Solution to Laplace’s Equation In Cartesian Coordinates

Lecture 6

1 Introduction

We wish to solve the 2\textsuperscript{nd} order, linear partial differential equation;

$$\nabla^2 V(x, y, z) = 0$$

We first do this in Cartesian coordinates. Thus the equation takes the form;

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

and we assume that we can find a solution using the method of separation of variables.

$$V(x, y, z) = X(x) Y(y) Z(z)$$

That is, the solution can be written as a product of functions of each of the separate variables. Clearly this is not the only possible functional form. Indeed this assumption is highly restrictive. But we have a uniqueness theorem which states that if we find a solution to the partial differential equation which also satisfies appropriate boundary conditions, then this is a representation of the unique solution.

2 Application of Separation of Variables

Substitute the above functional form into the differential equation. This gives;

$$Y(y) Z(z) \frac{d^2 X(x)}{dx^2} +$$

$$X(x) Z(z) \frac{d^2 Y(y)}{dy^2} +$$

$$X(x) Y(y) \frac{d^2 Z(z)}{dx^2} = 0$$

Note the partial derivative is replaced by the total derivative as the function only has one variable. Now divide by $V$ which gives;

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} + \frac{1}{Z(z)} \frac{d^2 Z(z)}{dz^2} = 0$$
Each term is a separate function of \( x, y, \) and \( z \). Since the variables can change in arbitrary ways we must expect that;

\[
\frac{1}{\mathcal{X}(x)} \frac{d^2 \mathcal{X}(x)}{dx^2} = - \left[ \frac{1}{\mathcal{Y}(y)} \frac{d^2 \mathcal{Y}(y)}{dy^2} + \frac{1}{\mathcal{Z}(z)} \frac{d^2 \mathcal{Z}(z)}{dz^2} \right] = -\alpha^2
\]

where \( \alpha \) is a constant. We also must have that;

\[
\frac{1}{\mathcal{Y}(y)} \frac{d^2 \mathcal{Y}(y)}{dy^2} = \alpha^2 - \frac{d^2 \mathcal{Z}(z)}{dz^2} = -\beta^2
\]

where \( \beta^2 \) is another constant. We define \( \gamma^2 = \alpha^2 + \beta^2 \). We then have to find a solution to ordinary differential equation (ode) of the form;

\[
\frac{d^2 \eta(x)}{dx^2} \pm c^2 \eta(x) = 0
\]

We studied such solutions in the last lecture finding that they can be represented by the Fourier series involving harmonic eigenfunctions. The constants \( \alpha \) and \( \beta \) are the separation constants which are determined by the boundary conditions. Write the general solution in the following form.

\[
\eta = \left( \begin{array}{c} A e^{i\alpha x} + B e^{-i\alpha x} \\ A e^{i\alpha x} + B e^{-i\alpha x} \end{array} \right)
\]

By combining solutions, \( V \) is written as;

\[
V \propto e^{\pm i\alpha x} e^{\pm i\beta y} e^{\pm \gamma z}
\]

In the case when \( \alpha = \beta = 0 \) the equation to be solved is;

\[
\frac{d^2 \mathcal{Z}}{dz^2} = 0
\]

which has solution, \( \mathcal{Z} = A + B z \)

### 3 Examples

We have already found the potential and field for an infinite set of parallel conducting plates, Figure 1. In this case by symmetry, the solution must be independent of the variables, \((x, y)\). From the last section this means that the solution for the potential must have the form;
$V(z) = A + Bz$

Apply the boundary conditions to determine the constants. At $z = 0$ $V = 0$ and at $z = a$ $V = V_0$. The solution is;

$$V = \left(\frac{V_0}{a}\right) z$$

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The field is obtained from the gradient of the potential;

$$\vec{E} = -\vec{\nabla} V = \frac{V_0}{a}$$

As another example, consider a hollow cube having sides of length, $a$. All sides are held at potential $V = 0$ except the side at $z = a$ which is held at potential, $V = V_0$. We look for the potential solving Laplace’s equation by separation of variables. The uniqueness theorem tells us that the solution must satisfy the partial differential equation and satisfy the boundary conditions within the enclosed surface of the cube - Dirichlet conditions on a closed boundary, Figure 2. We use the general form for the solution obtained above.
Figure 2: The geometry to find the potential within a conducting cube with a potential, \( V = V_0 \) placed on one side and the other sides grounded

\[
V \propto e^{\pm i\alpha x} e^{\pm i\beta y} e^{\pm \gamma z}
\]

Now we want the potential to vanish at the walls defined by \( x = 0, a \) and \( y = 0, a \). To do this we re-write the complex exponential above in terms of harmonic functions. For the \( x \) dimension, choose \( \sin(n\pi/a)x \). This makes the value of \( X \) vanish (and thus the potential) at \( x = 0 \). To vanish at \( x = a \) choose \( \alpha = n\pi/a \). These are eigenvalues with eigenvectors, \( \sin([n\pi/a]x) \). The eigenvalues and eigenfunctions for the potential between 2 parallel conducting plates held at different potentials \( \mathcal{Y} \) are similar. For the \( z \) direction, the complex exponentials are rewritten as hyperbolic \( \sinh \) and \( \cosh \) functions.

\[
Z = A \sinh(\gamma z) + B \cosh(\gamma z)
\]

