the bottom. The current $I = \lambda U$ is the same in all the segments, but the number of charges, N_t, at the top must move faster than number of charges, N_b, at the bottom since for a current constant there must be less charge at the top.

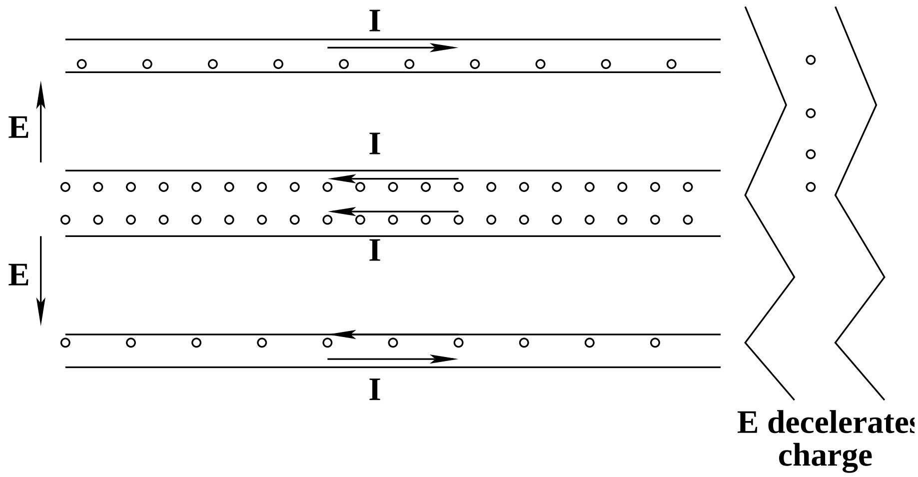

Figure 5: The movement of charges in the power flow in the coaxial cable to show the hidden momentum.

$$I = \frac{QN_t}{L} U_t = \frac{QN_b}{L} U_b$$

Classically the momentum of each charge carrier is mU for U the velocity. Therefore;

$$P_{classical} = mN_tU_t - mN_bU_b$$

Substitute for the number of charges;

$$P_{classical} = \frac{mIL}{Q} - \frac{mIL}{Q} = 0$$

However relativistically the momentum is $m\gamma U$. When we substitute for NU as above we obtain;

$$P_{rel} = \gamma_t mN_tU_t - \gamma_b mN_bU_b = \frac{mIL}{Q}[\gamma_t - \gamma_b]$$

There is work done. The field momentum provides the work to move the charges, and the momentum is canceled by the resistance to this flow.

10 Covarient stress tensor

The stress tensor developed above is a 3-dimensional rank 2 tensor. However, It is not covarient. To extend the tensor to 4-dimensional, covarient form, return to the equations representing both energy and momentum conservation. In the case when there is no charge or current density, these equations are;

$$\frac{\partial(c\mathcal{W})}{\partial t} + \vec{\nabla}\vec{S} = 0 \qquad\qquad \vec{S} = \vec{E} \times \vec{B}$$

$$\frac{\partial \vec{G}}{\partial t} + \vec{\nabla} \cdot \mathcal{T} = 0 \qquad\qquad \vec{G} = (1/c^2)\vec{S}$$

In the above equations, $\mathcal{W}$ is the energy density, $\vec{S}$ is the power flow out of the volume, $\vec{G}$ is the momentum density, and $\mathcal{T}$ is the 3-dimensional stress tensor. Thus with the help of the 4-vector gradient, a 4-dimensional tensor of rank 2 is obtained by requiring that the above equations are reproduced. This tensor has the form;

$$H^{\beta\alpha} = \begin{bmatrix} \mathcal{W} & cG_x & cG_y & cG_z \\ cG_x & \mathcal{T}^{xx} & \mathcal{T}^{xy} & \mathcal{T}^{xz} \\ cG_y & \mathcal{T}^{yx} & \mathcal{T}^{yy} & \mathcal{T}^{yz} \\ cG_z & \mathcal{T}^{zx} & \mathcal{T}^{zy} & \mathcal{T}^{zz} \end{bmatrix}$$

