You may NOT use any book or notes other than that supplied with this test. You will have 3 hours to finish. DO YOUR OWN WORK. Express your answers clearly and concisely so that appropriate credit can be assigned for each problem. There are 6 problems. You must do 5 for full credit. TURN IN ONLY 5 PROBLEMS - I WILL GRADE ONLY THE FIRST 5 PROBLEMS YOU SUBMIT. Full credit for each problem is 25 points.

1)

Find the approximate inductance per unit length of two long, straight, parallel wires. The wires are separated by a distance, \( d \), center to center and can be treated as thin, although each wire has a radius, \( a \). Assume only surface currents.

![Figure 1: The geometry of problem 1](image)

SOLUTION

The inductance is obtained by finding the magnetic flux through the area of the circuit and dividing by the current which generates the magnetic field. In this case, the magnetic field is a superposition of the field due to the two wires of the circuit. The magnetic field of each wire is obtained from Ampere’s Law. Choose the coordinate origin as the center of a wire.

\[
\oint \vec{B} \cdot d\vec{l} = \mu_0 I
\]

In the above, \( d\vec{l} \), is an element of the circular loop about the wire as a center, and \( I \) the current flowing in the wire which is carried along the axis of the loop. By symmetry, the magnetic field, \( \vec{B} \) only depends on the radius of the loop so;

\[
\oint \vec{B} \cdot d\vec{l} = B \oint dl = B(2\pi r) = \mu_0 I
\]

\[
B = \frac{\mu_0 I}{2\pi r}
\]
The flux due to one wire with \( a \leq r \leq d - a \), \( \sigma \) the area through which \( \vec{B} \) passes, and \( w \) a segment length along the wire is;

\[
\phi = \int_a^{d-a} \vec{B} \cdot d\vec{\sigma} = (\mu_0 I w) / 2\pi \int_a^{d-a} dr / r
\]

\[
\phi = \left( \frac{\mu_0 I w}{2\pi} \right) \ln[(d - a)/a]
\]

The flux increases by a factor of 2 to include the field of the other wire. Then the inductance per unit length is obtained by dividing the flux by the current and the length.

\[
L/w = 2\phi/Iw = \frac{\mu_0}{\pi} \ln[(d - a)/a]
\]

2) Show by a field transformation that \( E^2 - c^2B^2 \) is a Lorentz invariant.

**SOLUTION**

The field transformantions in Cartesian coordinates for a boost of velocity, \( V \), in the \( x \) direction are;

\[
\begin{align*}
E'_x &= E_x & B'_x &= B_x \\
E'_y &= \gamma(E_y - VB_z) & B'_y &= \gamma(B_y + (v/c)E_z) \\
E'_z &= \gamma(E_z + VB_y) & B'_z &= \gamma(B_z - (v/c)E_y)
\end{align*}
\]

Transform from the primed system to the unprimed system.

\[
\begin{align*}
E'^2 &= E'_x^2 + E'_y^2 + E'_z^2 & B'^2 &= B'_x^2 + B'_y^2 \\
E'^2 &= E_x^2 + \gamma^2(E_y - VB_z)^2 + \gamma^2(E_z + VB_y)^2 \\
B'^2 &= B_x^2 + \gamma^2(B_y + VB_z)^2 + \gamma^2(E_z - VB_y)^2
\end{align*}
\]

Collect terms and substitute into \( E'^2 - c^2B'^2 \) to show that the result for the unprimed system has the same form, \( E^2 - c^2B^2 \)

3) A two protons are connected by a rigid rod of length, \( d \), and spin with angular velocity, \( \omega \), about the \( x \) axis through the center of the rod. In another frame of reference, the rod moves with velocity, \( v_0 \), parallel to the axis of rotation. Find the velocity of the protons in this moving frame.
The protons have a velocity in the \((y,z)\) direction;

\[
\vec{V} = (d/2)\omega [\cos(\omega t) \hat{y} + \sin(\omega t) \hat{z}]
\]

\[V_0 = V_y^2 + V_z^2 = (d/2)\omega\]

For a boost in the \(x\) direction, the total velocity is \(V_x^2 + V_y^2 + V_z^2\)

\[
V_x = \frac{V_a}{1 - V_0V_a/c^2}
\]

\[
V_y = \frac{a\omega \cos(\omega t)}{\gamma(1 - V_0V_a/c^2)}
\]

\[
V_z = \frac{a\omega \sin(\omega t)}{\gamma(1 - V_0V_a/c^2)}
\]

