Hankel Transforms - Lecture 10

1 Introduction

The Fourier transform was used in Cartesian coordinates. Problems with cylindrical geometry need to use cylindrical coordinates. Thus suppose the Fourier transform of a function $f(x, y)$ which depends on $\rho = (x^2 + y^2)^{1/2}$. This is;

$$F(\alpha, \beta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, f(\rho)\, e^{i(\alpha x + \beta y)}$$

Change this to cylindrical coordinates using $x = \rho\, cos(\theta)$ and $y = \rho\, sin(\theta)$.

$$F(\alpha, \beta) = \frac{1}{2\pi} \int_{0}^{\infty} d\rho\, \rho\, f(\rho) \int_{0}^{2\pi} d\theta\, e^{i\rho\alpha\, cos(\theta - \theta')}$$

However the second integral is a representation of the cylindrical Bessel fuction.

$$\int_{0}^{2\pi} d\theta\, e^{i\rho\alpha\, cos(\theta-\theta')} = 2\pi J_0(\alpha\rho)$$

$$F(\alpha) = \int_{0}^{\infty} d\rho\, \rho\, f(\rho)\, J_0(\alpha\rho)$$

There is an inverse which is demonstrated by the completeness of the Bessel functions which can be used to expand the delta function. Thus an inverse transformation can be defined.

$$f(\rho) = \int_{0}^{\infty} d\alpha\, \alpha\, F(\alpha)\, J_0(\alpha\rho)$$

This represents the transform for the 0^{th} order Bessel function. Higher orders can be obtained by increasing the dimension of the Fourier transformations. However, generalize this result by the delta function expansion and applying this to the integral equation;

$$\int_{0}^{\infty} \rho\, d\rho\, J_m(k\rho)\, J_m(k'\rho) = \delta(k - k')/k$$

$$\int_{0}^{\infty} k\, dk\, J_m(k\rho)\, J_m(k\rho') = \delta(\rho - \rho')/\rho$$

$$f(\alpha) = \int_{0}^{\infty} \rho\, d\rho F(\rho)\, J_n(\alpha\rho)$$

$$F(\rho) = \int_{0}^{\infty} \alpha\, d\alpha\, F(\rho)\, J_n(\alpha\rho)$$

Substitute $F(\rho)$ into the first equation above, interchange the order of integration and use orthogonality to develop the delta function.

$$f(\alpha) \;=\; \int_0^\infty \rho\, d\rho \int_0^\infty \alpha'\, d\alpha'\, J_n(\alpha'\rho)\, J_n(\alpha\rho)\, f(\alpha')$$

The above transforms show that a Parseval's theorem relation exists for the Hankel transforms. They are used for cylindrical geometry problems.

$$\int_0^\infty \alpha\, d\alpha\, g(\alpha) f(\alpha) \;=\; \int_0^\infty \rho\, d\rho\, G(\rho) F(\rho)$$

2 Hankel transformations of the derivative

Suppose that;

$$F(\alpha) \;=\; \int_0^\infty \rho\, d\rho\, f(\rho)\, J_n(\alpha\rho)$$

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Then write;

$$F(\alpha) \;=\; \int_0^\infty \rho\, d\rho\, f(\rho)\, J_n(\alpha\rho)$$

$$I \;=\; \int_0^\infty \rho\, d\rho\, \frac{df(\rho)}{d\rho}\, J_n(\alpha\rho)$$

Integrate by parts. This results in ;

$$I \;=\; [\rho\, df(\rho) J_n(\alpha\rho)]_0^\infty \;-\; \int_0^\infty d\rho\, f(\rho) \frac{d(\rho\, J_n)}{d\rho}$$

Use the recurrence relations to write;

$$\frac{(d\rho\, J_n)}{d\rho} \;=\; J_n + \alpha\rho J_{n-1} - n J_n$$

Assume that $[\rho\, f(\rho)]_0^\infty \;=\; 0$. The integral then is;

$$I \;=\; (n-1) \int_0^\infty d\rho\, f\, J_n \;-\; \alpha \int_0^\infty \rho\, d\rho\, f\, J_{n-1}$$

For the 2^{nd} derivative, provided the surface terms vanish is;

$$L \;=\; \int_0^\infty \rho\, d\rho\, \frac{d^2 f(\rho)}{d\rho^2}\, J_n(\alpha\rho) \;=\; -\int_0^\infty d\rho\, \frac{df(\rho)}{d\rho}\, \frac{d(\rho\, J_n)}{d\rho}$$

