1 Introduction

We have found that the electric potential is a solution of the partial differential equation;

$$\nabla^2 V = \rho/\varepsilon_0$$

The above is Poisson’s equation where $\rho$ is the charge density and $V$ the electric potential. If there are a set of various charges in space, these create a potential and an electric field everywhere. In the case of Poisson’s equation the solution is;

$$V = \kappa \int d\tau \frac{\rho(\hat{r}^{'})}{|\vec{R} - \vec{r}|}$$

However this solution is not as useful as one might suppose as one usually has the value of the potentials on the surfaces of various charge distributions, but not the charge densities that created the potentials in the first place. Thus the equation is really not a solution but an integral equation equivalent to the differential equation above.

In many cases we will be concerned with determining the field between the charge distributions in free space where $\rho = 0$. This leads to finding the solution to Laplace’s equation, $\nabla^2 V = 0$. What we will need to do is find a solution to the differential equation subject to potentials that are placed on various surfaces in space. This is called a boundary value problem, because we are to find the value of $V$ given its value on various boundaries in the space.

2 Boundary conditions

We look first at possible boundary conditions. Previously we know that the electric field must be perpendicular to a conducting surface, and must vanish within the enclosed volume. This means that the potential has constant value on the surface. Figure 1 shows a section of an arbitrary conducting surface with a cylindrical Gaussian surface enclosing a section of the conductor. There is no field within the conductor so no flux penetrates the Gaussian surface within the volume. Outside the conductor the $E$ field is perpendicular to the surface at the interface. We shrink the dimensions of the Gaussian cylinder so that the outer end cap approaches the surface. The field out of this surface is perpendicular to the area vector and equal to the perpendicular value of $E$ at that point. The flux is then $E \text{Area} = Q/\varepsilon_0$;

$$\vec{E} = (\sigma/\varepsilon_0) \hat{n}$$
where \( \sigma \) is the surface charge density and \( \hat{n} \) is the outward normal. Then the surface charge on a conductor is given by;

\[
\sigma = -\varepsilon_0 \nabla V \cdot \hat{n}
\]

The above technique will also be useful to develop the boundary condition at non-conducting surfaces, so it will be revisited later. We already know that the electric field tangent to a conducting surface vanishes. We now look at this more closely. In figure 1 there is a drawing of a closed loop that encloses the surface. This loop is called an Amperian loop and we apply reasoning similar to that used for a closed Gaussian surface. In this case we consider the circulation, \( \Gamma \), of the field around the loop as the loop is reduced to lie near, both above and below, the surface. Thus contributions to the circulation from the edges of the loop vanish as the side length approaches zero. This leaves the result;

\[
\Gamma = E_{\parallel \text{above}} dl - E_{\parallel \text{below}} dl
\]

For static charge \( \Gamma = 0 \) so that \( E_{\parallel \text{above}} = E_{\parallel \text{below}} \), and for a conductor both equal zero. As with the Gaussian surface, this will be useful later and will be revisited at that time.

### 3 Uniqueness

We will be looking for solutions to a second order partial differential equation. This solution can be determined in several ways, and will usually be represented in the form of a series of special functions obtained from the solutions to a set of eigenvalue equations. Thus the solution may take different forms, and an important question will arise. How do we know that the solution we find is unique? After all, a proper physics solution should have only one answer, (one numerical value when evaluated). There are mathematical proofs which
demonstrate unique solutions to various second order differential equations if they satisfy
the differential equation and have specified values on a set of boundaries in the geometric
space in which the equation applies. These conditions are specified in table 1, and finding
a proper solution is called a boundary value problem.

Table 1: Boundary conditions required for unique solutions to various
2\textsuperscript{nd} order partial differential equations

<table>
<thead>
<tr>
<th></th>
<th>Poisson’s Eqn</th>
<th>Wave Eqn</th>
<th>Diffusion Eqn</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\nabla^2 V = \rho/\epsilon$</td>
<td>$\nabla^2 V = (1/c^2) \frac{\partial^2 V}{\partial t^2}$</td>
<td>$\nabla^2 V = (1/a) \frac{\partial V}{\partial t}$</td>
</tr>
<tr>
<td>Dirichlet</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Open Surface</td>
<td>not enough</td>
<td>not enough</td>
<td>unique</td>
</tr>
<tr>
<td>Closed Surface</td>
<td>unique</td>
<td>too much</td>
<td>too much</td>
</tr>
<tr>
<td>Neumann</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Open Surface</td>
<td>not enough</td>
<td>not enough</td>
<td>unique</td>
</tr>
<tr>
<td>Closed Surface</td>
<td>unique</td>
<td>too much</td>
<td>too much</td>
</tr>
<tr>
<td>Cauchy</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Open Surface</td>
<td>unstable</td>
<td>unique</td>
<td>too much</td>
</tr>
<tr>
<td>Closed Surface</td>
<td>too much</td>
<td>too much</td>
<td>too much</td>
</tr>
</tbody>
</table>

