

# Solution to Laplace's Equation in Cylindrical Coordinates

Lecture 8

## 1 Introduction

We have obtained general solutions for Laplace's equation by separation of variables in Cartesian and spherical coordinate systems. The last system we study is cylindrical coordinates, but remember Laplace's equation is also separable in a few (up to 22) other coordinate systems. As you know, choose the system in which you can apply the appropriate boundary conditions. It is only through application of the boundary conditions (Dirichlet or Neumann on a closed surface) that one finds a unique solution to the problem studied. In cylindrical coordinates apply the divergence of the gradient on the potential to get Laplace's equation.

$$\nabla^2 V(\rho, \phi, z) = \rho \frac{\partial^2 V}{\partial \rho^2} + \frac{\partial V}{\partial \rho} + (1/\rho) \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

We look for a solution by separation of variables;

$$V = \mathcal{R}(\rho)\Psi(\phi)\mathcal{Z}(z)$$

As previously, this yields 2 separation constants,  $k$  and  $\nu$ , which will lead to 2 eigenfunction equations. The three separated ode equations are;

$$\frac{d^2 \mathcal{Z}}{dz^2} - k^2 \mathcal{Z} = 0$$

$$\frac{d^2 \Psi}{d\phi^2} + \nu \Psi = 0$$

$$\frac{d^2 \mathcal{R}}{d\rho^2} + (1/\rho) \frac{d\mathcal{R}}{d\rho} + (k^2 - (\nu/\rho)^2) \mathcal{R} = 0$$

The later 2 equations can be set up as eigenfunction equations. The solutions are;

$$\mathcal{Z} \propto e^{\pm kz}$$

$$\Psi \propto e^{\pm i\nu\phi}$$

$$\mathcal{R} \propto J_\nu(k\rho), \text{ or/and } N_\nu(k\rho)$$

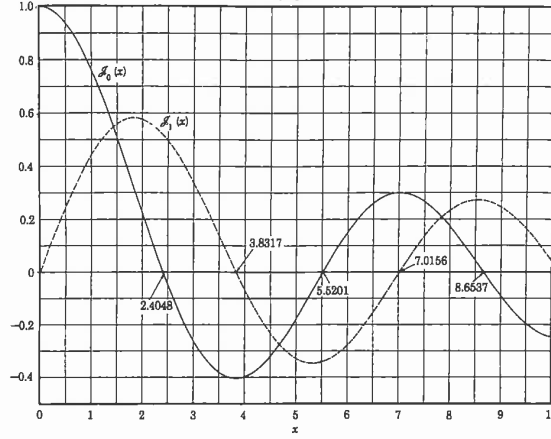


Figure 1: An example of the Cylindrical Bessel function  $J(x)$  as a function of  $x$  showing the oscillatory behavior

## 2 Bessel Functions

In the last section,  $J_\nu(k\rho)$ ,  $N_\nu(k\rho)$  are the 2 linearly independent solutions to Bessel's equation. Bessel functions oscillate but not harmonically, see Figure 1. Thus we expect that the harmonic function solutions for  $\Psi$  and the Bessel function solutions for  $\mathcal{R}$  will be the eigenfunctions when the boundary conditions are imposed. The Bessel functions,  $J_\nu(x)$ , are regular at  $x = 0$ , while the Bessel functions,  $N_\nu(x)$ , are singular at  $x = 0$ .

The limiting values of the Bessel functions are;

$$\lim_{x \rightarrow 0} J_\nu(x) \rightarrow \left(\frac{x}{2}\right)^\nu$$

$$\lim_{x \rightarrow 0} N_\nu(x) \rightarrow \begin{cases} (2/\pi) \ln(x) & \nu = 0 \\ (2/x)^\nu \frac{\Gamma(\nu)}{\pi} & \text{otherwise} \end{cases}$$

$$\lim_{x \rightarrow \infty} J_\nu(x) \rightarrow \sqrt{\frac{2}{\pi x}} \cos(x - \nu\pi/2 - \pi/4)$$

$$\lim_{x \rightarrow \infty} N_\nu(x) \rightarrow \sqrt{\frac{2}{\pi x}} \sin(x - \nu\pi/2 - \pi/4)$$

