Dielectric Problems and Electric Susceptability

Lecture 10

1 A Dielectric Filled Parallel Plate Capacitor

Suppose an infinite, parallel plate capacitor with a dielectric of dielectric constant ϵ inserted between the plates. The field is perpendicular to the plates and to the dielectric surfaces. Thus use Gauss' Law to find the field between the plates in the dielectric. For a cylindrical Gaussian surface surrounding an area of the plate surface;

$$\oint \vec{D} \cdot d\vec{A} = q_{free}$$

Note the field is only between the plates with a value obtained by superposition of the field from both plates.

$$E = \sigma_{free}/\epsilon$$

The potential between the plates is therefore;

$$V = \int \vec{E} \cdot d\vec{l} = \sigma d/\epsilon$$

where d is the plate separation. The capacitance is;

$$C = Q/V = (Area)\epsilon/d$$

For a given value of V, the dielectric reduces the total field between the plates, so.the capacitor stores additional charge on the plates for the same applied voltage. This is developed further in the sections below. However for the moment, you should consider how energy is conserved if it is determined using the square of the field intensity.

2 A Dielectric Sphere in a Uniform Electric Field

In a previous lecture we considered a conducting sphere in a uniform electric field. The field caused charge to move so that there was no $\vec{E}$ component parallel to the surface and no field inside the conductor. In the case of a dielectric sphere with dielectric constant $\epsilon = \epsilon_0 \epsilon_r$, Figure 1, charge cannot move but polarization in the material occurs. Therefore we expect to find a field within a polarized dielectric. There is no free charge in, or on, the sphere, so apply Laplace's equation (vanishing free charge density) with appropriate boundry conditions.

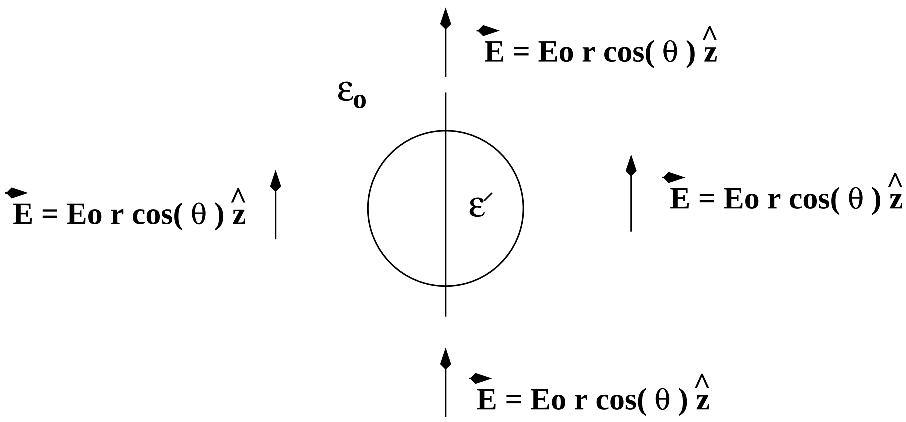

Figure 1: The geometry of a dielectric sphere placed in a uniform field

$$\nabla^2 V \;=\; \rho/\epsilon$$

Solve this equation using separation of variables with the boundary conditions;

$$V \;=\; -E_0\, z \;=\; -E_0\, r\, cos(\theta) \text{ as } r \;\rightarrow\; \infty;$$

In the above, E is finite as $r = 0$; and

$$E_\perp \;=\; (\epsilon'/\epsilon_0)\, E'_\perp, \qquad E_\parallel \;=\; E'_\parallel \text{ at } r = a$$

In spherical coordinates the solution to Laplaces's equation using separation of variables with azmuthal symmetry has the form;

For $r < a$

$$V \;=\; \kappa \sum A_l\, r^l\, P_l(x)$$

For $r > a$

$$V \;=\; \kappa \sum B_l\, r^{-(l+1)}\, P_l(x)$$

Then, $x = cos(\theta)$. Apply the boundry conditions to obtain the equation;

$$V \;=\; (\kappa)[A_0 + A_1\, r\, cos(\theta)] \text{ for } r < a$$

$$V \;=\; (\kappa/r)[B_0 + B_1/r]\; cos(\theta) - V_0\, r cos(\theta) \text{ for } r > a$$