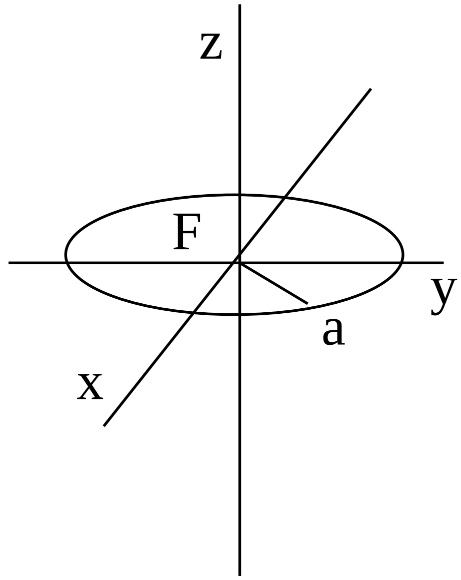

Figure 1: The geometry of a flat, conducting disk of radius a held at a pootnetial F

However, since the Bessel function satisfies the Bessel ode;

$$\frac{d}{d\rho}[\rho\, J'(\alpha\rho)] \;=\; -(\alpha^2 - (n/\rho)^2)\rho\, J_n(\alpha\rho)$$

Integrating each term by parts, Bessel's equation is obtained for f so the result is just, $-\alpha^2 F_n(\alpha)$.

3 Example - Conducting Disk held at a Potential

The equation for the potential is $\nabla^2 V \;=\; 0$. In cylindrical coordinates, Figure 1;

$$(1/\rho)\,\frac{\partial}{\partial\rho}[\rho\,\frac{\partial V}{\partial\rho}] \;+\; (1/\rho^2)\,\frac{\partial^2 V}{\partial\theta^2} \;+\; \frac{\partial^2 V}{\partial z^2} \;=\; 0$$

Asume a solution of the form;

$$V \;=\; \sum_{n=0} V_n(z,\rho)\, sin(n\theta)$$

$$V_n(z,\rho) \;=\; 1/\pi \int_0^{2\pi} d\theta\, V(z,\rho,\theta)\, sin(n\theta)$$

This results in the pde for V_n.

$$(1/\rho)\,\frac{\partial}{\partial\rho}[\rho\,\frac{\partial V_n}{\partial\rho}] \;-\; (n^2/\rho^2)\,V_n \;+\; \frac{\partial^2 V_n}{\partial z^2} \;=\; 0$$

Apply a Hankel transform;

$$\int_0^\infty \rho\,d\rho\,J_n(\lambda\rho)\,[(1/\rho)\,\frac{\partial}{\partial\rho}[\rho\,\frac{\partial V_n}{\partial\rho}] \;-\; (n^2/\rho^2)\,V_n \;+\; \frac{\partial^2 V_n}{\partial z^2}] \;=\; 0$$

$$\overline{V}_n(\lambda, z) \;=\; \int_0^\infty \rho\,d\rho\,V_n\,J_n(\lambda\rho)$$

The solution is symmetric for $z \to -z$, so choose $z > 0$. Use eigenfunctions for $J_n(\lambda\rho)$ and $sin(n\theta)$. This means the functional form for z is $e^{-\lambda z}$. Choose $n = 0$ to simplify the exposition below.

$$V_{n=0}(\rho, z) \;=\; \int_0^\infty \lambda\,d\lambda\,A(\lambda)\,e^{-\lambda z}\,J_0(\lambda\rho)$$

the boundary conditions are applied in two steps.

Step 1 - $z = 0$ and $\rho < a$

$$V_0(\rho, 0) \;=\; F(\rho) \;=\; \int_0^\infty \lambda\,d\lambda\,A(\lambda)\,J_0(\lambda\rho)$$

Step 2 - $z = 0$ and $\rho > a$

$$\frac{\partial V_0}{\partial\rho}|_{z=0} \;=\; \int_0^\infty \lambda^2\,d\lambda\,A(\lambda)\,J_0'(\lambda\rho)$$

The above forms a pair of integral equations which must be simultaneously solved.