From the requirement that the solution be single valued as  $\phi \rightarrow 2\pi$ , *ie* the solution is must not change when  $\phi$  is replaced by  $\phi + 2\pi$ , the values of  $\nu$  are integral and this produces eigenfunctions of  $\Psi$ . The series solution for the Bessel function  $J_\nu$  can be found by the method of Frobenius. However, the second linearly independent equation is not easily obtained when  $n$  is an integer, and another technique is required. The Bessel function takes the form;

$$J_\nu(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(\nu+s)!} (x/2)^{\nu+2s} = \frac{x^\nu}{2^\nu n!} - \frac{x^{\nu+2}}{2^{\nu+2}(\nu+1)!} + \dots$$

For integral values of  $\nu$  one can show  $J_{-\nu} = (-1)^\nu J_\nu$ . The Bessel functions also satisfy the recurrence relations;

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{\nu}{x} J_\nu(x)$$

$$J_{\nu-1}(x) = \frac{x}{\nu} J_\nu(x) + \frac{dJ_\nu}{dx}$$

$$\frac{d}{dx}[x^\nu J_\nu(x)] = x^\nu J_{\nu-1}$$

At times an integral representation is useful.

$$J_\nu(x) = (1/\pi) \int_0^\pi d\theta e^{ix \cos(\theta)} \cos(\nu\theta)$$

The Bessel functions are orthogonal;

$$\int_0^\infty k dk J_\nu(k\rho) J_\nu(k\rho') = \delta(r - r')/r$$

$$\int_0^a \rho d\rho J_\nu(\alpha_{\nu k}\rho/a) J_\nu(\alpha'_{\nu k}\rho/a) = (a^2/2)[J_{\nu+1}(\alpha_{\nu k})]^2 \delta_{kk'}$$

In the above  $\alpha_{\nu k}$  are the zeros of the Bessel function of order  $\nu$  where  $k$  orders these zeros. As the Bessel functions form a complete set, any function may be expanded in a Bessel series or integral for an infinite space.

$$F(\rho) = \int k dk A(k) J_\nu(k\rho)$$

### 3 Examples

We find the solution for the interior of a cylindrical shell with the top end cap held at a potential  $V = V_0(\rho)$  and all the other surfaces grounded, Figure 2. The solution we seek has the form;

$$V = \sum_{\nu n} A_{\nu k_n} J_\nu(k_n \rho) \sinh(k_n z) e^{i\nu\phi}$$

In this case the solution is independent of the angle  $\phi$  so we take  $\nu = 0$ . Note that we have not included  $N_\nu$  in the solution because we want it to be finite as  $\rho = 0$ . Also we have chosen  $\sinh(k_n z)$  to satisfy the boundary condition at  $z = 0$ . The reduced solution is;

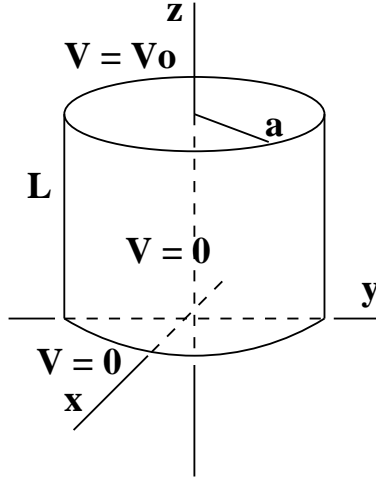


Figure 2: The geometry of a cylinder with one eddcap held at potential  $V = V_0(\rho)$  and the other sides grounded

$$V = \sum_n A_n J_0(k_n \rho) \sinh(k_n z)$$

Now  $V = 0$  for  $\rho = a$ . This means that;

$$J_0(k_n a) = 0$$

The values of  $k_n a$  are the zeros of the bessel function  $J_0(k_n a)$ . The first few are,  $\alpha_{0n} = 2.4048, 5.5201, 8.6537, \dots$ . Then at  $Z = L$  we find  $A_n$  using the orthogonality of the Bessel functions.

$$A_n = \frac{1}{a^2 [J_1(k_n a)]^2 \sinh(k_n L)} \int_0^a \rho d\rho J_0(k_n \rho) V_0(\rho)$$

The graphic form of the solution is shown in figure 3.

