Bessel Functions - Lecture 7

1 Introduction

We study the ode;

$$x^2 f_\nu'' + x f_\nu' + (x^2 - \nu^2) f_\nu = 0$$

This is a Sturm-Liouville problem where we look for solutions as the variable ν is changed. The equation has a regular singular point at $z = 0$. Substitution of the form; $f_\nu(z) = z^\nu e^{iz} F(\nu + 1/2 | 2\nu + 1 | 2iz)$ yields the confluent hypergeometric function, F. This is the same function we used in studying the Coulomb wave functions. The solution of the above ode which remains finite as $z \to 0$ is called a Bessel function of the 1^{st} kind.

The equation can put in self-adjoint form;

$$x \frac{d}{dx}[x f_\nu'] = -(x^2 - \nu^2) f_\nu$$

Look for a solution to this equation in terms of a series. We already know from previous development that we can easily find one of the two solutions. The second will require more work.

$$f_\nu = \sum a_p x^{p+s}$$

$$f_\nu' = \sum a_p (p+s) x^{p+s-1}$$

$$f_\nu'' = \sum a_p (p+s)(p+s-1) x^{p+s-2}$$

Substitute to get;

$$\sum (p+s)(p+s-1) a_p x^{p+s} + \sum (p+s) a_p x^{p+s} + \sum a_p x^{p+s+2} - \nu^2 \sum a_p x^{p+s} = 0$$

The recurrance relation is;

$$a_{p+2} = -a_p [(p+s+2)^2 - \nu^2]^{-1}$$

The indicial equation is obtained when $p = 0$

$$[s(s-1) + s - \nu^2] a_0 = 0$$

or when $p = 1$

$$[(s+1)s + (s+1) - \nu^2]a_1 \ = \ 0$$

Then choose $a_1 \ = \ 0$ so that $s^2 = \nu^2$, and $s = \pm\nu$. Choose $s = \nu$. The recursion relation is

$$a_{p+2} \ = \ -\frac{a_p}{p^2 + 2p(\nu+2) + 4(\nu+1)}$$

The series solution has the form with a_0 chosen to satisft the normalization condition;

$$f_\nu \ = \ \sum_{p=0}^{\infty} \frac{(-1)^p}{p!(p+\nu)}(\frac{x}{2})^{2s+\nu}$$

The second solution could be obtained with the choice $s = -\nu$ if ν is not integral. However if ν is integral, then this choice gives a solution which is not linearly independent.

2 Convergence and Recursion Relations

Consider the solution;

$$f_\nu \ = \ \sum_{p=0}^{\infty} \frac{(-1)^p}{p!(\nu+p)!}(x/2)^{\nu+2p}$$

The ratio of terms from the recursion relation is;

$$|\frac{a_{p+2}}{a_p}| \ = \ \frac{x^2}{(p+\nu+2)^2 - \nu^2}$$

This is < 1 as $p \to \infty$ for a given x and ν. Thus by the ratio test, the series converges for $0 < x < \infty$. One can use the series to demonstrate the recursion relation between Bessel functions of different order. Here we choose to use $f_\nu \to J_n$ which is the standard convention for the regular, cylindrical Bessel function where n is integral.

$$J_{n-1} + J_{n+1} \ = \ (2n/x)J_n$$

One can also demonstrate;

$$J_{n-1} - J_{n+1} \ = \ 2J_n'$$

Finally the recursion relations above can be combined to reproduce the Bessel equation

3 A second solution

Return to the ode of the Bessel equation.

$$x^2 f_\nu'' + x f_\nu' + (x^2 - \nu^2) f_\nu = 0$$

We previously looked for a solution of the form;

$$f_\nu = \sum s_p x^{p+s}$$

The solution we obtained was regular at the origin, *ie* we expanded about $x = 0$. If we had chosen $s = -1$ the series would have taken the form;

$$f_\nu = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!(p-\nu)!} (x/2)^{2p-\nu}$$

Now $(p-\nu)! \to \infty$ so the series starts for $p = \nu$. Then replace $p \to p + \nu$ which reproduces the series when $p = \nu$. The series is singular at $x = 0$.

4 Integral Representation

Now let;

$$f_n = (1/\pi) \int_0^\pi d\omega \, cos(x \, sin(\omega) - n\omega)$$

Then look at the relations for f_{n+1} and f_{n-1}. Subtracting these we obtain;

$$f_{n+1} - f_{n-1} = (1/\pi) \int_0^\pi d\omega \, [cos(x \, sin(\omega) - n\omega) \, cos(\omega) + sin(x \, sin(\omega - n\omega) \, sin(\omega) -$$
$$cos(x \, sin(\omega) - n\omega) \, cos(\omega) + sin(x \, sin(\omega - n\omega) \, sin(\omega)$$

$$f_{n+1} - f_{n-1} = (2/\pi) \int_0^\pi d\omega \, sin(\omega - n\omega) \, sin(\omega)$$