4 Example - Hydrodynamic Problem

For any closed surface in an fluid having no sources or sinks (divergences), then the increase (decrease) in mass is due to the mass flowing into (out of) the volume. Let ρ be the density of the fluid. The total mass is $M \;=\; \int \rho\,d\tau$. The rate of mass flow is given by;

$$\frac{\partial M}{\partial t} \;=\; \int_{surface} (\vec{V}\cdot\hat{n})\rho\,d\sigma$$

In the above, $\vec{V}$ is the fluid velocity, and $\hat{n}$ is the surface normal. By Gauss' law

$$\int \vec{\nabla}\cdot(\rho\vec{V})d\tau \;=\; \int_{surface} (\vec{V}\cdot\hat{n})\rho\,d\sigma$$

$$\int \vec{\nabla} \cdot (\rho \vec{V}) d\tau \; = \; -\frac{\partial}{\partial t} \int \rho \, d\tau$$

This gives the equation of continuity.

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{V}) \; = \; 0$$

Suppose the rate of change of ρ due to convection on the fluid. A element of fluid at $\vec{r}$ and time t will move to the point, $\vec{r} + \vec{V}\, \delta t$ at a time $t + \delta t$. Thus;

$$\delta\rho \; = \; \rho(r + v\delta t, t + \delta t) - \rho(r, t)$$

Use this to re-write;

$$\frac{\partial \rho}{\partial t} \; + \; \rho \vec{\nabla} \cdot \vec{V} + \vec{V} \cdot \vec{\nabla} \rho \; = \; 0$$

$$\frac{d\rho}{dt} \; + \; \rho \vec{\nabla} \cdot \vec{v} \; = \; 0$$

For a perfect fluid $\vec{\nabla} \times \vec{V} \; = \; 0$ and the density is constant. Choose a velocity potential, $\vec{V} \; = \; -\vec{\nabla}\phi$. Thus Laplace's equation is obtained.

$$\nabla^2 \phi \; = \; 0$$

Suppose the flow of a perfect fluid through a circular aperature (a screen). The flow satisfies Laplace's equation and has the boundary conditions;

$$V \; = \; g(r) \quad r < a \text{ and } z = 0$$

$$\frac{\partial V}{\partial z} \; = \; 0 \quad r > a \text{ and } z = 0$$

Apply the Hankel transform $J_0(\alpha r)$

$$\Phi(\alpha) \; = \; \int_0^\infty r\, dr\, \phi\, J_0(\alpha r)$$

$$\phi(r) \; = \; \int_0^\infty \alpha\, d\alpha\, \Phi\, J_0(\alpha r)$$

Using the form of the transformation for the derivatives, note the boundary conditions must be satisfied;

$$\frac{d^2 \Phi}{dz^2} \; - \; \alpha^2 \Phi \; = \; 0 \qquad\qquad F(\alpha) \; = \; \int_0^\infty r\, dr\, f(r) \,|_{z=0}$$

This has solution;

$$\Phi \,=\, A\, e^{-\alpha z} \,+\, B\, e^{\alpha z}$$

Choose $B = 0$ which results in the dual integral equations. Let a $\rho = r$ and $\alpha = u/a$.

$$G(\rho) \,=\, \int_0^\infty u\, du\, F(u)\, e^{-\alpha z}\, J_0(u\rho/a) \quad 0 < \rho < 1$$

$$0 \,=\, \int_0^\infty u\, du\, F(u)\, e^{-\alpha z}\, J_0(u\rho/a) \quad \rho > 1$$

5 Dual integral equations

Dual integral equations occur when different boundary conditions need to be applied over different regions. We have seen how they are developed in the above examples. A general form for these equations is ;

$$\int_0^\infty dy\, y^\alpha\, f(y)\, J_\nu(xy) \,=\, g(x) \;\; 0 < x < 1$$

$$\int_0^\infty dy\, f(y)\, J_\nu(xy) \,=\, 0 \;\; x > 1$$

These equations can be solved in particular cases by change of variable, particularly taking advantage of any symmetries in the problem. More generally, thay can be solved by a Mellin transform, which takes the form;

$$F(s) \,=\, \int_0^\infty dx\, f(x)\, x^{s-1}$$

$$f(s) \,=\, \frac{1}{2\pi 1} \int_{c-i\infty}^{c+i\infty} ds\, F(s)\, x^s$$

The development is complicated, requiring the result to be found by integration in the complex plane. Because of the detail required, and the scope of this class, this is not pursued further here. However, if your need to solve such equations your know where to begin.