The final boundary conditions are that \( V = 0 \) for \( z = 0 \) and \( V = V_0 \) for \( z = a \). To satisfy these conditions we choose,

\[
Z = A \sinh(\gamma z)
\]

with \( \gamma = \sqrt{\alpha^2 + \beta^2} \). Then use the product form to write the potential;

\[
V = \sum_{n,m=1}^{\infty} A_{nm} \sin([n\pi/a]x) \sin([m\pi/a]y) \sinh(\gamma z)
\]
Note that we have developed the solution in terms of the two eigenvectors, \( \sin\left(\frac{n\pi a}{x}\right) \) and \( \sin\left(\frac{m\pi a}{y}\right) \), which were obtained by choosing the two separation constants so that the eigenvectors satisfy the boundary conditions. One sees that there are 2 separation constants because there are (3 - 1) dimensions. However, we still need to find the values of the expansion coefficients of the 2-D Fourier series. The remaining boundary condition at \( z = a \) is used along with the orthogonality of the eigenfunctions in \( x \) and \( y \). Multiply both sides of the above equation by \( \sin\left(\frac{j\pi a}{x}\right) \sin\left(\frac{n\pi a}{y}\right) \) and integrate over \( x \) and \( y \), using orthogonality. For example integration over \( x \) gives:

\[
\int_0^a dx \sin\left(\frac{j\pi a}{x}\right) \sin\left(\frac{n\pi a}{x}\right) = \begin{cases} \left(\frac{a}{2}\right) & \text{if } j = n \\ 0 & \text{if } j \neq n \end{cases}
\]

Also the solution must have \( n, m \) odd otherwise the \( \sin \) function vanishes. The value of the expansion coefficients is then:

\[
A_{nm} = \frac{16 V_0}{n \, m \, \pi^2 \, \sinh(\gamma a)} \quad m, n \text{ odd}
\]

The final solution is:

\[
V = \sum_{n,m=1}^{\infty} \frac{16 V_0}{n \, m \, \pi^2 \, \sinh(\gamma a)} \sin\left(\frac{n\pi a}{x}\right) \sin\left(\frac{m\pi a}{y}\right) \sinh(\gamma z)
\]

Convergence to the potential \( V = V_0 \) is shown in Figure 3 which is a typical Fourier convergence pattern.

How could one find a solution to the same problem, but with sides in addition to that at \( z = a \) held at \( a \), potential other than \( 0 \)? This may be done by superposition. Look at Figure 4. The solution for the problem is obtained by addition of solutions of the same form as for Figure 2 above. Laplace’s equation is linear and the sum of two solutions is itself a solution. In his case the boundary conditions of the superimposed solution match those of the problem in question.

As a final example we find the solution for the potential within the infinite slit shown in Figure 5 where the three sides are held at different potentials. In this case, Laplace’s equation must be independent of \( z \). From the general solution above, we choose eigenfunctions in \( y \) and \( z \), but choose \( \beta = 0 \) to remove the \( z \) dependence. We want exponential functions for the \( z \) dependence to match the boundary condition at \( x \to \infty \). Thus choose:

\[
\mathcal{Y}(y) = A \sin\left(\frac{nk}{a}\right) y
\]

Here \( k = \pi/2 \) and \( n \) must be odd. In this case \( \mathcal{Y} \) is zero at \( y = 0 \) and \( A \) at \( y = a \). Now choose \( \mathcal{X} = B e^{-\gamma x} \). Then \( \mathcal{X} \) equals \( B \) at \( x = 0 \) and goes to zero as \( x \to \infty \). The solution has the form:
Figure 3: An example of convergence at the boundary to produce the potential, $V = V_0$.

Figure 4: The geometry to find the potential within a conducting cube with a potential, $V = V_1$ and $V = V_2$ placed on two sides with the other sides grounded.

Figure 5: The geometry of a slit infinite in the $z$ direction and extending to $+\infty$ for positive $y$, with different potentials on the conducting surfaces.
\[ V = \sum_{n \text{ odd}}^\infty C_n \sin([nk/a]y) e^{-\gamma x} + [V_1/a]y \]

Here we have added a term \([V_1/a]y\) which represents the potential of 2 infinite parallel plates held at potentials \(V_1\) and \(V_0\), which would be the asymptotic condition as \(x \to \infty\). Then we must match the boundary condition at \(x = 0\).

\[ V_2 = \sum_{n \text{ odd}}^\infty C_n \sin([nk/a]y) + [V_1/a]y \]

Use orthogonality to obtain;

\[ C_n = (a/2) \int_0^a dy (V_2 - [V_1/a]y) \sin([nk/a]y) \text{ n odd} \]

4 An example of separation of the Schrodinger Equation

This example illustrates aspects of the separation of variables technique. The Schrodinger equation \((-\hbar^2/2m)\nabla^2 \Psi = E \Psi\) is not Laplace’s equation. It is really a time suppressed version of the wave equation but it illustrates several points. We first assume separation in the form;

\[ \Psi = XYZ \]

Substitute into Schrodinger’s equation and divide by \(\Psi\) as previously. This gives;

\[ \frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = -c^2 \]

Here \(c^2 = [2mE/\hbar^2]\). then proceed to separate the equations obtaining the two separation constants \(\alpha\) and \(\beta\) as previously.

\[ \frac{d^2 X}{dx^2} + \alpha^2 X = 0 \]

\[ \frac{d^2 Y}{dy^2} + \beta^2 X = 0 \]

\[ \frac{d^2 Z}{dx^2} - \gamma^2 X = 0 \]

Here \(\gamma^2 = \alpha^2 + \beta^2 - c^2\) The form of the solutions for the first two equations are harmonic eigenfunctions as previously. However the last equation can be either exponential or harmonic depending on whether \(\gamma^2\) is positive or negative. Now \(E\) is always positive, and for a finite solution must be sufficient large so that a harmonic solution develops. Thus the energy cannot equal zero. Indeed if we put the particle in a box of radius \(a\) and require the
solution to vanish at the box surfaces, the minimum value of $E$ is;

$$E = \frac{\pi^2 \hbar^2}{2m} (3/a^2)$$