The term on the right is twice f_n'. Thus the recursion relation for the Bessel function is reproduced. We identify $f_n \to J_n$. The integral representation is;

$$J_n(x) = (1/\pi) \int_0^\pi d\omega \, cos(x \, sin(\omega) - n\omega) = (1/2\pi) \int_{-\pi}^{\pi} d\omega \, e^{i(x \, sin(\omega) - n\omega)}$$

This implies that the Bessel function, J_n, is the n^{th} Fourier coefficient of the expansion; $e^{i2 \, sin(\omega)} = \sum_{n=-\infty}^{\infty} J_n e^{in\omega}$. This allows the following expressions for the generating function;

$$cos(z\,sin(\omega)) \;=\; \sum_{n=-\infty}^{\infty} J_n\,cos(n\omega)$$
$$sin(z\,sin(\omega)) \;=\; \sum_{n=-\infty}^{\infty} J_n\,sin(n\omega)$$

Observe that $J_{-n} \;=\; (-1)^n\,J_n$

5 Generating function

Begin with the series obtained in the last section;

$$cos(z\,sin(\omega)) \;=\; \sum_{n=-\infty}^{\infty} J_n\,cos(n\omega)$$
$$sin(z\,sin(\omega)) \;=\; \sum_{n=-\infty}^{\infty} J_n\,sin(n\omega)$$

If we let $\omega = \pi/2$ we then obtain;

$$cos(z) \;=\; J_0 - 2J_2 + 2J_4 + \cdots$$

$$sin(z) \;=\; 2J_1 - 2J_3 + \cdots$$

Let $e^{i\omega} \;=\; t$ so that $e^{i\omega} - e^{-i\omega} \;=\; 2i\,sin(\omega)$. Then;

$$e^{iz\,sin(\omega)} \;=\; e^{z(t-1/t)/2} \;=\; \sum_{n=-\infty}^{\infty} J_n\,t^n$$

This is the generating function for J_n.

$$G(z,t) \;=\; e^{z(t-1/t)/2} \;=\; \sum_{n=-\infty}^{\infty} J_n(z)\,t^n$$

One can obtain the value of $J_n(x)$ by determining the coefficient of the n^{th} power in the series. Thus;

$$\frac{d^n}{dt^n}[G(z,t)]|_{t=0} \;=\; n!\,j_n(z)$$

The generating function can be used to establish the Bessel power series, and the recursion relations.

6 Limiting values

Limiting Values for the Bessel fns.

$$J_n(x)\lim_{x\to 0} \;\to\; \frac{(x/2)^\nu}{\Gamma(\nu+1)}$$

$$N_0(x)\lim_{x\to 0} \;\to\; (2/\pi)ln(z)$$

$$N_\nu(x)\lim_{x\to 0} \;\to\; (1/\pi)\Gamma(\nu)(z/2)^{-\nu}$$

$$J_\nu(x)\lim_{x\to\infty} \;\to\; \sqrt{2/\pi z}\,cos(z-\nu\pi/2-\pi/4)$$

$$N_\nu(x)\lim_{x\to\infty} \;\to\; \sqrt{2/\pi z}\,sin(z-\nu\pi/2-\pi/4)$$

$$H^1_\nu(x)\lim_{x\to\infty} \;\to\; \sqrt{2/\pi z}\,e^{i(x-\nu\pi/2-\pi/4)}$$

$$H^2_\nu(x)\lim_{x\to\infty} \;\to\; \sqrt{2/\pi z}\,e^{-i(x-\nu\pi/2-\pi/4)}$$

7 A second solution for integral order

Begin by observing that the asymptotic form for the Bessel function is;

$$\lim_{x\to\infty} J_n(x) \;\to\; \sqrt{2/\pi x}\,cos(x-n\pi/2-\pi/4)$$

Now we suppose a 2^{nd} solution would have the asymptotic form;

$$\lim_{x\to\infty} N_n(x) \;\to\; \sqrt{2/\pi x}\,sin(x-n\pi/2-\pi/4)$$

Not only are these forms linearly independent, but are out of phase by $\pi/2$. Also note that if we multiply J_n by $cos(nx)$ and N_n by $sin(nx)$ and look at the asymptotic form, we have;

$$sin(nx)\,sin(x-n\pi/2-\pi/4) \;=\; [-cos(x-n\pi/2-\pi/4+n\pi)+cos(x-n\pi/2-\pi/4-n\pi)]/2$$

$$cos(nx)\,cos(x-n\pi/2-\pi/4) \;=\; [cos(x-n\pi/2-\pi/4+n\pi)+cos(x-n\pi/2-\pi/4-n\pi)]/2$$