6 Laplace Transform

It was pointed out that the Fourier transforms provides a solution for the steady-state problem. However, in many cases we look for transient solutions, so we need a different technique. In addition, the Fourier transformation of $f(x)$ requires that $\int_{-\infty}^{\infty} dx\, |f(x)|$ converge. Thus, suppose that we have the condition that $f(x) = 0$ for $x < 0$, and use;

$$f(x) \rightarrow \begin{bmatrix} e^{-\gamma x} f(x) & x > 0 \\ 0 & x < 0 \end{bmatrix}$$

In the above $\gamma > 0$. Then look at the Fourier transformation.

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-\gamma x} f(x)\, e^{-i\alpha x}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\alpha\, e^{\gamma x} F(\alpha)\, e^{i\alpha x}$$

Use these equations to write;

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha\, e^{\gamma x}\, e^{i\alpha x} \int_{0}^{\infty} dx'\, e^{-\gamma x'} f(x')\, e^{-i\alpha x'}$$

Define;

$$p = \gamma + i\alpha \quad dp = i d\alpha$$

$$\phi(p) = \int_{0}^{\infty} dx'\, f(x')\, e^{-px'}$$

$$f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} dp\, \phi(p)\, e^{px}$$

The function $\phi(p)$ is the Laplace transform of the function $f(x)$. In almost cases the transform will now converge. Problems with convergence and integration have been pushed into the inverse transform. Evaluation of the inverse requires integration in the complex plane. However, the inverse transform can be handled in many cases by the convolution theorem. Thus supppose the integrand in the inverse transform is a product of 2 known Laplace transforms.

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} dp\, \phi(p)\, \Psi(p) e^{px} =$$

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} dp\, \phi(p)\, e^{px} \int_{0}^{\infty} g(y)\, e^{-py}$$

Or;

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} dp\, \phi(p)\, \Psi(p) e^{px} =$$

$$\frac{1}{2\pi i} \int_{0}^{\infty} g(y) \int_{\gamma - i\infty}^{\gamma + i\infty} dp\, \phi(p)\, e^{p(x-y)}$$

Since $f(x-y) = 0$ if $(x-y) < 0$, the above is written;

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} dp\, \phi(p)\, \Psi(p) e^{px} \;=\; \int_0^{\infty} dy\, g(y)\, f(x-y)$$

7 Transformation of a derivative

Consider the transformation;

$$\phi^n(p) \;=\; \int_0^{\infty} dx\, \frac{d^r f}{dx^r}\, e^{-px}$$

Integration by parts yields;

$$[\frac{d^{r-1} f}{dx^{r-1}}\, p\, e^{-px}]_0^{\infty} \;+\; p \int_0^{\infty} dx\, \frac{d^{r-1} f}{dx^{r-1}}\, e^{-px}$$

$$\phi^{(r)}(p) \;=\; -\sum_{n=0}^{r-1} p^n\, f^{r-n-1}(0) \;+\; p^r\, \phi(p)$$

8 Laplace transform of the δ function

Suppose;

$$\int_0^{\infty} dx\, \delta(x-\epsilon)\, e^{-px} \;=\; e^{-p\epsilon}$$

The inverse transformation is;

$$\delta(x-\epsilon) \;=\; \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} dp\, e^{(\epsilon - x)p}$$

Let $i\alpha = p - \gamma$ and $id\alpha = dp$. Substitution;

$$\delta(x-\epsilon) \;=\; \frac{1}{2\pi}\, e^{-(\epsilon - x)\gamma} \int_{-\infty}^{\infty} d\alpha\, e^{-i(\epsilon - x)\alpha}$$

$$\delta(x-\epsilon) \;=\; e^{-(\epsilon - x)\gamma}\, \delta(x-\epsilon) \;=\; \delta(x-\epsilon)$$

9 Transformation of the step function

Suppose the function;