In the above $\mathcal{T}^{\alpha\beta}$ is the 3-dimensional stress tensor. Note that H^{00} is the energy density and H^{0i} the momentum density of the fields. In absence of charge and current densities the conservation of energy and momentum in 4-vector notation is ;

$$\frac{\partial H^{\alpha\beta}}{\partial x^{\alpha}} = 0 \; .$$

Unfortunately, $H^{\alpha\beta}$ as defined above, is not symmetric and gauge invarient. However, as noted previously the contraction of the field tensor with itself is $F^{\alpha\beta}F_{\alpha\beta} = 2(B^2 - E^2/c^2)$. This is similar to the expresion for H^0_0, and further manipulation beyond the scope of this lecture can obtain a 4-dimensional stress tensor which is truly invarient, symmetric and defined by;

$$H^{\alpha}_{\beta} = (1/2\mu_0)[-F^{\alpha\nu}F_{\beta\nu} - (1/4)F^{\mu\nu}F_{\mu\nu}\,\delta^{\alpha}_{\beta}].$$

11 Angular momentum

There is also angular momentum in the EM field. This is obtained from;

$$\vec{L}_{field} = (1/c^2) \int d^3x\,[\vec{x} \times (\vec{E} \times \vec{H})]$$

In terms of a 4-dimensional tensor formulation;

$$\mathcal{M}^{\alpha\beta\gamma} = H^{\alpha\beta}x^{\gamma} - H^{\alpha\gamma}x^{\beta}$$

Conservation of angular momentum requires that a divergence equation demonstrate the conservation of angular momentum;

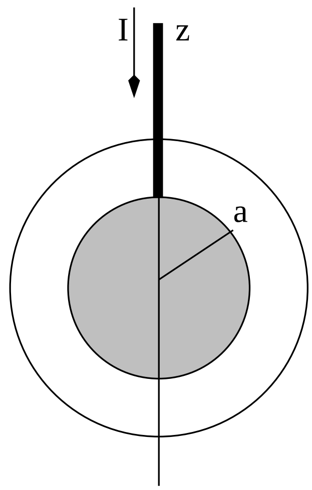

Figure 6: An example showing power flow around a magnetic sphere, representing angular momentum in the field required to cancel the rotation of the spheres

$$\partial_\alpha M^{\alpha\beta\gamma} = 0$$

This requires that $H^{\alpha\beta}$ is symmetric, and thus the symmetric form for the stress tensor is required for a 4-dimensional representation.

12 Example

Now investgate an example of angular momentum in a static electromagnetic field. The geometry of the example is shown in Figure 6. There is a spherical permanent magnet inside a spherial shell. The magnet has a non-conducting interior with the exception of a thin layer on its surface which allows it to serve as the inner conductor of a spherical capacitor. The outer shell is non-magnetic, but conducting, and acts as the outer surface of the spherical capacitor. A charge $\pm Q$ is placed on the capacitor surfaces by applying a potential between the spherical surfaces. There is an electric field between these surfaces given by;

$$\vec{E} = \kappa \frac{Q}{r^2} \hat{r}$$

Both spheres are free to rotate about the z axis. The inner magnetic sphere has uniform magnetization M in its interior pointed in the $\hat{z}$ direction. A dipole magentic field is produced having the value;

$$\vec{B} = \frac{Ma^3}{3r^3}[2\, cos(\theta)\, \hat{r} + sin(\theta)\, \hat{\theta}]$$

In the above equation a is the radius of the magnetic sphere. Then find the Poynting vector.

$$\vec{S} = (1/\mu_0)\vec{E} \times \vec{B}$$

$$\vec{S} = \frac{QMa^3 sin(\theta)}{12\pi\epsilon_0\mu_0 r^5}\,\hat{\phi}$$

Thus in the space between the shell and the magnetic sphere, energy flows in a circular motion around the $\hat{z}$ axis. To understand why, consider the initial state of the system with the capacitor uncharged and the spheres at rest. There is no Poynting vector because $\vec{E} = 0$. To charge the capacitor charge flows down a conductor along the z axis to the surface of the magnetic sphere. These currents interact with the magnetic field to produce an equal but opposite angular momentum in the spheres. The difference in this angular momentum between the spheres is equal to that in the fields.