Then subtract these forms to obtain;

$$cos(n\pi)J_n - sin(n\pi)N_n \;=\; J_{-n}$$

Using the Wronskian, the second solution would be expressed as ;

$$N_n(x) \;=\; \frac{cos(n\pi)\, J_n(x) - j_{-n}(x)}{sin(n\pi)}$$

For $x \rightarrow 0$ the second solution has the form;

$$J_{-\nu}(x) \;=\; \sum_n \frac{(-1)^\nu}{n!(n-\nu)!}(x/2)^{2n-\nu}$$

Use the expression for the gamma function;

$$z!(-z)! \;=\; \frac{\pi z}{sin(\pi z)}$$

$$\frac{1}{(-z)!} \;=\; \frac{z!\, sin(\pi z)}{\pi z}$$

Substitute this in the expression above in order to obtain the first term in the series as $x \rightarrow 0$.

$$\lim x \rightarrow 0\, [N_\nu(x)] \;=\; \frac{-(\nu-1)!}{\pi}(2/x)^\nu$$

When $\nu \rightarrow 0$ one must be more careful as presented in the text. A series expansion shows logrithmatic behavior. The second solution, N_ν whether ν is an integer or not, is called the Neumann function. It satisfies the Bessel ode and has the same recursion relations as J_ν. The Wronskian is;

$$J_\nu\, N_{\nu+1} - J_{\nu+1}\, N_\nu \;=\; -\frac{2}{\pi x}$$

8 Example of Fraunhofer Diffraction

A circular aperature is uniformly illuminated by a beam of light as ashown in Figure 1. The incident light at all points within the aperature are in phase. By Huygen's principle, the amplitude of the diffracted light at point P is the sum of amplitudes from all points within the aperature. The amplitudes at the aperature are equal at equal points but propagate unequal distances so are out of phase at P. Thus an element of amplitude will have the form;

$$A_n \;=\; C\, \frac{sin(kr - \omega t)}{r}\, \rho\, d\rho\, d\phi$$

The distances are;

$$r^2 \;=\; d^2 + (h - \rho\, cos(\phi))^2 + \rho^2\, sin^2(\phi)$$

$$R^2 \;=\; d^2 + h^2$$

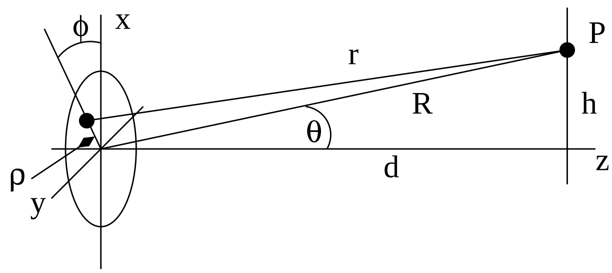

Figure 1: Fraunhofer difraction through a circular aperature

Combine these to obtain;

$$r^2 \,=\, R^2 - 2\rho R\, sin(\theta)\, cos(\phi) + \rho^2$$

Let $R \gg \rho$ and take the Binomial expansion for ρ/R.

$$r \,=\, R[1 - (\rho/R)\, sin(\theta)\, cos(\phi)\, \cdots]$$

The amplitude at P is the integral over the aperature of the above contribution.

$$A \,=\, [C\frac{sin(kx-\omega t)}{R}][\int_0^{2\pi} d\phi \int_0^a d\rho\, \rho\, [cos(k\rho\, sin(\theta)\, cos(\phi)) \,-\, sin(k\rho\, sin(\theta)\, cos(\phi))]]$$

Use the integral expression for the Bessel function;

$$J_n(x) \,=\, (1/pi) \int_0^{\pi} d\omega\, cos(x\, sin(\omega) - n\omega)$$

Change variable by using $\omega = \pi + t$ and let $n = 0$

$$J_0(x) \,=\, (1/2\pi) \int_0^{2\pi} dt\, cos(xsin(t))$$

$$\int_0^{2\pi} d\phi\, sin(x\, cos(\phi)) \,=\, 0$$

Finally;

$$A \,=\, [(2\pi C/R)\, sin(kx - \omega t)/R] \int_0^a d\rho\, \rho\, J_0(k\rho\, sin(\theta))$$

The integral can be evaluated using the Bessel recursion relations, or the generating function.

$$A \,=\, \frac{2\pi Ca}{kR\, sin(\theta)}\, J_1(ka\, sin(\theta))$$

9 Numerical evaluation of the Bessel function

The determination of the value of a Bessel function using the recursion relations is a fast and efficient method. However, the recursive equation;

$$J_{n-1}(x) \; = \; (2n/x)\, J_n(x) - J_{n+1}(x)$$

is stable only upon downward interation. The Neumann function is stable upon upward iteration. One can assume for starting values where $n \gg 1$ and $n \gg x$ that;

$$J_{n+1} \approx 0 \quad \text{and } J_n = \epsilon$$

where ϵ is small and will be determined later. Use these starting values and recurse downward to obtain all J_n for a specific x. The Normalization (determination of ϵ) is obtained from the Wronskian or it may be determined from the generating function with the value of $t = 1$.