The Dirichlet boundary conditions require specification of the value of the solution on
the boundary. The Neumann boundary conditions require the specification of the derivative
of the solution on the boundary. The Cauchy boundary conditions require the specification
of both the value of the solution and its normal derivative on the boundary. Here we are
interested in the solution to Laplace’s (or Poisson’s) equation which requires specification of
the value of the solution or its derivative on a closed surface. Later we consider the wave
equation (hyperbolic equation) when we study dynamics.

As a very simple example here we write the potential between the plates of a parallel
plate capacitor as;

$$V = -V_0(z/d) + \text{constant}$$

Here $d$ is the distance between the plates and $z$ is the coordinate in this direction so that
$z = \text{constant}$ represents the potential on lower plate and $z = d$ the potential on the up-
per one. By this we have satisfied the boundary condition that $\vec{E}$ is perpendicular to the
plates. This is a Neumann condition applied on a closed boundary. The space between
the plates extends to $\infty$ in both directions in $(x,y)$. The above function for $V$ also satisfies
Laplace’s equation, so by the uniqueness theorem, this is the unique solution to the problem.

4 Images

Now that we have a uniqueness theorem we can attempt to find solutions by applying symmetries to guess a solution. Then if the uniqueness theorem is satisfied for the proposed function, we have found a true representation of the solution. We first take as an example the solution of a point charge, $-q$, above an infinite conducting plane. This is shown in figure 2. In this example there is no field below the $(x, y)$ conducting plane. The boundary conditions are that the field (normal derivative of the potential) must be perpendicular to the plane, the field at the point must be given by Coulomb’s law, and the field at infinity must decrease to zero. This forms a closed surface within which we are to find the potential and the field. The uniqueness theorem states that if we specify either the value of the potential or its derivative on the surfaces, we obtain the unique solution to the problem. To do this we remove the conducting plane and replace it by an image charge as shown in figure 2. Note that by doing this the field is perpendicular to the $(x, y)$ plane, and it has the form of Coulomb’s law as it gets close to the charge, $q$. It also satisfies the boundary condition at $z \to \infty$. This must give the solution we seek in the region $z \geq 0$. Of course we must ignore the solution for $z < 0$ where we already know that the field must vanish. The solution for the potential at point, $P$ is:

$$V = \kappa q \left[ \frac{1}{|\vec{r} + d|} - \frac{1}{|\vec{r} - d|} \right]$$
Figure 3: An image charge solution for a charge $q$ outside a conducting spherical shell

The field is then obtained by $\vec{E} = -\vec{\nabla} V$. We also see that when $\vec{r} = 0$ the evaluation point lies on the conducting surface and the potential equals zero. Note that the surface charge, $\sigma$, on the plane is not uniform, but it is symmetric. It is found from Gauss’ law as developed in a previous section.

$$\sigma = \epsilon_0 \vec{E} \cdot \hat{n} = -\epsilon_0 \vec{\nabla} V \cdot \hat{n}$$

Now consider a point charge above a conducting spherical surface. In this example we want to remove the surface and replace it by an image charge so that the potential on the spherical surface which will be removed, has a constant value of zero. This is shown in figure 3. In the image problem we have a charge $q$ a distance of $z_1 > R$ above the center of the sphere, and $-q'$ a distance $z' < R$ from the center of the sphere. Then we evaluate the potential at a point, $\vec{r}$, on the spherical surface which we must force to be zero.

$$V = \kappa \left[ \frac{q}{|z - \vec{r}|} - \frac{q'}{|r - z'|} \right]$$

Then we write;

$$\frac{q}{|z - \vec{r}|} = \left( \frac{q}{z} \right) \frac{1}{\left[ 1 + (r/z)^2 - 2(r/z) \cos(\theta) \right]^{1/2}}$$

$$\frac{q'}{|r - z'|} = \left( \frac{q'}{r'} \right) \frac{1}{\left[ 1 + (z'/r)^2 - 2(z'/r) \cos(\theta) \right]^{1/2}}$$