The solutions match the boundary conditions when $r \to \infty$ and $r \to 0$. All other cofficients, A_l ,B_l, must vanish. Then match the potential and field when $r = a$. Use $\vec{E} = -\epsilon \vec{\nabla} V$ so that;

$$\epsilon \frac{\partial V}{\partial r}_{in} = \epsilon_0 \frac{\partial V}{\partial r}_{out}$$

By definition, $\epsilon_r = \epsilon/\epsilon_0$ which gives;

$$\epsilon_r \sum A_l \, l \, a^{l-1} \, P_l = -\sum B_l \, (l+1) \, a^{-(l+2)} \, P_l - E_0 \, cos(\theta)$$

The requirement that tangential E is continous is equivalent to the continuity of the potential.

$$\sum A_l \, a^l \, P_l = \sum B_l \, a^{-(l+1)} \, P_l - E_0 \, P_l$$

Equate the constants for each value of l.

$$A_0 = B_0/a \text{ and } B_0 a^{-2} = 0$$

$$A_1 a = B_1 a^{-2} - E_0 a \text{ and } -\epsilon_r A_1 = 2B_1 a^{-3} + E_0$$

$$A_2 a^2 = B_2 a^{-3} \text{ and } -\epsilon_r a A_2 = -3B_2 a^{-4}$$

This means that ;

$$B_0 = 0; \; A_0 = 0$$

$$A_1 = B_1/a^3 - E_0$$

All other values of A_l and B_l are zero. Finally;

$$B_1 = \frac{(\epsilon_r - 1)}{(\epsilon_r + 2)} \, a^3 E_0$$

$$A_1 = -\frac{3}{(\epsilon_r + 2)} \, E_0$$

The potential is then;

$r < a$

$$V = -\frac{3 \, E_0}{(\epsilon_r + 2)} \, r \, cos(\theta)$$

$r > a$

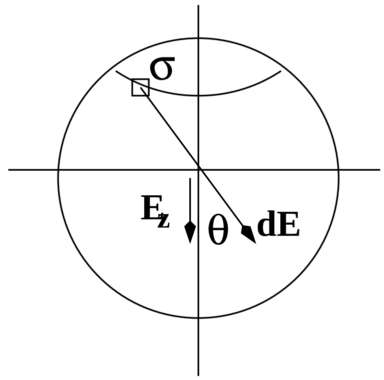

Figure 2: The geometry used to find the field at the center of the polarized sphere

$$V = -E_0\, r cos(\theta) + \frac{(\epsilon_r - 1)}{(\epsilon_r + 2)}\, E_0\, (a/r)^3\, r\, cos(\theta)$$

3 Polarization of the Dielectric Sphere in a Uniform Electric Field

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The field inside the dielectric sphere, as obtained in the last section, is;

$$\vec{E} = -\vec{\nabla} V$$

with $V_{in} = -\frac{3E_0}{(\epsilon_r + 2)}\, r\, cos(\theta)$. Thus $\vec{E} = \frac{3E_0}{(\epsilon_r + 2)}\, \hat{z}$

The field is uniform and in the $\hat{z}$ direction. The volume charge density is given by $\rho = -\vec{\nabla} \cdot \vec{P} = 0$, since the field within the sphere is constant. The Polarization is given by;

$$\vec{P} = (\epsilon - \epsilon_0)\vec{E}$$

Thus the volume charge density vanishes, but there is a surface charge density given by;

$$\sigma = \vec{P} \cdot \hat{n} = P\, cos(\theta)$$

where $\hat{n}$ is the outward surface normal. The field inside the sphere is due to the surface charge which forms a dipole field, Figure 2, in addition to the applied field. Calculate this field at the center of the sphere 2. The field due to a small element as shown in the figure is;