13 Symmetry

A symmetry is "observed" when some feature of a system remains unchanged when the system changes. Such changes can be described as either continuous or discrete. An example of a continuous change is a rotation about some axis, and a discrete change is represented by a reflection of a coordinate. Symmetries are important because they can be used to mathematically deduce solutions to many physical problems. For example, the homogenous property of spacetime allowed us to formulate the mathematics of the Lorentz transformations.

All symmetries lead to invariance (conservation) of some quantity. Thus invariance under time translation yields conservation of energy, while invarience under spatial translation yields conservation of momentum, and invariance under rotations yields conservation of angular momentum. Inversely, corresponding to each conserved quantity there is a symmetry. However, these symmetries do not have to be geometrical as those used in the above examples.

Time reversal is an example of a discrete symmetry, $t \rightarrow -t$. This is demonstrated in the mathematics describing a physical system through the use of the square of the time parameter, or in conjunction with other parameters which change sign under time reversal. Thus consider Ampere's law;

$$\vec{\nabla} \times \vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{B} = \mu_o I \int \frac{d\vec{l} \times \vec{r}}{r^3}$$

When $t \rightarrow -t$ both t and $\vec{B}$ change sign since $I = \frac{dq}{dt}$. Macroscopically T is not a good

symmetry. However, in Quantum Mechanics used to describe microscopic systems;

$$\begin{aligned} T^{\dagger}\mathrm{H}T &= \mathrm{H}; \\ \mathrm{H}\psi &= i\hbar\frac{\partial}{\partial t}\psi; \\ T\mathrm{H}\psi &= [T\psi]; \\ \mathrm{H}[T\psi] &= -i\hbar\,[T\psi]. \end{aligned}$$

Thus Ψ and $[T^{\dagger}\psi]$ are not equivalent, and $T^{\dagger}$ requires $t \to -t$ and $i \to -i$. One constructs observables in Quantum Mechanics by bilinear forms, (*i.e.* by products two operators and wave functions) so that microscopic time reversibility holds.

14 Parity

Another example of a discrete symmetry is spatial inversion which leads to conservation of parity. The normal modes of a string have either even or odd symmetry.
This also occurs for stationary states in Quantum Mechanics. The transformation is called parity exchance. In the QM description of the harmonic oscillator there are 2 distinct types of wave function solutions characterized by the selection of the starting integer in their series representation. This selection produced a series in odd or even powers of the coordiante so that the wave function was either odd or even upon reflections about the origin, $x = 0$. Since the potential energy function depends on the square of the position, x^2, the energy eigenvalue was always positive and independent of whether the eigenfunctions were odd or even under reflection. In 1-D, parity is a symmetry operation, $x \to -x$. In 3-D, the strong interaction is invarient under the symmetry of parity.

$$\vec{r} \to -\vec{r}$$

Parity is a mirror reflection plus a rotation of 180°, and transforms a right-handed coordinate system into a left-handed one. Our Macroscopic world is clearly “handed”, but “handedness” in fundamental interactions is more involved.

Vectors (tensors of rank 1), as illustrated in the definition above, change sign under Parity. Scalars (tensors of rank 0) do not. One can then construct, using tensor algebra, new tensors which reduce the tensor rank and/or change the symmetry of the tensor. Thus a dual of a symmetric tensor of rank 2 is a pseudovector (cross product of two vectors), and a scalar product of a pseudovector and a vector creates a pseudoscalar.