4) Find the power transmission coefficient for a plane EM wave incident normally from a material with permittivity, \(\varepsilon_0\) and permeability, \(\mu_0\) into a transparent dielectric with permittivity, \(\varepsilon\), and permeability, \(\mu_0\).

\[
\varepsilon\mu\]

Material II | \(\mu_0\varepsilon\)
---|---

Material I | \(\mu_0\varepsilon_0\)

---|---

Figure 2: The geometry of problem 4

SOLUTION

In region I;

The incident plane wave is;

\[
E_i = E_{i0}e^{i(k_i z - \omega t)} \quad B_i = \sqrt{\mu_0\varepsilon_0} E_i
\]
The reflected plane wave is:
\[ E_r = E_{r0} e^{i(k_r z + \omega t)} \quad B_r = \sqrt{\mu_0 \varepsilon_0} E_r \]

In region II

The transmitted wave is:
\[ E_t = E_{t0} e^{i(k_t z - \omega t)} \quad B_t = \sqrt{\mu_0 \varepsilon_0} E_t \]

Apply the boundary conditions at \( z = 0 \). 1) The frequencies are identical for each of the waves. 2) The parallel component of \( \vec{E} \) to the surface is continuous. 3) The power flow at the surface is conserved.

For parallel \( \vec{E} \) continuous at the surface;
\[ E_i + E_r = E_t \]

The average incident power in the wave is:
\[ \langle S \rangle = 1/(2\mu_0)|\vec{E} \times \vec{B}| = (1/2)\sqrt{\varepsilon_0/\mu_0}|E_i|^2 \]

For conservation of power flow at the surface;
\[ \sqrt{\varepsilon_0/\mu_0}(1/2)|E_i|^2 - \sqrt{\varepsilon_0/\mu_0}(1/2)|E_r|^2 = \sqrt{\varepsilon/\mu_0}(1/2)|E_t|^2 \]

Solve these equations to obtain the amplitude transmission coefficient.
\[ R_t = \frac{E_t}{E_i} = \frac{2\sqrt{\varepsilon_0}}{\sqrt{\varepsilon_0 + \varepsilon}} \]

The Power transmission coefficient, \( R_p \), is given by \( R_t^2 \)

5) A wire of length, \( d \), carries a harmonic current of frequency, \( \omega \), which has position dependence, \( I = I_0 \cos^2(\pi x/d) \). Ignore phase differences due to position and use non-relativistic expressions.

Find the effective dipole moment, \( p \).

Use the equation of continuity:
\[ \nabla \cdot \vec{J} = -\frac{\rho}{\partial t} = i\omega \rho. \]

Then use Ampere’s law,

\[ I = \int \vec{J} \cdot d\vec{A} = \int (\nabla \cdot \vec{J}) d\tau = -\omega \int dz \int \rho dA. \]

This results in (suppressing the time dependence);

\[ \frac{\partial I}{\partial z} = 2I_0 [\sin(\pi x/d) \cos(\pi x/d)] (\pi/d) = i\omega \lambda \]

\[ \lambda = -\frac{i2\pi I_0}{\omega d} \sin(\pi x/d) \cos(\pi x/d) \]

\[ p = \int_{d/2}^{d/2} dx x\lambda \]

\[ p = \frac{iI_0}{\omega} \]

Find the power radiated.

The radiated power is obtained from the Lamor equation for harmonic time dependence.

\[ P = \frac{\mu_0 \omega^4 p^2}{12\pi c} \]

Find the radiation resistance.

\[ (1/2)I_0^2 R = P \]

\[ R = 2P/I_0^2 \]

**SOLUTION**

6)
Show Gauss’s law holds for a point charge moving in a straight line with constant velocity, \( v \).

**SOLUTION**

The E field for a point charge in uniform motion is;

\[ \vec{E} = \left( \frac{q}{4\pi \epsilon_0} \right) \frac{(1 - \beta^2) \hat{r}}{\gamma^2 (1 - \beta^2 \sin^2(\theta))^{3/2}} \]
Then evaluate $\int \vec{E} \cdot d\vec{A}$ to show the result is $\frac{q}{\epsilon_0}$

The integral is:

$$I = \frac{q(1 - \beta^2)}{4\pi \epsilon^2 \gamma^2} \int \frac{r^2 \sin(\theta) d\theta d\phi}{(1 - \beta^2 \sin^2(\theta))^{3/2} r^2}$$

Careful attention to the algebra, substitution for $\alpha = \cos(\theta)$, with $\sin(\theta) d\theta = d\alpha$, and integration over the solid angle yields the expected result.