Phys 6321 Final Exam - Solutions
May 3, 2013

You may **NOT** use any book or notes other than that supplied with this test. You will have 3 hours to finish. **DO YOUR OWN WORK**. Express your answers clearly and concisely so that appropriate credit can be assigned for each problem. There are 6 problems. You must do 5 for full credit. **TURN IN ONLY 5 PROBLEMS - I WILL GRADE ONLY THE FIRST 5 PROBLEMS YOU SUBMIT**. Full credit for each problem is 25 points.

1)

A square loop of wire lies in the (x, y) plane (see the figure below). In the rest frame it carries a current, I_0. Assume the wire has square cross sectional area with negligible thickness. It is boosted with a velocity, V, parallel to the $\hat{x}$ direction. Find the linear charge densities in the wire on the top and bottom of the loop in the moving frame.

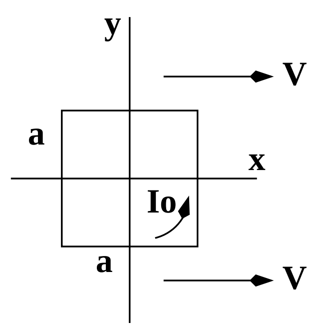

Figure 1: The geometry of problem 1

Solution 1

In the rest frame, the top and bottom of the loop have vanishing linear charge density since the positive and negative charge on the wire are equal.

$$\lambda_t - \lambda_+ - |\lambda_-| = \delta Q_+/\delta l - |\delta Q_-/\delta l| = 0.$$

$$\delta Q_+ = |\delta Q_-| = \delta Q$$

In the moving frame, the above equation equating the charge is also true since charge transforms as a scalar. However, length is contracted in the moving frame as compared to the rest frame by a factor of $\gamma = \sqrt{\frac{1}{1-\beta^2}}$ with $\beta = V/c$. Thus if the negative charge moves for provide the current, I, we obtain in the moving frame;

$$\text{For postiive charge } \delta Q/\delta l_{mov} = Q\gamma_{+}/\delta l_{rest}$$

$$\text{For negative charge } \delta Q/deltal_{mov} = Q(\gamma_{+}/\gamma_{-})/\delta l$$

Note that the ratio of the γ factors above, (γ_{-}/γ_{u}) transforms the length in the frame moving with the current to the system rest frame before the boost, and then transforms this length to the moving frame of the system after the boost. Then we need values for gamma. For the rest frame to the moving frame; $\gamma_{+} = \sqrt{\frac{1}{1-\beta^2}}$. For the transformation of the negative charge to the rest frame the velocity using U_1 as the velocity due to the current in the rest frame.

$$U'_{-} = \frac{V \pm U}{1 \pm VU/c^2}$$
$$\gamma_{-} - \sqrt{\frac{1}{1 - U'^2_{-}/c^2}}$$

Insert the value for U^{pr}_{-} into the above equation for γ and work through the algebra. This results in;

$$\gamma_{-} = \gamma_{u}\gamma_{+}(1 \pm \beta\beta_{u})$$

Therefore the linear charge density after subtraction of the negative charge density form the positive charge density, is ;

$$\lambda_{mov} = \mp\beta\beta_{u}\gamma_{+}\,\lambda_{rest}$$

2)

A conducting cube with side lengths, a, has the upper side ($z = a$) held at a potential, $V = Vo$. All other sides are held at a potential, $V = 0$. Find;

1) The resulting electric field in the interior of the cube;

2) The force on the upper side ($z = a$) of the cube using the Maxwell stress tensor.

Solution 2

Solve Laplace's equation for the electric potential using separation of variables in Cartesian

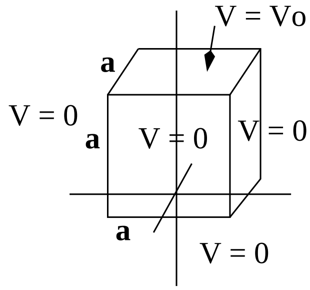

Figure 2: The geometry of problem 2

coordinates to get the electric field inside the cube.