As another example we find the potential inside a cylinder when the potential is specified on the end caps and the cylindrical wall is at zero potential, figure 4. The boundary conditions are that;

$$V = V_0 \sin(\phi) \quad z = L$$

$$V = -V_0 \sin(\phi) \quad z = -L$$

$$V = 0 \quad \rho = a$$

The solution must have the form;

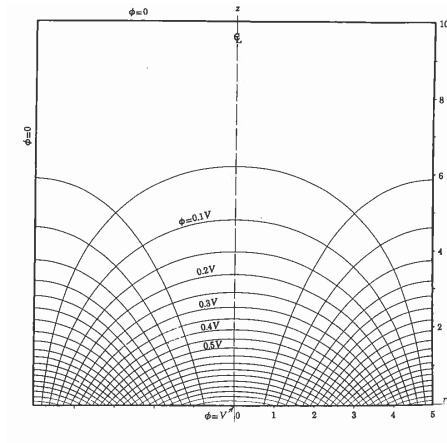


Figure 3: A graphical representation of the above solution

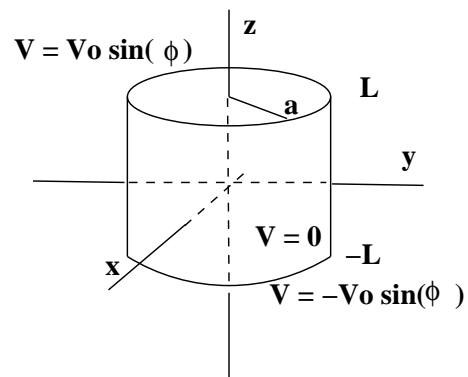


Figure 4: The geometry of the problem with endcaps held at potential  $V = V_0 \pm \sin(\phi)$

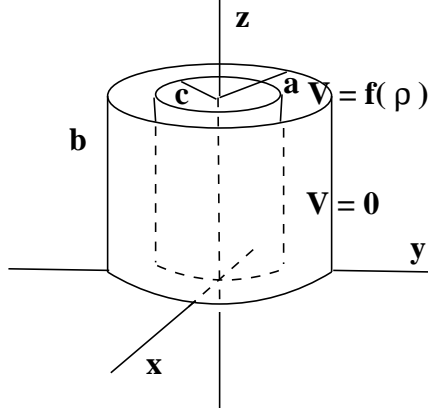


Figure 5: The geometry for the problem of two concentric cylinders

$$V = \sum_{\nu n} A_{\nu n} J_{\nu}(k_n \rho) \sin(\nu \phi) \sinh(k_n z)$$

Here we have discarded solutions in  $N_{\nu n}(k\rho)$  which are infinite at the origin. To match the boundary at  $z = \pm L$  we need to have a term  $\sin(\nu\phi)$  which requires  $\nu = 1$ . Then we require that the Bessel function,  $J_{1n}(k_n a) = 0$  which determines the zeros of the Bessel function of order 1. We write these as  $\alpha_{1n}$  so that  $k_n = \alpha_{1n}/a$ . The solution then has the form;

$$V = \sum_n A_n J_1(\alpha_{1n} \rho/a) \sinh(\alpha z/a) \sin(\phi)$$

Finally we match the boundary condition at  $z = \pm L$  where  $V = V_0 \sin(\phi)$ . Use orthogonality to obtain;

$$(L/2)[J_1(\alpha_{1n})]^2 A_n = \frac{1}{\sinh(\alpha L/a)} \int_0^a \rho d\rho J_1(\alpha_{1n} \rho/a) V_0$$