$$J_0(x) + 2 \sum_{n=1}^{\infty} J_{2n}(x) \; = \; 1$$

10 Hankel functions

The cylindrical Hankel function is defined as;

$$H_\nu^{(1)} \; = \; J_\nu + iN_\nu$$

$$H_\nu^{(2)} \; = \; J_\nu - iN_\nu$$

In the above, the superscript indicates the Hankel function type and is not a derivative. The asymptotic forms for $r \to \infty$ is extremely useful to satisfy traveling wave boundary conditions. These were given earlier in this lecture. The asymptotic form for large argument ($\nu \to \infty$ through real values) are obtained using the above definition from;

$$J_\nu(z) \; = \; \frac{1}{\sqrt{2\pi\nu}} [\frac{2z}{2\nu}]^\nu$$

$$N_\nu(z) \; = \; -\frac{2}{\sqrt{\pi\nu}} [\frac{2z}{2\nu}]^{-\nu}$$

11 Helmholtz equation in cylindrical coordinates

Helmholtz equation in cylindrical coordinates expressed in wave vector space is;

$$[\nabla^2 + k^2]\psi \,=\, (1/\rho)\frac{\partial}{\partial\rho}[\rho\frac{\partial\psi}{\partial\rho}] + (1/\rho^2)\frac{\partial^2\rho}{\partial\phi^2} + \frac{\partial^2\psi}{\partial^2 z} + k^2\psi \,=\, 0$$

The variables are;

$$x \,=\, \rho\, cos(\phi)$$

$$y \,=\, \rho\, sin(\phi)$$

$$z \,=\, z$$

Apply the separation of variables, $\psi \,=\, J(\rho)\Phi(\phi)\mathcal{Z}(z)$

Substitution in the the pde and using separation of variables gives;

$$\frac{d^2\Phi}{d\phi^2} \,=\, -\alpha^2\Phi$$

$$\frac{d^2\mathcal{Z}}{dz^2} \,=\, (\beta^2 - k^2)\mathcal{Z}$$

$$\rho\frac{d}{d\rho}[\rho\frac{dJ}{d\rho}] \,+\, (\beta^2\rho^2 - \alpha^2)J \,=\, 0$$

The boundary condition that ϕ is single valued as $\phi \to 2n\pi$ reqiures that $\alpha = m$ where m is an integer. This results in the solution;

$$\Phi \,\to\, sin(m\phi);\;\; cos(m\phi)$$

Also we obtain;

$$\mathcal{Z} \,=\, e^{\pm\omega} \;\;\omega^2 \,=\, (\beta^2 - k^2)$$

Finally let $x = \beta\rho$. The radial equation becomes;

$$\frac{dJ}{dx}[x\frac{dJ}{dx}] \,+\, (1 - (\alpha/x)^2)J \,=\, 0$$

When the self-adjoint form is expanded, one finds that the weighting factor for the orthogonality relation is ρ. This is Bessel's equation with solutions;

$$J \,\to\, \left\{ \begin{array}{c} J_\alpha(k\rho) \\ N_\alpha(k\rho) \end{array} \right\}$$

The boundary conditions create a complete set of eigenfunctions. These are the cylindrical Bessel functions. The orthogonality integral is;

$$\int_0^a \rho\, d\rho\, J_\alpha(k_{\alpha n}\rho/a)\, J_{\alpha m}(k_{\alpha m}\rho/a) \;=\; (a^2/2) J^2_{\alpha+1}(k_{\alpha n})\, \delta_{nm}$$

Note in the above that the order of the Bessel functions, α must be the same. Since the Bessel functions are complete, any function for $0 < \rho < a$ can be expanded in an infinite series of Bessel functions. The eigenvalues for a given α are labled by n and written αn.

$$F(\rho) \;=\; \sum_{n=1}^{\infty} A_{\alpha n}\, J_\alpha(k_{\alpha n}\rho/a)$$

$$A_{\alpha n} \;=\; \frac{2}{a^2 J^2_{\alpha+1}(k_{\alpha n})} \int_0^a \rho d\rho\, F(\rho)\, J_\alpha(k_{\alpha n}\rho/a)$$

As with Fourier transforms, there are a set on integral transforms using the Bessel functions. These are;

$$\int_0^\infty \rho\, d\rho\, J_m(k\rho)\, J_m(k'\rho) \;=\; \delta(k - k')/k$$

$$\int_0^\infty k\, dk\, J_m(k\rho)\, J_m(k\rho') \;=\; \delta(\rho - \rho')/\rho$$

12 Example

Consider the the problem of heat flow in a metal cylinder. The bottom and top are kept at temperature $T = 0$ while the side is held at temperature, $f(z)$, as shown in Figure 2.

