1 Klein-Gordon equation

We have studied the Dirac equation, mainly because it described the motion of Fermions. It is a relativistic equation, naturally containing spin and anti-particles. In particular, we noted that one could describe a conserved probability density using the Dirac wave functions, meaning that Baryon and Lepton number is conserved. However, the first proposed relativistic wave equation was the Klein-Gordon equation. It was initially rejected as it contained both positive and negative energy solutions, did not have a conserved probability current and when applied to the hydrogen atom, it did not correctly predict the atomic fine structure. We now use this equation for spin-zero particles (Mesons), where number conservation is not conserved. While the equation includes negative energy solutions, these are interpreted as in the Dirac equation, as a description of anti-particles.

The equation is derived by inserting the de Broglie operators for energy and momentum in the relativistic expression for a free particle, $E^2 = (pc^2 + (mc^2)^2)$

$$(1/c^2) \frac{\partial^2}{\partial t^2} \psi = \nabla^2 \psi - (mc/\hbar)^2 \psi$$

Solutions have the form $\psi = Ae^{i(k \cdot x \pm \omega t)}$. The equation is identical to the Proca equation of electrodynamics which describes the potential of a massive photon. The covariant equation has the form,
\[ \partial_\mu \partial^\mu \psi + m^2 \psi = 0 \]

where natural units are used. If the Klein-Gordon particle is placed in an electromagnetic field the momentum and energy are replaced by their conjugate values.

\[-(\partial_t - iq\phi)^2 \psi + (\partial_{xi} - iqA_{xi})^2 \psi = m^2 \psi \]

The Langrange density is;

\[ \mathcal{L} = (\partial_\mu + iqA_\mu)\psi^\dagger (\partial^\mu - iqA^\mu)\psi \]

2 Exchange Forces

In 1935 Hideki Yukawa proposed that, based on the electromagnetic interaction, the strong force between nucleons was due to the exchange of a heavy particle which he called a meson. The electromagnetic potential decreases as the inverse distance between two charges. The exchanged particle is the zero mass photon as shown in the figure below. If the exchanged particle has mass, the range of the interaction is determined by the uncertainty relation, \( \Delta E \Delta t = \hbar \). Write this as

\[ \Delta E(\Delta t c) = \hbar c = 197 \text{ MeV-fm} \]

Let \( \Delta E = \) meson mass and \( \Delta c = \) the range of the force which is \( \approx 1 \text{fm} \). Thus the meson mass is on the order of 100-200 MeV.
Recall the equation for the electromagnetic scalar potential in the Lorentz gauge:

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon}$$

In the static limit the solution is;

$$\phi = \frac{1}{4\pi \epsilon} \int d^3 x \frac{\rho}{r}$$

If the charge density is localized at a point $\rho = Q \delta(\vec{r})$ then

$$\phi = \frac{1}{4\pi \epsilon} \frac{1}{r}$$

The factor $Q/\epsilon$ is gives the strength of the coupling and the potential decreases as $1/r$.

Now suppose a nucleon field is coupled to a meson field. The Lagrangian density is composed of the meson Lagrangian, the nu-
cleon Lagrangian, and the interaction Lagrangian. The nucleon is a Fermion and is a solution of the Dirac equation. The Lagrange density has the form;

\[ \mathcal{L} = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{mesom}} + \mathcal{L}_{\text{int}} \]

Where the variation of the action gives the free particle Dirac and mesonic equations, respectively. The meson wave function is a solution of the Klein-Gordon equation.

\[ \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = \nabla^2 \phi - \left( \frac{mc}{\hbar} \right)^2 \phi \]

We choose to assume that the meson field moves rapidly in a quasi-static nucleon field. Thus when a variation of the action of the Lagrangian density is taken the nucleon field is assumed static. This leads to an interaction term of the scalar form \( g_s \bar{\psi} \phi \psi \) or the pseudoscalar form \( g_p \bar{\psi} \phi \gamma^5 \psi \) where \( G \) is the coupling constant which is equivalent to the charge for electromagnetic coupling. Averaging over the nucleon wave function, the interaction term in this case acts as a source of the mesons - mesonic charge density. This results in a form identical to the Proca equation, which is the classical equation for an electromagnetic potential with a non-zero mass term. The static limit takes the form;

\[ \nabla^2 \phi - \left( \frac{mc}{\hbar} \right)^2 \phi = \rho \]

which has a solution ;
\[ \phi = q \frac{e^{-\mu r}}{r} \]

where \( \mu = mc/\hbar \)