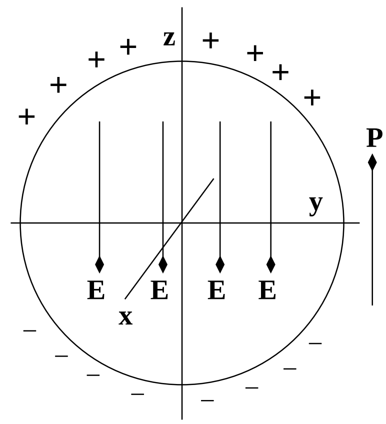

Figure 3: The geometry used to find the field at the center of the polarized sphere

$$dE_z \,=\, -\kappa \frac{P\, cos(\theta)}{r^2}\, cos(\theta)\, r^2 d\Omega$$

Integrate over the solid angle $d\Omega$;

$$E_z \,=\, -\frac{\vec{P}}{3\epsilon_0}$$

Although this field was found at the center of the sphere, it is the same for all points in the sphere, since the field inside the sphere is constant as obtained in the solution using separation of variables. Note this solution also shows the polarization field is directed opposite to the applied field, Figure 7.

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4 Connection between the Electric Susceptibility and Atomic Polarizability

An applied field induces a polarization in a dielectric material. To better understand this process consider the polarization at the center of a polarized spherical dielectric. Then combine this polarization with the applied field to obtain for a Class A dielectric;

$$\vec{D} \,=\, \epsilon_0 \vec{E} \,+\, \vec{P} \,=\, \epsilon \vec{E}$$

In this case the polarization is independent of surface effects. Assume the applied field is $\vec{E}_0$, and $\vec{E}'$ is the field due to the polarized material. The total field inside the dielectric is the superposition of these fields. Now the total field (applied plus induced) causes the polarization, so the effect is non-linear due to this self interaction. The field in the dielectric is, $\vec{E}_{total}$.

$$E_{total} \; = \; \vec{E}_0 \; + \; \vec{E}_{pol}$$

The field at the center of the dielectric sphere, as solved above is;

$$E_z \; = \; -\frac{P}{3\epsilon_0}$$

The dipole moment of the sphere is obtained from the atomic polarizability, α.

$$\vec{p} \; = \; \alpha\vec{E}_{total} \; = \; \alpha(\epsilon_0\vec{E}_{total} + \frac{\vec{P}}{3})$$

The polarization is the dipole moment per unit volume, $\vec{P} \; = \; N\vec{p}$, where N is the number density of dipoles. Now solve for the polarization.

$$\vec{P} \; = \; \frac{N\alpha}{1 \; - \; N/(3)} \, \epsilon_0\vec{E}_{total}$$

The electric susceptibility χ_e is defined by;

$$\chi_e \; = \; \frac{N\alpha}{(1 - N/3)}$$

The field inside the sphere as previously obtained is $\frac{3E_0}{\epsilon_r + 2}$. The applied field is E_0. Remove the vector directions, as all fields are along the z axis.

$$P \; = \; \frac{3E_{in}}{\epsilon_r + 2} \; - \; E_{in} \; = \; -3\,\frac{\epsilon_r - 1}{\epsilon_r + 2}\,\epsilon_0 E_0$$

The polarization is then;

$$\vec{P} \; = \; \epsilon_0\chi_e\vec{E}_{total} \; = \; (\epsilon - \epsilon_0)\vec{E}_{total}$$

which can be used to obtain, $\chi_e \; = \; (\epsilon_r - 1)$

Now suppose we replace the total field in the material, E_{in}, with applied field, $\vec{E}_0$. We the obtain in first order, a polarization, P_1 .

$$P_1 \; = \; \epsilon_0\chi_e E_0 \; = \; (\epsilon_r - 1)\epsilon_0 E_0$$

However, this does not equal the above value for the final polarization, *ie* $P \neq P_1$. Thus polarization acts to create new polarization (*ie* a non-linear effect). Iterate the above, first order polarization, to obtain a first order electric field, E_1 due to the polarization. This gives the next order in the polarization iteration.

$$E_1 = \frac{P_1}{3\epsilon_0} = -\frac{(\epsilon_r - 1)}{3} E_0$$

This creates an incremental polarization, P_2 ;

$$P_2 = (\epsilon_r - 1)\epsilon_0 E_1 = \frac{(\epsilon_r - 1)^2}{3} \epsilon_0 E_o$$

and this creates an additional E field;

$$E_2 = -(\frac{\epsilon_r - 1}{3})^2 E_0$$

Continuing the iterations;

$$\vec{P} = 3 \sum_n (-\frac{\epsilon_r - 1}{3})^n \epsilon_0 \vec{E}_0 = \frac{3(\epsilon_r - 1)}{\epsilon_r + 2} \epsilon_0 \vec{E}_0$$