The equation for steady state temperature is ontained as follows. Let Q be the heat, and experimentally one finds that the heat flow rate (energy flow) is given by;

$$Q/t \;=\; k(T_0 - T_1)s/L$$

In this equation, t is the time, k is the thermal conductivity, $(T_0 - T_1)$ is the temperature difference across the surfaces, s, and L is the distance between the surfaces. Thus the heat flow per unit time through a differential surface perpendicular to the surface, dx, is ;

$$q \;=\; -k\frac{dT}{dx}$$

The heat flow per unit time into a unit volume is the divergence of the above, so that in 3-D this becomes;

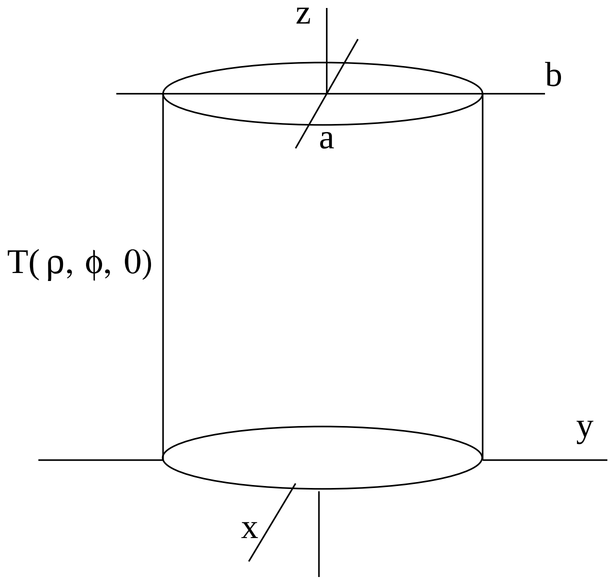

Figure 2: The problem of heat flow in a metal cylinder with side ketp at temperature, f(z)

$$\rho c \frac{\partial T}{\partial T} = k \vec{\nabla} \cdot \vec{\nabla} T$$

$$\frac{\partial T}{\partial t} - \kappa \nabla^2 T = 0$$

The first equation above expresses the specific heat (heat/(mass-Temp)). The second is the diffusion equation. In steady state $\frac{\partial T}{\partial t} = 0$, so we wish to solve Laplaces equation.

$$\nabla^2 T = 0$$

Since the boundary conditions are in cylindrical coordinates, we choose to separate the equation in this system. There is axmuthal symmetry, so the solution is independent of ϕ. Look for a solution of the form $R(\rho)\, \mathcal{Z}(z)$. The solutions are ;

$$\frac{d^2 \mathcal{Z}}{dz^2} = k^2 \mathcal{Z} = 0$$

$$\frac{d^2 R}{d\rho^2} + (1/\rho) \frac{dR}{d\rho} - k^2 R = 0$$

$$\mathcal{Z} = A\, sin(kz) + B\, cos(kz)$$

Note that we need harmonic functions for $\mathcal{Z}$ which requires an imaginary component for k. Thus replace $k \rightarrow ik$. For the solution to vanish at $z = 0, b$ choose $k = n\pi/b$ and use $\sin(kz)$ above. The radial solution which is finite at $\rho = 0$ is $J_0(ik\rho)$. Because of azmuthal symmetry the order of the Bessel function is 0. The solution then has the form;

$$T \; = \; \sum A_n \, J_0([i\pi n/b]r) \, sin([n\pi/b]z)$$

Then to match the last boundary condition on the surface $\rho = a$ the coefficients A_n are obtained using completeness.

$$f(z) \; = \; \sum_{n=0}^{\infty} \; = \; A_n J_0([in\pi/b]a) \, ain([n\pi/b]z)$$

$$A_n \; = \; \frac{2}{bJ_0([in\pi/b]a)} \int_0^b dz \, f(z) \, sin([n\pi/b]z)$$

The Bessel function of inaginary argument is the modified Bessel function written as;

$$J_0([in\pi/b]\rho) \; \rightarrow \; i^i I([n\pi/b]\rho)$$

For a different configuration of the boundary conditions choose the temperature on the cylindrical surface and the lower end cap to be zero. Choose the temperature on the upper end cap to be $f(\rho)$. Again the solution is azmiuthally symmetric, so the soluton is independent of ϕ. However, we choose to satisfy the boundary conditions when $\rho = a$ by setting the Bessel functions to equal zero at the surface. Thus;

$$J_0(ka) \; = \; 0$$

The Bessel functions oscillate, passing through zero an infinite numner of times, although these zero's are not equally spaced as are the zeros of the harmonic functions. As previously written, these are labled $\alpha_{\nu n}$ where ν is the order of the Bessel function. In this case $\nu = 0$. The solution for $\mathcal{Z}$ is no longer harmonic, and is not and eigenfunction. The solutions are hyperbolic sine and cosines. To match the solution for $z = 0$ choose $sinh(kz)$. Thus all boundary conditions with the exception of the condition at $z = b$ are satisfied by;

$$T \; = \; \sum_{n=0}^{\infty} A_n \, J_0(\alpha_{0n}\rho/a) \, sinh(\alpha_{0n} z)$$