As another example we look at a solution for concentric cylinders with the boundary conditions;

$$r = a, c \text{ and } z = 0 \quad V = 0$$

$$z = b \quad V = f(\rho)$$

This geometry is shown in Figure 5. We choose a solution to have the form;

$$V = \sum_n^{\infty} A_n \sinh(k_n z) G_0(k_n \rho)$$

Here we have written a superposition of the Bessel and Neumann functions;

$$G_0 = \left[ \frac{J_0(k_n \rho)}{J_0(k_n c)} - \frac{N_0(k_n \rho)}{N_0(k_n c)} \right]$$

So that at  $\rho = c$ , the cylindrical surface of the inner cylinder,  $G_0$  vanishes. Note we have chosen  $\nu = 0$  because the potential is independent of  $\phi$ , *ie* the problem is aximuthally symmetric. Now we must choose the values of  $k_n$  to make  $G_0 = 0$  when  $\rho = a$ . This will select a set of zeros,  $\alpha_{\nu n}$ , of the function,  $G_0$ , and in fact make the functions,  $G_0$ , a complete orthogonal set. This points out that we separated the solutions of the radial ode into a form which was regular at  $\rho = 0$  and one which was not. But we could have separated the solutions so that,  $G_0$ , was one of the two linearly independent solutions, and thus it would have similar oscillating properties as the function  $J_\nu$ . Of course the location of the zeros would be different. Use orthogonality to obtain the coefficients in the above equation.

$$H A_n = [1/\sinh(k_n b)] \int_c^a \rho d\rho V(\rho) G(\alpha_n \rho/a)$$

Here;

$$H = \int_c^a \rho d\rho G^2(\alpha_n \rho/a)$$

Finally consider the problem with the cylindrical wall held at potential  $V = f(z)$  and the endcaps grounded. This geometry is shown in figure 6. The boundry conditions are;

$$z = 0, b \quad V = 0$$

$$\rho = a \quad V = f(z)$$

In this case we cannot use the hyperbolic function in  $z$  to match the boundry conditions. However if we let  $k \rightarrow ik$  then the hyperbolic function becomes harmonic at the expense of making the argument of the Bessel function complex. Note here that the problem is 2-D so we expect only one eigenfunction and this now occurs for the  $z$  coordinate. Then the radial ode with complex Bessel function solutions cannot be eigenfunctions. The eigenvalue of  $k = n\pi/b$  is determined by the harmonic form;

$$\sin(n\pi z/b) \quad n \text{ integral}$$

The solution has the form;

$$V = \sum_{n=1}^{\infty} A_n \sin(n\pi z/b) J_0(in\pi \rho/b)$$

In this case we use the orthogonality of the harmonic functions rather than the Bessel functions. The value of the coefficients are;

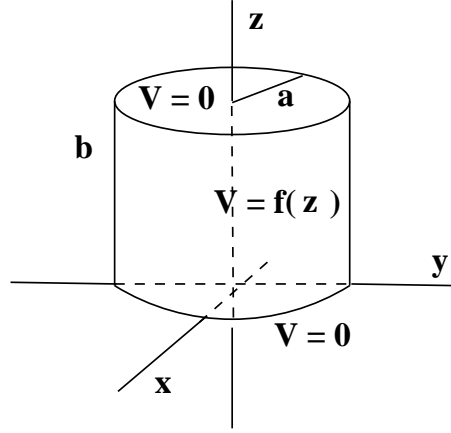


Figure 6: The geometry for the problem where a potential is placed on the cylindrical surface and the end caps grounded

$$A_n = \frac{2}{b J_0(in\pi a/b)} \int_0^b dz f(z) \sin(n\pi z/b)$$

## 4 Numerical Solutions

Separation of variables provides an analytic solution when the boundaries of the potential surfaces are the same as those obtained by taking each variable of the separation geometry as constant. Of Laplace's equation also must separate into separate equations each involving only one of these variables. While analytic solutions provide insight into more realistic problems, they are not always useful in obtaining detailed information which is needed for detailed design and engineering work. Thus we require techniques to obtain accurate numerical solution of Laplace's (and Poisson's) equation.