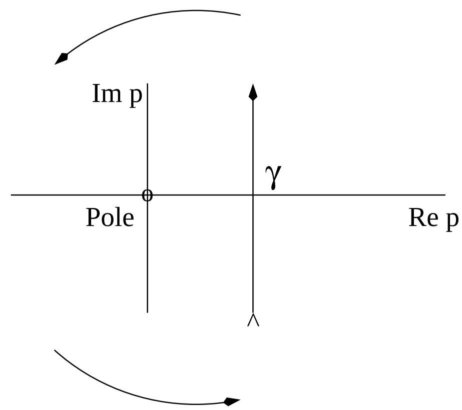

Figure 2: The contour to evaluate the inverse transformation of the step function

$$\lim_{t\to\infty} \frac{sin(tx)}{t} \;=\; 2\pi\delta(x)$$

The above is the derivative of the step function, so its integral should be the step function.

$$F(t) \;=\; \int_0^\infty dx\, \frac{sin(xt)}{x}$$

This is an improper integral since it does not converge. However, consider the Laplace transform;

$$\int_0^\infty dt\, e^{-pt}\, F(t) \;=\; \int_0^\infty dt\, e^{-pt} \int_0^\infty dx\, \frac{sin(xt)}{x}$$

Interchange the order of integration to obtain, $\frac{\pi}{2p}$. Now apply the inverse transformation.

$$f(t) \;=\; \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} dp\, (\pi/2)\, \frac{e^{pt}}{p}$$

Figure 2 shows the contour for the integration. The value of γ is moved so that the contour includes the singulatities, in this case at $p = 0$. For $t > 0$ close the contour in the negavive p plane as shown.

The calculus of residues is used to obtain;

$$f(t) \;=\; 2\pi i[(\pi/2)\, \frac{1}{2\pi i}] \;=\; \pi/2 \quad t > 0$$

For $t < 0$ replace p by $-p$. The result is $-\pi/2$ for $t < 0$. Note that;

$$\frac{dF(t)}{dt} \;=\; 2\pi\delta(t) \;=\; \pi \int_0^\infty dx\, cos(tx)$$

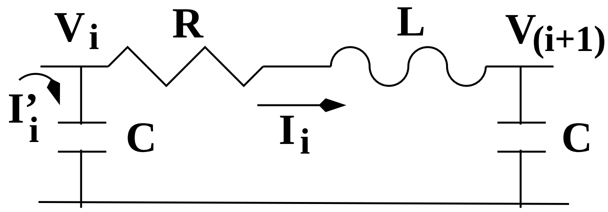

Figure 3: An incremental element of a resistive transmission line

10 Resistive transmission line

Suppose the i^{th} increment of a transmission line appears as in Figure 3. Use Kirchoff's circuit laws to write the equations for the voltage and currents which flow and then let the length of the element $\delta x \to 0$. Define a capacitance per unit length, inductance per unit length, and resistance per unit lenght as, $c = C/\delta x$, $l = L/\delta x$, $r = R/\delta x$, respectively.

The voltage equations are;

$$(V_{i+1} - V_i) = L_i \frac{dI_i}{dt} + I_i R$$

$$V_i = Q_i'/C$$

The current equations are;

$$\frac{dQ_i'}{dt} = I_i'$$

$$I_{i+1} = I_{i+1}' + I_i$$

Combine these, use the expressions for the components per unit length, c, l, r, and let the length of the element $\delta x \to 0$. This results in the pde for the wave equation previously obtained. Let $J = I$ in what follows.

$$\frac{\partial^2 J}{\partial x^2} = cl \frac{\partial^2 J}{\partial t^2} + cr \frac{\partial J}{\partial t}$$

Then apply the Laplace transform;

$$\mathcal{J}(x,p) = \int_0^\infty dt\, e^{-pt}\, J(x,t)$$

The initial conditions are set so that $J(x,0) = 0$. This removes the surface terms in the transform of the derivative.

$$\frac{d^2\mathcal{J}}{dt^2} = cl\,p^2\,\mathcal{J} + cr\,p\,\mathcal{J}$$

The solution is;

$$\mathcal{J} = Ae^{-\alpha x} + Be^{\alpha x}$$

Use the step function for the initial conditions;

$$J(0,t) = \int_0^\infty ds\,\frac{sin(st)}{s} = \begin{bmatrix} \pi/2 & t > 0 \\ 0 & t = 0 \\ -\pi/2 & t < 0 \end{bmatrix}$$