To summarize, we find the following solution in the interior of a dielectric sphere in a uniform electric field.

$$\vec{E}_{in} = \frac{3E_0}{\epsilon_r + 2}$$

From the definition of the electric displacement, $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$, so that $\vec{P}$ is;

$$\vec{P} = \epsilon_0 (\epsilon_r - 1) \vec{E}$$

For the dielectric sphere;

$$P = 3\frac{(\epsilon_r - 1)}{\epsilon_r + 2} \epsilon_0 E_0$$

5 Energy in a Dielectric

Return to a parallel plate capacitor filled with a dielectric constant, ϵ, and plate separation, d. The capacitance is ;

$$C = Q/V$$

$$V = Ed$$

$$C = Q/Ed$$

Use Gauss's Law to get E;

$$\oint \vec{D} \cdot d\vec{A} = Q_{free}$$

$$E = \sigma/\epsilon$$

$$C = \epsilon(Area)/d$$

Without the dielectric the capacitance is

$$C_0 = \epsilon_0(Area)/d$$

Therefore;

$$C = \epsilon_r C_0 = \frac{\epsilon_0 E_t + P}{E_t} C_0$$

In the above, E_t is the total field in the capacitor. From this we obtain;

$$\vec{D} = \epsilon_r \epsilon_0 \vec{E}_t = (\epsilon_0 \vec{E}_t + \vec{P})$$

Then $\vec{P}$ points in opposition ot $\vec{E}$

$$\epsilon_r = (\epsilon_0 + P/E_t) = \epsilon_0(1 + \chi_e)$$

The above equation connects the permittivity (dielectric constant) to the susceptibility. The energy of a parallel plate capacitor is obained by;

$$W = 1/2\, CV^2 = 1/2\, \epsilon_r C_0 V^2$$

$$W = (\epsilon/2) \int d\tau\, E^2$$

When one keeps the same voltage across the capacitor, there is an increase in energy $W = \epsilon_r W_0$ in a dielectric filled capacitor. Look at this additional energy. The differential energy to polarize a dipole $\vec{p}$ in the direction of the applied field is ;

$$dW = \vec{F} \cdot d\vec{x} = q\vec{E} \cdot d\vec{x} = Edp$$

$$dW = E\, dp$$

Then use the dipole density, N, to obtain the potential energy per unit volume due to the polarization, $\vec{P} = N\vec{p}$.

$$\int d\mathcal{W} = \int dP\, E = \int dE\, \epsilon_0(\epsilon_r - 1)E$$

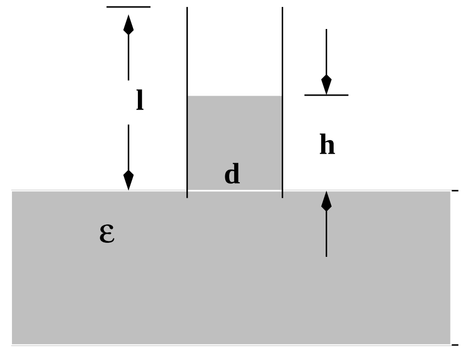

Figure 4: A parallel plate capacitor dipped into a dielectric liquid

$$\mathcal{W} = (1/2)\epsilon_0(\epsilon_r - 1)E^2$$

This is to be added to the energy density of the vacuum field $(\epsilon_0/2)E^2$ which gives the expected result $(\epsilon/2)E^2$. Thus the additional energy is stored in the polarization of the material.

6 Example of Energy in a Dielectric

Suppose a charged parallel plate capacitor is dipped into a dielectric liquid. The liquid is pulled up into the capacitor. The final position of the liquid can be determined by minimizing the system energy. The geometry is shown in figure 4. In this problem the voltage is disconnected from the capacitor so the charge remains constant, but the voltage changes as the liquid fills the volume between the plates. On the other hand, if the voltage supply remains connected to the capacitor, then the voltage remains constant, but the charge changes. In this case the battery charging the plates continues to insert energy into the system and so the energy in the capacitor is not constant.