15 Gauge invarience

Symmtries can not only be continuous or discrete, but they can be global or local. A global symmetry requires a universal change of all spacetime points, while a local symmetry is valid near one spacetime point, but the symmetry cannot be extended to nearby points. Local symmetries form the basis of all gauge theories.

Gauge symmetry describes a system which is invarient under a continuous, local transformation. Historically, gauge symmetry and gauge transformations were explored in classical electrodynamics. Thus the static electric field is determined from the scalar potential, $\vec{E} = -\vec{\nabla} V$. Now if we allow the potential to be changed by a scalar, C, such that $V \rightarrow V + C$, the field is not changed. Note that the force and energy, which are observables, depend only on the field, so that Maxwell's equations are invarient under such a change. In addition, in electrodynamics;

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

In the above equations we could insert a function $\lambda(\vec{x}, t)$ such that;

$$\vec{A} \rightarrow \vec{A} + \vec{\nabla} \lambda$$

$$V = \rightarrow V - \frac{\partial \lambda}{\partial t}$$

This change to the potentials does not change Maxwell's equations. Thus this represents a mathematical symmetry leaving Maxwell's equations invarient. Previously Maxwell's equations in covarient form are;

$$\frac{\partial F^{\alpha\beta}}{\partial x^{\alpha}} = \mu_0 \, J^{\alpha}$$

In this equation, $F^{\alpha\beta}$ is the field tensor, and J^{α} is the charge/current density 4-vector. We also considered the Proca equation which is a form of Maxwell's equations with a non-zero photon mass, μ_{γ}. This was written;

$$\frac{\partial F^{\alpha\beta}}{\partial x^{\beta}} + \mu_{\gamma}^{2} = \mu_0 J^{\alpha}$$

Take the divergence of the above equation;

$$\frac{\partial^2 F^{\alpha\beta}}{\partial x^{\alpha}\,\partial x^{\beta}} + \mu_{\gamma}^{2} = \mu_0 \frac{\partial J^{\alpha}}{\partial x^{\alpha}}$$

The first term on the left vanishes as $F^{\alpha\beta}$ is antisymmetric in (α, β). Now, only when $\mu_{\gamma}^2 = 0$ does one find charge conservation, $\frac{\partial J^{\alpha}}{\partial x^{\alpha}} = 0$. Note that if this does not occur, the mass term depends on the potential A^{α} so the equation would not be gauge invarient.

The Lorentz condition expresses this symmetry as;

$$\vec{\nabla} \cdot \vec{A} + (1/c)^2 \frac{\partial V}{\partial t} = 0$$

In covarient form this is;

$$\partial_{\alpha} A^{\alpha} = 0$$

The choice which satisfies the Lorentz condition results in applying the Lorentz gauge, and is a relativistic invarient. One could also choose a gauge in which $\vec{\nabla} \cdot \vec{A} = 0$. This is the Coulomb gauge, and in this gauge;

$$\nabla^2 V = -\rho/\epsilon$$

with solution;

$$V = \frac{1}{4\pi\epsilon} \int d\tau' \, \frac{\rho(\vec{x}', t)}{|\vec{x}' - \vec{x}|}$$

In this gauge the scalar potential is the instantaneous Coulomb potential, but the vector potential is much more complicated, as it is the solution of the pde obtained after substituting, $\vec{\nabla} \cdot \vec{A} = 0$

$$\nabla^2 \vec{A} - \epsilon\mu \frac{\partial^2 \vec{A}}{\partial t^2} = \mu \vec{J} + \epsilon\mu \vec{\nabla}(\frac{\partial \phi}{\partial t})$$

A gauge transformation does not change the value of the fields which determine the Lorentz force and energy conservation. The fields are the observables, so a gauge transformation is a symmetry of the electromagnetic interaction, which can be asssociated with the vanishing mass of the photon.