Use the orthogonality of the Bessel functions to satisfy the remaining boundary condition.

$$A_n \; = \; \frac{2}{C^2 \, sinh(\alpha_{0n} b/a)} \int_0^a \rho \, d\rho \, f(\rho) \, J_0(\alpha_{0n}\rho/a)$$

$$C \; = \; a \, J_1(\alpha_{0n})$$

13 Green's function

To obtain the Green's function in cylindrical coordinates when applying the Dirichlet boundary condition $G = 0$ suppose the geometry as shown in Figure 4. The solutions in this case

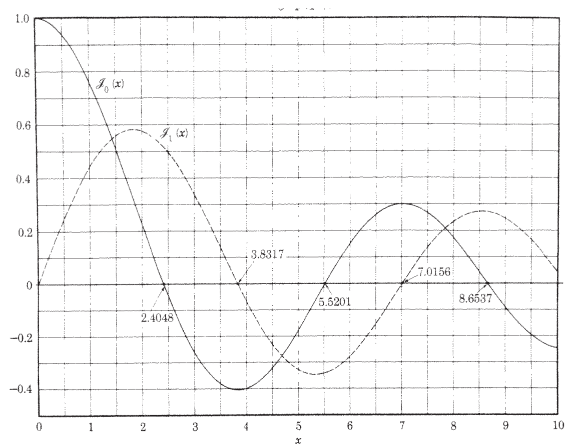

FIG. 5.03 Bessel functions of the first kind, $\mathscr{J}_0(x)$ and $\mathscr{J}_1(x)$.

Figure 3: An example of the Bessel function J_1 showing oscillation

are constructed from;

$$\begin{bmatrix} sin(kz) \\ cos(kz) \end{bmatrix} \begin{bmatrix} e^{im\phi} \\ e^{-im\phi} \end{bmatrix} \begin{bmatrix} I_m(k\rho) \\ k_m(k\rho) \end{bmatrix}$$

In the above, $I(k\rho)$ and $K(k\rho)$ are the two cylindrical modified Bessel functions obtained by inserting $\rho \rightarrow i\rho$ in J_n and N_n. Note as pointed out in an earlier section we could have chosen k to be imaginary which would have made the eigenfunction the solution of the equation for ρ rather than that for the equation for z. In this later case, the solution involves ordinary bessel functions with ρ as argument, and hyperbolic sine cosine functions with z as argument.

For the problem at hand, the eigenfunctions in the z direction satisfying the boundary condition $G = 0$ at $z = 0, L$ are;

$$sin(kz) \quad k = n\pi/L$$

Expand the Green's function in the complete set of eigenfunctions go that;

$$\nabla^2 G = -4\pi\delta(\vec{r} - \vec{r'})$$

The components of the Green's function in ϕ and z are ;

$$sin([n\pi/L]z)sin([n\pi/L]z')$$

$$e^{im\phi}e^{im\phi'}$$

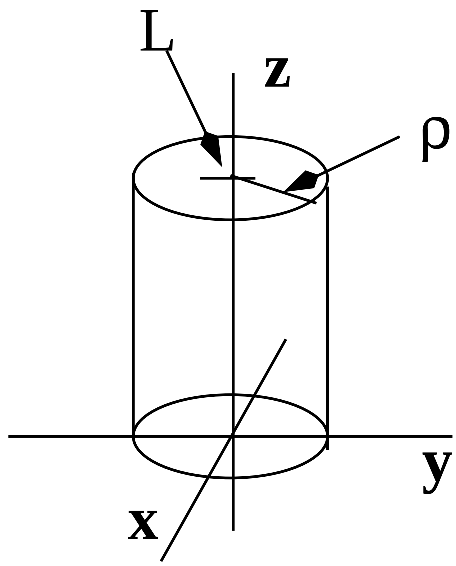

Figure 4: The boundary condition for the Green's function in a cylindrical coordinate system

Ignore the azimuthal dependence to simplify the development as this only adds an additional component to the solution. Thus we consider a solution which has the form;

$$G = \sum_{n=1}^{\infty} A_n \, sin([n\pi/L]z) \, sin([n\pi/L]z) g(\rho, \rho')$$

The problem is to solve for the δ function in ρ. The solution is constructed from the two linearly independent functions satisfying the Bessel equation ($m = 0$);

$$\frac{d^2 g}{d\rho^2} + (1/\rho)\frac{dg}{d\rho} - [(n\pi/L)^2]g = (1/L\pi)\delta(\rho - \rho')/\rho$$