As we have seen the Laplace transform of this initial condition is;

$$\mathcal{J}(0,p) = \pi/2p$$

Thus from above the transform of the solution takes the form;

$$\mathcal{J}(x,p) = (\pi/2p)e^{-\alpha x} \quad \alpha^2 = cl\,p^2 + cr\,p$$

Now let $cl = v$ the signal velocity, and re-arrange the terms in the solution.

$$\mathcal{J}(x,p) = (\pi/2p)e^{-[\sqrt{(p^2+(r/l)p)}](x/v)}$$

The inverse transform has the form;

$$J(x,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} dp\mathcal{J}(x,p)e^{pt}$$

The inverse transformation is accomplished using the convolution theorem. Let $p = u - r/2l$ and break the integrand into 3 components with inverse as given in the equations below.

$$[e^{\sqrt{u^2-(r/2l)^2}x/v} - e^{-u\,x/v}] \rightarrow A(t) =$$

$$\begin{bmatrix} & 0 & 0 < t < x/v \\ \dfrac{(r/2l)(x/v)}{\sqrt{t^2-(x/v)^2}} & I_1((r/2l)\sqrt{t^2-(x/v)^2}) & t > x/v \end{bmatrix}$$

$$\frac{e^{-u(x/v)}}{u-(r/2l)} \rightarrow \begin{bmatrix} 0 & 0 < t < x/v \\ 1 & t > x/v \end{bmatrix}$$

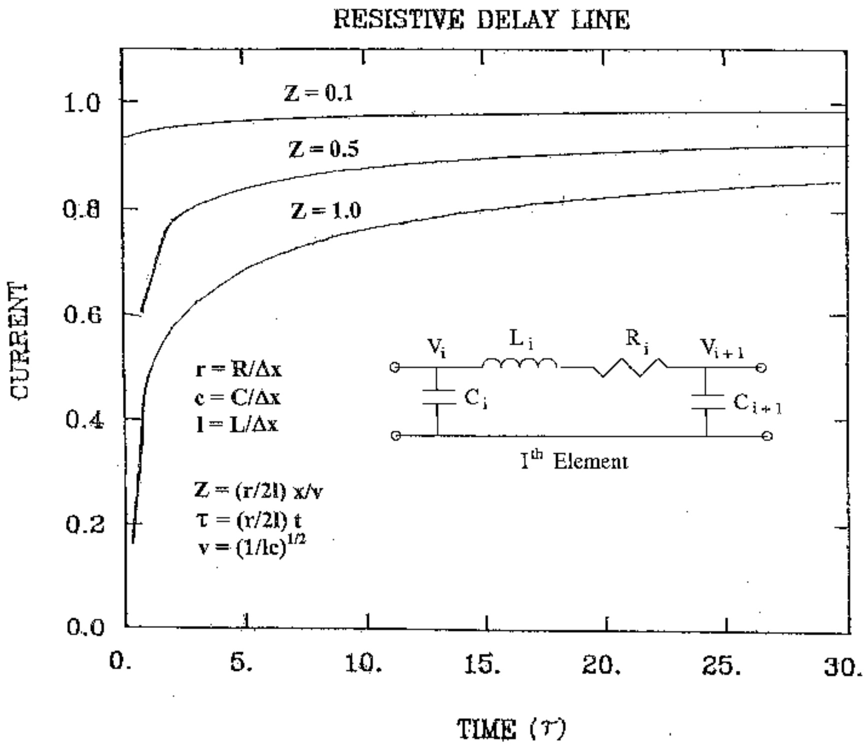

Figure 4: The solution to the resisttve line for various resistances. The curves all begin at the time when the signal appears. All parameters are normalized to be unitless

$$\frac{1}{u-(r/2l)} \rightarrow e^{(r/2l)t}$$

In the above, I_1 is the modified bessel function of order 1. Then combining using the convolution theorem.

$$J(x,t) = (\pi/2) \int_0^t d\tau\, e^{(r/2l)(t-\tau)}\, A(\tau) + (\pi/2)\, e^{-(r/2l)(x/v)}\, \Theta(t - x/v)$$

Θ vanishes for negative argument and equals 1 for postitive argument. The solution is shown in Figure 4 where all parameters are normalized to unitless dimensions. The curves show the change in slope as the step function travels through the resistive line and are ploted at the time point when the signal arrives. The result clearly obeys causality being zero until the leading edge of the signal travels to the point x in time t at velocity v.