First consider a result of Gauss' theorem. Integrate Laplace's equation over a volume where we want to obtain the potential inside this volume.

$$\int d\tau \nabla^2 V = \int \vec{\nabla} V \cdot d\vec{\sigma} = 0$$

In the above  $\vec{\sigma}$  is the surface which encloses the volume  $\tau$ . In the case of a spherical surface,  $d\vec{\sigma} = R^2 d\Omega \hat{r}$  which we substitute in the above to write;

$$R^2 \frac{d}{dR} \int d\Omega V = 0$$

This equation means that  $\int d\omega V$  is a constant. Now in Cartesian coordinates we divide space into a grid with cells of the dimensions  $(\delta x, \delta y, \delta z)$ . From the above analysis we know that the potential at the center of the cell will approximately be the average of the



potential over the enclosing surface. In Cartesian coordinates this means that the potential at a point is approximately the average of the sum of the potentials over its nearest neighbors.

$$V_{l,m,n} = 1/6[V_{l-1,m,n} + V_{l+1,m,n} + V_{l,m-1,n} + \\ V_{l,m+1,n} + V_{l,m-1,n} + V_{l,m,n-1} + V_{l,m,n+1}]$$

One begins by taking the exact potential values on the surface and assigning initial values to the potential at all the grid points. The initial values can be any guesses. The average values at each point are then obtained, keeping the correct potential on the surface. The process is iterated to convergence. The technique is called the relaxation method. It is stable by iteration and converges rapidly to the potential within a volume. This technique (finite element analysis) is generally applied to any process which is described by Laplace's equation, and this includes a number of physical processes in addition to electrostatics.

If charge is present, we must have a solution to Poisson's equation. For a sphere of radius,  $r$ , the potential at the center relative to the surface is;

$$\Delta V = \rho r^2 / (6\epsilon_0)$$

This would be included in the equation above when computing the average. As an example, Figure 7 shows a numerical valuation of a potential at the center of a set of grounded metal boundaries and wires which are held at constant potential.

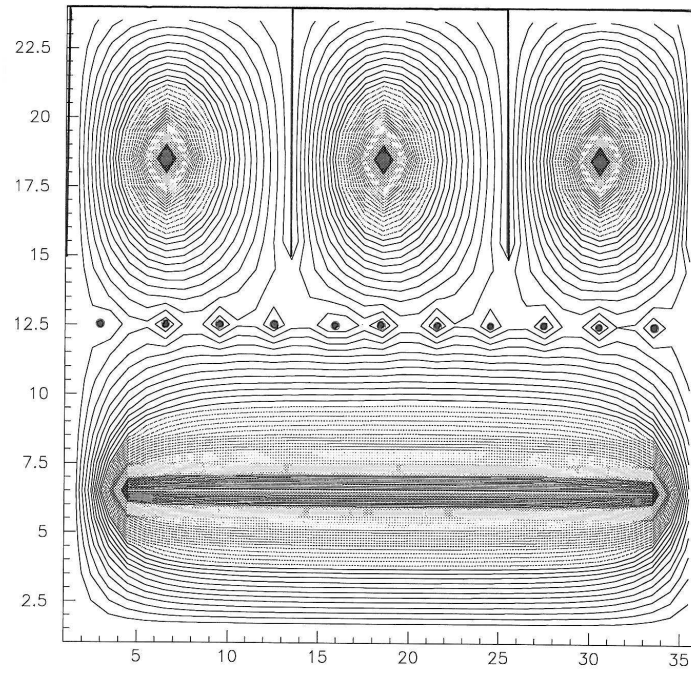


Figure 7: An 3-D numerical example showing contour lines of constant potential of a geometry having grounded metal boundaries and wires held at potential