For these 2 terms to cancel we must choose;
Therefore we choose the image to be placed at:


g' = r^2/z and q' = -(r/z)q

Finally look at figure 4. This represents two infinite, conducting cylinders which are chosen to have potentials $V_0$ and $-V_0$ on their surfaces. We want to find the potential everywhere in space, and to do this we can apply the method of images. We replace the cylinders by image line charges $\lambda$ and $-\lambda$ as shown. This problem can be worked in the 2-D of the figure because of the symmetry. Recall that the potential of a line charge is:

\[ V = \frac{\lambda}{2\pi\epsilon_0} \ln(a) \]

Here $a$ is the distance from the line charge to the field point, $P$. From the figure, the potential at the position of the cylindrical surface of the left cylinder is:

\[ V_T = \frac{\lambda}{2\pi\epsilon_0} [\ln(a) - \ln(a')] = \frac{\lambda}{2\pi\epsilon_0} [\ln(a/a')] \]

From the location of the center point of the left cylinder;

\[ \vec{r}' = \vec{w} + \vec{a}' \text{ and } \vec{a}' = \vec{r}' - \vec{w} \]

\[ \vec{r}' = (2\vec{d} - \vec{w}) + \vec{a} \text{ and } \vec{a} = \vec{r}' - (2\vec{d} - \vec{w}) \]

Then;

\[ a'^2 = r'^2 + w^2 - 2r'w \cos(\theta) = r'^2[1 + (w/r')^2 - 2(w/r') \cos(\theta)] \]
\[
a^2 = r'^2 + (2d - w)^2 - 2r'(2d - w)\cos(\theta) = (2d - w)^2[1 + (r'/2d - w)]^2 - 2(r'/2d - w)\cos(\theta)
\]

Substitution gives for the potential gives

\[
V_T = \frac{\lambda}{2\pi\epsilon_0}[\ln[(2d - w)/r']
\]

\[
(1/2)\ln[\frac{1 + (r'/2d - w)^2 - 2(r'/2d - w)\cos(\theta)}{1 + (w/r')^2 - 2(w/r')\cos(\theta)}]
\]

The second term on the right vanishes if;

\[
(r'/2d - w) = w/r'
\]

\[
w = d \pm \sqrt{d^2 - r'^2}
\]

The potential over the cylinder defined by the radius \(r'\) is a constant value given by;

\[
V_T = \frac{\lambda}{2\pi\epsilon_0}[\ln[(2d - w)/r']
\]

The potential on the right cylindrical surface is the negative on this value.

5 Separation of Variables

We wish to find a solution to Laplace’s equation which is a 2\textsuperscript{nd} order, linear, homogeneous, partial differential equation. The standard technique for finding a solution to this type of equation is to use separation of variables. That is to assume a solution in the form of \(X(x)Y(y)Z(z)\) - a product of separate functions of \(x\) and \(y\) and \(z\). Clearly not all solutions can be put in this form. The technique is useful if there is a coordinate system that can match the boundaries of the problem. That is, if surfaces of a constant variable are the surfaces on which the boundary conditions are applied. Thus in this case, the solution can at least be placed in separable form at the boundary. For Laplace’s equation the possibility of obtaining a separable solution occurs for a small number of coordinate systems. These are listed in Table 2. However, just because a separable solution can be found does not mean separable boundary conditions are possible.

There are a few other systems not listed in the table that allow \(R\) separation, which basically means they separate for one or two of the variables. We found a solution by images for a system of two long conducting cylinders. The solution to this problem can be found in Bi-cylindrical Coordinates which is \(R\) separable. Such coordinates can be used if symmetry can be applied to automatically remove one or two of the coordinate variables from
Table 2: Coordinate systems in which Laplace’s equation separates. To find a solution one must apply the boundary condition on a surface defined when one variable is constant.

<table>
<thead>
<tr>
<th>Coordinate System</th>
<th>Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cartesian Coordinates</td>
<td>(x, y, z)</td>
</tr>
<tr>
<td>Circular Cylindrical Coordinates</td>
<td>(r, φ, z)</td>
</tr>
<tr>
<td>Elliptic Cylindrical Coordinates</td>
<td>(η, φ, z)</td>
</tr>
<tr>
<td>Parabolic Cylindrical Coordinates</td>
<td>(µ, φ, z)</td>
</tr>
<tr>
<td>Spherical Coordinates</td>
<td>(r, θ, φ)</td>
</tr>
<tr>
<td>Prolate Spheroidal Coordinates</td>
<td>(η, θ, φ)</td>
</tr>
<tr>
<td>Oblate Spheroidal Coordinates</td>
<td>(η, θ, φ)</td>
</tr>
<tr>
<td>Parabolic Spheroidal Coordinates</td>
<td>(µ, θ, φ)</td>
</tr>
<tr>
<td>Conical Coordinates</td>
<td>(r, θ, λ)</td>
</tr>
<tr>
<td>Ellipsoidal Coordinates</td>
<td>(r, θ, λ)</td>
</tr>
<tr>
<td>Paraboloidal Coordinates</td>
<td>(µ, ν, λ)</td>
</tr>
</tbody>
</table>