These solutions are the modified Bessel functions $I_{m=0}([n\pi/L]\rho)$ and $K_{m=0}([n\pi/L]\rho)$. Choose to find the solution when $\rho < \rho'$ in terms of I_0 and when $\rho > \rho'$ in terms of K_0. As previously make the solution continuous at $\rho = \rho'$ and match the discontinuity in the derivative at this point. Therefore;

$$A\rho' \, I_0([n\pi/L]\rho') = \rho \, K_0([n\pi/L]\rho')$$

$$[n\pi/L][B \, \frac{K_0([n\pi/L]\rho)}{d\rho}|_{\rho=\rho'} - A \, \frac{I_0([n\pi/L]\rho)}{d\rho}|_{\rho=\rho'}] = -4/(L\rho')$$

Solve for the coefficients and substitute to obtain

$$G(\vec{r}, \vec{r'}) = (4/L) \sum_{n=1}^{\infty} sin([n\pi/L]z) \, sin([n\pi/L]z')$$

$$K_0([n\pi/L]\rho')\, I_0([n\pi/L]\rho) \quad \rho < \rho'$$

$$G(\vec{r},\vec{r}') \;=\; (4/L) \sum_{n=1}^{\infty} sin([n\pi/L]z)\, sin([n\pi/L]z')$$

$$I_0([n\pi/L]\rho')\, K_0([n\pi/L]\rho) \quad \rho > \rho'$$

14 Helmholtz equation in spherical coordinates

The Helmholtz equation when the time coordinate has been suppressed is ;

$$\nabla^2\psi + k^2\psi \;=\; 0$$

In the above, the value of $k = \omega/v$ where ω is the frequency of the wave and v is its velocity. Separation of variables yields harmonic functions of the azimuthal angle, associated Legendre functions of the polar angle, and a radial equation of the form;

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + [k^2r^2 - l(l+1)]R \;=\; 0$$

The separation constant, $l(l+1)$, provides the order of the Legender polynomial. The above equation is similar to Bessel's equation and can be put into the form of Bessel's equation by the change of variable, $R = f(r)/\sqrt{kr}$.

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + [k^2r^2 - (l+1/2)]R \;=\; 0$$

This is Bessel's equation of order $(n+1/2)$ whose solutions are given by the spherical Bessel functions.

$$j_n(x) \;=\; \sqrt{\pi/(2x)}J_{n+1/2}(x)$$

$$n_n(x) \;=\; \sqrt{\pi/(2x)}N_{n+1/2}(x)$$

$$h_n^{(1)}(x) \;=\; j_n + n_n$$

$$h_n^{(2)}(x) \;=\; j_n - in_n$$

Limiting Values for the spherical Bessel fns.

$$j_n(x)\lim_{x\to 0} \;\to\; \frac{2^n n!}{(2n+1)!}\, x^n$$

$$n_n(x)\lim_{x\to 0} \;\to\; -\frac{(2n)!}{2^n n!}\,(1/x)^{n+1}$$

$$j_n(x)\,\lim_{x\to\infty} \,\to\, \frac{sin(x - n\pi/2)}{x}$$

$$n_n(x)\,\lim_{x\to\infty} \,\to\, \frac{cos(x - n\pi/2)}{x}$$

$$h_n^1(x)\,\lim_{x\to\infty} \,\to\, -i\frac{e^{i(x-n\pi/2)}}{x}$$

$$h_n^2(x)\,\lim_{x\to\infty} \,\to\, i\frac{e^{-i(x-n\pi/2)}}{x}$$

15 Energy levels of the Klein-Gordon equation

Suppose the relativistic wave equation for spin zero particles;

$$(pc)^2 \,+\, (M + W)^2 \,=\, E^2$$

Use the momentum operator $p \to i\hbar\vec{\nabla}$. In the above, p is the momentum, W is the energy of the potential well, E the relativistic energy, and M is the particle mass in energy units, Mc^2. Substitution gives the wave equation;

$$-(c\hbar)^2\nabla^2\psi \,+\, (M + W)^2\psi \,=\, E^2\psi$$

Then define;

$$k^2 \,=\, \frac{E^2 - (M + W)^2}{(c\hbar)^2}$$
$$\nabla^2\psi \,+\, k^2\psi \,=\, 0$$

Attempt a solution in spherical coordinates using separation of variables, $\psi \,=\, R(r)\Theta(\theta)\Phi(\phi)$. This results in eigenfunction equations for Θ and Φ, with integral eigenvalues of m and l.

$$\Phi(\phi) \,=\, e^{\pm im\phi}$$

$$\Theta(\theta) \,=\, L_l^m(\theta)$$

In the above $L_l^m(\theta)$ is an associated Legendre polynomial to be studied later. The radial equation has the form;

$$r^2\frac{d^2R}{dr^2} \,+\, 2r\frac{dR}{dr} \,+\, [k^2r^2 - l(l+1)]R \,=\, 0$$

Make the substitution $R(kr) \,=\, \frac{f(kr)}{\sqrt{kr}}$. This results in the form of Bessel's equation of half integral order.