### 3 Interaction terms for a Lagrangian

We recall that a Dirac wave function has 4-components, and that \( \gamma \), \( \alpha \) and \( \beta \) are \( 4 \times 4 \) matrices used in the dirac equation. As an aside note that the current \( \vec{j} = \overline{c \psi \alpha} \psi \) leads to an expectation value of the velocity. We write the following bilinear forms that have the various listed transformation properties;

These forms are then coupled with another field with like transformation properties in order to form a scalar which is then inserted in the Lagrangian. For example, in the section above we assumed that the interacting fields were a Dirac field which was written as either a scalar or a pseudoscalar coupled to a meson which had either a scalar or pseudoscalar transformational properties.
4 Symmetry breaking

There are many examples of broken symmetry. Perhaps the most familiar is symmetry breaking due to the application of a magnetic field. We suppose a Hamiltonian that to first order $H_0$, is rotationally symmetric. In this case the particle angular momentum, $\vec{L}$, commutes with $H_0$ and the particle energy is independent of spatial orientation. However, suppose that a magnetic field is applied. This field will interact with the magnetic moment $\vec{\mu} = (q/2mc)\vec{L}$ so that, $E_{int} = -\vec{\mu} \cdot \vec{B}$. $\vec{L}$ does not commute with the full Hamiltonian and the energy depends on the value of $\vec{L}$.

In this case, the symmetry breaking term is small compared to the symmetric term so that the symmetry breaking term adds only a small non-symmetric energy.

5 Spontaneous symmetry breaking

Spontaneous symmetry breaking is different than the symmetry breaking described in the last section. In spontaneous symmetry breaking the dynamics of the interaction break the symmetry in the system ground state. The classical example is a ferromagnet. All spatial directions of a ferromagnet are equivalent. However, in its ground state the ferromagnetic field is polarized in some direction. This state is a state of minimum energy, selected independently by the system. As the energy is increased, symmetry can be restored as the individually aligned atoms break their ferromagnetic bonds. Consider a Lagrangian density for a complex scalar field, $\phi = (1/\sqrt{2})(\phi_1 + \phi_2)$;
Figure 2: An example of spontaneous symmetry breaking

\[ \mathcal{L} = \partial_\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi \]

Now for the static case the potential energy term has the form;

\[ \vec{\nabla} \phi^\dagger \cdot \vec{\nabla} \phi + m^2 \phi^\dagger \phi \]

This has a minimum when \( \phi_1 = \phi_2 = 0 \). Now consider the following functional form for the mass term which is inserted in the Lagrange density

\[ \mathcal{L} = \partial_\mu \phi^\dagger \partial_\mu \phi - (m^2/2\phi_0^2)[\phi^\dagger \phi - \phi_0^2] \]

At \( \phi \), the energy is not zero but decreases as \( \phi \to \phi_0 \) where it vanishes. As \( \phi \gg \phi_0 \) the energy increases to infinity. This is shown in the figure above.

The example has a global \( U(1) \) symmetry. However, the system is still symmetric as the lowest energy state lies anywhere in the ring when \( \phi = \phi_0 \) as shown in the figure. When a specific phase angle
is selected, $\alpha$ the choice breaks the $U(1)$ invarience. The Massless Particles which arise as a result of global symmetry breaking are called Goldstone Bosons.

6 Higgs Bosons

Gauge invariance requires that the field be massless. At first sight it appears that it would not be possible to use gauge invariance in the electroweak interaction because the bosons that are required must be (and are observed to be) massive. However, we can preserve the symmetry through spontaneous symmetry breaking. This occurs by an interaction of at least one particle called a Higgs boson. To outline the development, suppose we choose a massless gauge field, $A_\mu$ which we put into a Lagrange density with an interaction $V$.

$$
[(\partial_\mu - i q A_\mu)\phi^\dagger][(\partial_\mu + i q A^\mu)\phi] - F_{\mu\nu}F^{\mu\nu} - V(\phi^\dagger\phi)
$$

Here $F^{\mu\nu}$ is the field tensor of the gauge field, and we choose $V(\phi^\dagger\phi) = (m^2/2)\phi^\dagger\phi - \phi^2_0|^2$, as we did above. The Lagrangian
is invariant under a local gauge transformation. A minimum energy occurs when the field $A_\mu$ vanishes and $\phi$ is constant. However once we choose a gauge transformation on $\phi$ the symmetry is broken. This broken symmetry inserts a mass into the Lagrangian equat to $(mh/2\phi_0)^2[\sqrt{2}\phi_0 + (1/4)h^2]$, where $h$ is the Higgs field.