$$\nabla^2 V = 0$$

After application of the boundary conditions on all sides but the top, the solution takes the form;

$$V = \sum_{nm} A_{nm}\, cos(n\pi x/a)\, cos(m\pi y/a)\, sinh(\gamma z)$$

In the above, n, m must be odd for the potential to vanish at $x, y = \pm a/2$. Also $\gamma^2 = (n\pi/a)^2 + (m\pi/a)^2$. Now use orthogonality of the cosine functions to find A_{nm} so that $V = V_0$ when $z = a$.

$$A_{nm} = \frac{4}{\pi^2 nm} (V_0/sinh(\gamma a)) \int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy\, cos(n\pi x/a)\, cos(m\pi y/a)$$

The electric field is $\vec{E} = -\vec{\nabla} V$

$$\vec{E} = -\sum_{nm} A_{nm}[(n\pi) sin(n\pi x/a)\, cos(m\pi y/a)\, sinh(\gamma z)]\hat{x} + [(m\pi) cos(n\pi x/a)\, sin(m\pi y/a)\, sinh(\gamma z)]\hat{y} + [(\gamma\pi) cos(n\pi x/a)\, cos(m\pi y/a)\, cosh(\gamma z)]\hat{z}$$

The field tensor is obtained from only the electric field components. By symmetry, only the force in the z direction is non-zero, F_z.

$$T_{zz} = (1/2)\epsilon_0[E_z^2 - E_x^2 - E_y^2]$$

$$F_z = \int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy\, T_{zz}\,|_{z=a}$$

3)

A cylindrical wave guide is constructed of perfect conductors in a coaxial geometry. Although the TEM is the lowest mode, the geometry also supports both TE and TM modes. Find an expression for the lowest frequency of the TM mode. The inner conductor has radius, a, and the outer conductor radius, b

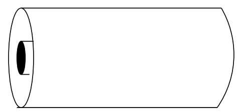

Figure 3: The geometry of problem 3

Solution 3

The equation for a wave traveling in the z direction in the wave guide is;

$$[\nabla^2 + \mu_0\epsilon_0\omega^2 - k^2]E_z =$$

For the TM mode the magnetic field in the z direction vanishes. Thus we solve for E_z and apply the boundary condition that $E_z = 0$ for $\rho = a, b$ using cylindrical coordinates. Separation of variables gives;

$$E_z = \sum_\nu A_\nu D_o(\gamma_\nu \rho)\, e^{ikz}$$

In the above, we choose the zeroth order cylindrical Bessel function, $D_0(\gamma\rho)$ to give the lowest mode and the solution in independent of the azimuthal angle. To match the boundary conditions at $\rho = a, b$ the Bessel function takes the form;

$$D_0(\gamma\rho) = \frac{J_0(\gamma\rho)}{J_0(\gamma a)} - \frac{N_0(\gamma\rho)}{N_0(\gamma a)}$$

Here, J_0 and N_0 are the cylindrical Bessel and Neumann functions, respectively. Then α_ν are the zeros of $D_0(\alpha_{nu}) = 0$. Thus, $\gamma_{nu} b = \alpha_\nu$ and the dispersion relation is;

$$(\alpha_\nu/b)^2 = \nu_0\epsilon_0\omega^2 - k_z^2$$

From this choose the lowest zero, α_0, to get the lowest frequency.

4)

Show that the equation of continuity (charge conservation) results directly from Maxwell's equations.

Solution 4

Maxwell's equations are;

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \vec{\nabla} \times \vec{B} = \mu \vec{J} + 1/c^2 \, \frac{\partial \vec{E}}{\partial t}$$

Then consider;

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0 = \mu \vec{\nabla} \cdot \vec{J} + (1/c^2) \vec{\nabla} \cdot \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{J} + \epsilon \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{E}$$

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t}$$

5)

A charge falls from rest under the influence of gravity. Using an approximation as guided below, find the approximate time it takes the charge to fall a distance, d

1) Write the equation for the system energy including radiation as a function of time.

2) Write an equation for the energy balance at the time when the charge reaches the distance, d.

3) Assume the charge falls without radiation, and write the equation in (2) above using the time to reach d without radiation.

4) Solve the equation in (3) for the time.