$$(kr)^2 f'' + (kr)f' + [(kr)^2 - (l+1/2)^2]f \;=\; 0$$

The solutions are spherical Bessel functions, $j_{l+1/2}(kr)$ and $n_{l+1/2}(kr)$. To simplify the solution in what follows, choose a spherically symmetric solution, *i.e.* $l = 0$ which also sets $m = 0$. Then substitute $\rho = kr$ and in the original radial equation, $R(\rho) \;=\; U(\rho)/\rho$. This gives the equation;

$$U'' + U \;=\; 0$$

$$U \;=\; C\, sin(\rho)$$

where we choose to have the solution vanish as $r \to 0$ in order for $U(\rho)/\rho$ to remain finite. Now choose ;

$$W \;=\; \left\{ \begin{array}{ll} -M & r \leq a \\ 0 & r > a \end{array} \right\}$$

These boundary conditions will be set to reproduce those of a quark bound inside a hadronic particle. Choose to subsume the factor $\hbar c$ into the energy which takes energy into inverse units of length. The solutions are then;

$$U \;=\; \left\{ \begin{array}{ll} A\, sin(Er) & r \leq a \\ B\, e^{-\sqrt{M^2-E^2}\,r} & r > a \end{array} \right\}$$

Now make the solution continuous at $r = a$ so;

$$B \;=\; A\, \frac{sin(Ea)}{Ea}\, e^{\sqrt{M^2-E^2}}$$

Also make the derivative continuous and substitute for the value of A found from the above equation;

$$cos(Ea)/a \;-\; sin(Ea)/(Ea^2) \;=$$
$$-\sqrt{M^2 - E^2}/(Ea)\, sin(Ea) \;-\; sin(Ea)/(Ea^2)$$

Then take $M \to E$ so that we represent masless quarks for r less than radius a and allows the quarks to be free with mass $\to \infty$ for $r > a$. The eigenvalue equation takes the form;

$$j_0(Ea) \;=\; j_1(Ea)$$

The energy levels are,

$$E_n \;=\; w_n/R$$

16 Mathematical Theory of Scattering

The propagation of a scalar wave in free space is given by;

$$\nabla^2\psi \,-\, (1/c^2)\frac{\partial^2\psi}{\partial t^2} \,=\, 0$$

which for a specific frequency ω is;

$$\nabla^2\psi \,-\, (\omega/c)^2\,\psi \,=\, 0$$

with $k \,=\, \omega/c$. The process of scattering is represented by the equation;

$$\nabla^2\psi \,-\, (\omega/c)^2\,\psi \,=\, U(\vec{r})$$

where $U(\vec{r})$ represents a 3-D scattering potential. There is an incident plane wave in the z direction which is a soluton to the wave equation far from the scattering center, $U(r) \rightarrow 0$.

$$\psi_I \,=\, A\,e^{ikr}$$

A time dependence of $e^{-\omega t}$ is suppressed. This plane wave is scattered from the potential $U(r)$. Thus we expect a solution to the inhomogeneous wave equation, ψ_S above with the complete solution such that;

$$\psi_T \,=\, \psi_I \,+\, \psi_S$$

The scattered solution must take the form;

$$\psi_S \,\rightarrow\, \lim_{r\rightarrow\infty} to\, f(\theta,\phi)\,e^{ikr}/r$$

The radial dependence in the above, e^{ikr}/r, represents a spherically outgoing wave, as found for the wave equation in spherical coordinates when $U \rightarrow 0$. The solution ψ_I is added to satisfy the initial boundary conditions. The function, $f(\theta,\phi)$ is the scattering amplitude and the differential scattering cross section is given by;

$$\frac{d\sigma}{d\Omega} \,=\, |f|^2$$

Substitute the above solution, ψ_T into the wave equation;

$$[\nabla^2 + k^2]\psi(T \,=\, [\nabla^2 + k^2]\psi_S \,=\, U\psi_T$$

Define $\mathcal{L} \,=\, [\nabla^2 + k^2]$ to write;

$$\mathcal{L}\psi_S = U\psi_T = U[\psi I + \psi_S]$$

Formally write an inverse operator $\mathcal{L}^{-1}$ to obtain the mathematical form;

$$\psi_S = \frac{1}{1 - \mathcal{L}^{-1}U}\,\mathcal{L}^{-1}(U\psi_I)$$

This can be expanded to produce;

$$\psi_S = [1 + \mathcal{L}^{-1}U + \mathcal{L}^{-1}U\,\mathcal{L}^{-1}U + \cdots]\mathcal{L}^{-1}U\psi_I$$

This is the Born series with the first term, the Born approximation.

$$\psi_S \approx \mathcal{L}^{-1}U\psi_I$$

The Born series is a perturbation expansion which converges for small values of the scattering potential U. The solution is written in terms of the inverse operator $\mathcal{L}^{-1}$ which actually develops the Green's function for the scattered wave. We return to this after discussing Legnedre's equation.