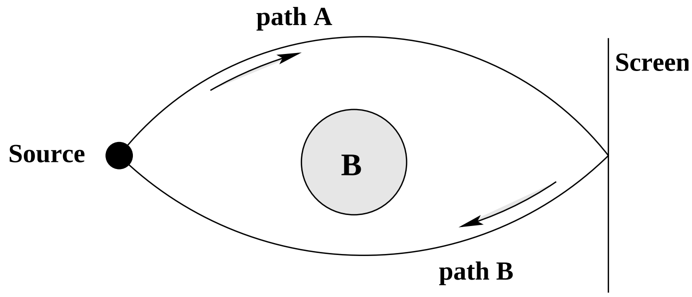

Figure 7: A path integral around a Magnetic field

16 Bohm-Aharanov effect

Classically the force on a charge is independent of the gauge. However, in QM the phase of the wave function depends on the gauge properties. In global symmetry the phase must be the same at all points and at all times and this connects the spatial values of the vector potential. Consider the plane wave function of a non-interacting particle wave.

$$\psi_0 = A\, e^{i(\vec{p}\cdot\vec{x})/\hbar}$$

In the presence of a static vector potential $\vec{B} = \vec{\nabla} \times \vec{A} = 0$. But the kinetic energy term in the Schrodinger equation has the form;

$$-(\hbar^2/2m)[\vec{\nabla} + \frac{iqA}{\hbar c}]^2\, \psi = E\psi$$

The solution is $\psi = e^{i\phi}\psi_0$

Now choose a path around the magnetic field as shown in the figure. Here $\phi = (q/\hbar c) \int\limits_{path} \vec{A}\cdot \vec{dl}$ As the phase depends on the path length it will not necessarily return to the same value. This can be related to a geometric effect of moving a vector around a surface in 3-D. Thus the potentials have an effect that is not contained in the fields.

17 Additive conservation numbers

Charge is an example of an additive conservation number. As a continuious distibrution, charge is conserved as expressed by the equation of continuity. However, charge is quantized in units of the electronic charge, e. Still the total charge obtained by adding the $\pm$ particle charge is conserved as would be conservation of a continuious distribution. Thus conservation of charge is additive.

There are other additive conservation relations. If we assign $+1$ to baryons and -1 to anti-baryons then baryon number is conserved. A similar relation provides an additive conservation of lepton number. Conservation of baryon and lepton number is related to the Pauli principle.

18 Continuous groups

We developed the concept of symmetry in terms of a unitary rotation in 3-D coordinate space. The infinitesmal operators of the rotational transformation were found to have an algebra defined by the commutation rule $[S_i, S_j] = i\,\epsilon_{ijk}\,S_k$. Because the group is continuous, an infinite number of such infinitesmal transformations must be applied to reproduce a finite transformation.

The lowest representation of this algebra, SU(2), is obtained by the 2-D Pauli matricies. There are 2 eigenvalues ± 1, and these matricies are unitary and traceless. Now there are N-dimensional representations of this group, *i.e.* we can find N dimensional matricies that satisfy the algebra, where N is an arbitrary integer > 0. If $N = 3$ we found that the regular representation of this group is a set of 3×3 matricies. Different representations can be specified by the eigenvalue, $S(S+1)$, of S^2. here S is the maximum eigenvalue of S_3. In the case of the irreducible representation $S_3 = 1/2$ so that the eigenvalue of S^2 is $3/4$. The value of S_3 for the regular representation is 1 so the eigenvalue of S^2 is 3. Therefore, rotational symmetry for various angular momentum values $\vec{L}$ is described by a representation of SU(2).

Remember that rotational symmetry requires that the system is independent of the spin projection on the z axis, m value. However, the energy depends on the total value of the angular momentum. Thus, while the interaction is independent of the projection, S_3, it can depend on S^2. Indeed, the higher the eigenvalue of S^2 the higher the self energy (mass) of the particle.