Solution 5

The radiative power loss is given by the Larmor equation (non-relativistic). The acceleration is $\dot{v} = g$.

$$P == \frac{E}{dt} = (2/3)(q^2/c^3)a^2$$

The energy loss due to the radiation is obtained by integration.

$$E_{total} = (2/3)(q^2/c^3) \int_0^{T_0} a^2 \, dt$$

The energy balance at position, d, is;

$$mgd = (1/2)mv_0^2 + E_t$$

In the first approximation, $a = g$, $T_0 = \sqrt{2d/g}$, and $v_0 = gT_0$. Substitution for T_0, a, and v_0 gives;

$$(gT_0)^2 + \alpha g^2 T_0 - 2gd = 0$$

Solving for T_0.

$$T_0 = (1/2)[-\alpha \pm \sqrt{\alpha^2 + 8d/g}]$$

Expansion yields for small α yields;

$$T_0 \approx \sqrt{2d/g} - \alpha/2 + \frac{\alpha^2 g}{4d} \sqrt{2dg}$$

In the above, neglect the term in α^2, as in general there are additional terms of this order which are not included in this expression.

6)

Two equal charges, each $Q/2$, are placed 180° apart, and lie in the (x, y) plane. The charges spin with angular velocity, ω, about the $\hat{z}$ axis keeping their radial distance, a, from the origin constant. Find the power radiated in the lowest multipoles for both the electric and magnetic radiation fields. (Note that you needt to write the charge motion in terms of $e^{i\omega t}$ in order to use the expressions for the radiation source components in the notes)

Solution 6

The charge density is;

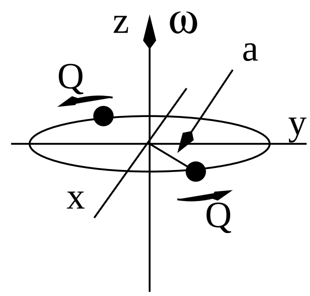

Figure 4: The geometry of problem 6

$$\rho \,=\, Q/2\,(\delta(r-a)/a^2)\,\delta(cos(\theta))\,[\delta(\phi-\phi_0)+\delta(\phi-(\phi+\pi))]$$

Integration over the spherical volume gives the total charge, Q, as it should. We let $\phi_0 = \omega t$ below. Now to write the charge density in a form with time dependence $e^{i\omega t}$, apply a Fourier time de-composition.

$$\rho \,=\, \sum_n \rho_n\, cos(n\omega t) \,=\, Re\sum_n \rho_n\, e^{in\omega t}$$

$$\rho_n \,=\, (\omega/2\pi)\int^{2\pi/\omega} dt\,\rho\, e^{-in\omega t}$$

Substitute in to the above equation the expression for ρ and integrate over time.

$$\rho_n \,=\, (Q\omega/2\pi)(\delta(r-a)/a^2)\,\delta(cos(\theta))e^{in\omega t}[1+(-1)^n]$$

Thus n must be even or 0, however 0 has no time dependence and the lowest possible value would be $n = 2$. Subsitute into the source term for the electric multipole. There is no magnetization term, $M = 0$.

$$Q_l^m \,=\, \int d^3x\, r^l\, Y_l^{*\,m}$$

Note that $Y_2^2 \,=\, (1/4)\sqrt{15/2\pi}\, sin^2\, e^{i2\phi}$ with $\phi = \omega t$. Thus;

$$Q_2^2 \,=\, \frac{Q\omega}{8\pi}\, sqrt15/32\pi\, \int d^3x\, r^2\, sin^2(\theta)\,\delta(r-a)/a^2\,\delta(cos(\theta))e^{-i2\phi}\, e^{i2\phi}$$

$$Q_2^2 \,=\, \frac{Q\omega}{8\pi}\,\sqrt{15/2\pi}$$

To obtain the magnetic radiation source component the current density is;

$$\vec{J} \,=\, \rho_n V\hat{\phi}$$

Then $\vec{r} \times \vec{J} = \rho a^2 \omega \hat{z}$. The divergence of this vanishes after converting to cylindrical coordinates or converting $\hat{z}$ to spherical coordinates. Thus there is no magnetic component.

The electric radiation component is;

$$a_E = \frac{ck^4}{i(5!!)} \sqrt{3/2} Q_2^2$$

The radiated power is;

$$P = \frac{Z_0}{2k^2} |a_E|^2 \quad Z_0 = \sqrt{\mu_0/\epsilon_0}$$