the solution. In the Bi-cylindrical coordinates, (r, θ, z), separation occurs when the solution is independent of z, see Figure 5. All of the coordinate systems listed in the table are 3-dimensional and orthogonal. However, in this course we deal with only Cartesian, Circular Cylindrical, and Spherical coordinates.

When separation of variables is applied, one obtains a set of 2nd order ordinary differential equations (ode) of the form;

\[
\frac{d}{dz} \left[ p(z) \frac{d\eta}{dz} \right] + \left[ q(z) + \lambda r(z) \right] \eta = 0
\]

here \( \lambda \) is a constant of separation and \( p, q, r \) are functions which depend on the coordinate system. The solution which represents a specific physical problem occurs by applying the boundary conditions. This specifies a discrete but infinite set of the separation constants, \( \lambda_n \) and solutions \( \eta_n(z) \). The solution to the ode we seek can then be constructed from this set of functions. The problem of finding a solution to the ode subject to boundary conditions is called the Sturm-Liouville problem.

6 The Sturm-Liouville Problem

The Sturm-Liouville solutions are eigenfunctions and the value of \( \lambda_n \) are the corresponding eigenvalues. These functions form an orthogonal set of functions in the space defined within the problem boundaries, \( a \leq z \leq b \). We have visited this previously. Thus any function, \( F(z) \), can be represented by a linear sum of eigenfunctions;
Figure 5: The geometry of Bi-cylindrical coordinates which is R separable

\[ F(z) = \sum_{n=0}^{\infty} A_n \eta_n(z) \]

so that;

\[ \lim_{m \to \infty} \int_a^b dz \left[ F(z) - \sum_{n=0}^{m} A_n \eta_n(z) \right]^2 r(z) = 0 \]

This represents convergence in the mean. In addition;

\[ \int dz r(z) \eta_n(z) \eta_m(z) = 0 \text{ when } m \neq n \text{ and } (\lambda_n - \lambda_m) \neq 0 \]

So that the eigenfunctions are orthogonal using a weighting factor \( r(z) \) which comes from the ode.

7 Example in Cartesian Coordinates

A simple example illustrates the eigenvalue solution to the ode which is used to find a representation to the function \( F(z) \) given in figure 6. We seek a solution to the equation below in Cartesian Coordinates;
Figure 6: The function to be represented in Cartesian eigenvalues for $0 \leq z \leq 2\pi/k$

$$\frac{d^2 \eta}{dz^2} + \lambda \eta = 0$$

Comparison to the ode of the general Sturm-Liouville problem written above, $p(z) = r(z) = 1$ and $q(z) = 0$ The solution has the form;

$$\eta(z) = A \begin{pmatrix} \sin(\lambda z) \\ \cos(\lambda z) \end{pmatrix}$$

Now we apply a boundary condition, for example choose $0 \leq z \leq 2\pi/k$ from the figure 6. Then $\lambda_n = nk$. We find the expansion coefficients, $A_n$, using the orthogonality of the eigenfunctions as follows. We ask for a solution;

$$F(z) = \sum A_n \sin(nkz)$$

Multiply both sides by the eigenfunction $\sin(mkz)$ and integrate over $0 \leq z \leq 2\pi/k$. The use orthogonality;

$$\int_0^{2\pi/k} dz \ F(z) \sin(mkz) =$$

$$\sum A_n \int_0^{2\pi/k} dz \sin(mkz) \sin(nk) = A_m \pi/k$$

Thus;

$$A_m = (k/\pi) \int_0^{2\pi/k} dz \ F(z) \sin(mkz)$$

Insert $F(z) = (k/2\pi)z - 1/2$ and integrate to obtain;

$$A_m = -(1/m\pi)$$

Figure 7 shows how the representation of the function is built up as the eigenfunctions are
Figure 7: Convergence to the function shown in figure 6. Note convergence to the mean value at the discontinuity.

added. Note in particular the convergence to the mean value at the discontinuity.