From the figure, the system can be considered as 2 capacitors connected in parallel. Assume the width of the capacitor plates is w. The capacitance values for each capacitor are;

$$C_1 = \frac{\epsilon_0 w(l-h)}{d}$$

$$C_2 = \frac{\epsilon w h}{h}$$

So that the system capacitance is $C_t = C_1 + C_2$.

$$C_t = C_0[1 + (h/l)(\epsilon_r - 1)]$$

Where we have used $C_0 = \dfrac{\epsilon_0 l w}{d}$. Write the energy stored in the capacitor as a function of h in terms of the stored charge, $W = (1/2)Q^2/C$.

$$W = \frac{Q^2}{2C_0[1 + (h/l)(\epsilon_r - 1)]}$$

Since $\epsilon_r > 1$ the energy decreases as h increases. The difference in the energy goes into raising the liquid. The system energy is then;

$$W_S = W + mg(h/2) = W + g(h/2)\rho(wdh)$$

In the above ρ is the mass density and the second term on the left represents the potential energy of the raised liquid. The minimum in the energy is then found which provides the equilibrium position.

$$\frac{\partial W_S}{\partial h} = 0$$

$$-\frac{Q^2 l(\epsilon_r - 1)}{2C_0[l + h(\epsilon_r - 1)]^2} + 2\rho w d(h/2) = 0$$

Solve for h to find the equilibrium position. This results in a cubic equation for h.

$$h^3 + \frac{2\,l}{(\epsilon_r - 1)}h^2 + \frac{l^2}{(\epsilon_r - 1)^2}h - \frac{2Q^2 l}{4\rho w^2 \epsilon_0(\epsilon_r - 1)C_0} = 0$$

There is one real root of the equation if $q^3 + r^2 > 0$ where;

$$q = -(1/3)[\frac{l}{\epsilon_r - 1}]^2$$

$$r = (1/9)(\frac{l}{(\epsilon_r - 1)})^3 - (3/4)\frac{2Q^2 l}{\rho w^2 g \epsilon_0(\epsilon_r - 1)}$$

This will be the case in all physical situations. The solution is obtained as follows.

$$a_2 = \frac{2l}{\epsilon_r - 1}$$

$$s_1 = [r + (q^3 + r^2)^{1/2}]^{1/3}$$

$$s_2 = [r + (q^3 - r^2)^{1/2}]^{1/3}$$

$$h = (s_1 + s_2 - a_2/3$$

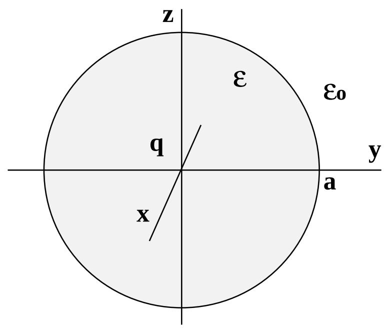

Figure 5: The geometry of a problem with a point charge q placed at the center of a spherical tank of water

7 Point Charge placed at the center of a Spherical Tank of Water

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The geometry of the problem is shown in figure 5. Use Gauss' law to get the electric displacement in the water. The electric displacement (and electric field) is radial and independent of angle. Thus assume a small spherical shell centered on the charge. Because the field is radial, the electric displacement, D' equals the electric diaplacement in the water, D. This means;

$$\oint \vec{D} \cdot d\vec{A} = Q_{free} = q$$

Thus because of symmetry;

$$\vec{D} = \frac{1}{4\pi} \frac{q}{r^2} \hat{r}$$

and $\vec{D} = \epsilon\vec{E}$. Then the polarization is,

$$\vec{P} = \epsilon_0(\epsilon_r - 1)\vec{E} = \epsilon_0(\epsilon_r - 1)\frac{1}{4\pi\epsilon} \frac{q}{r^2} \hat{r}$$

The volume charge density is ;

$$\rho = -\vec{\nabla} \cdot \vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}[r^2 P_r] = 0$$

Thus there is no volume charge density. The surface charge density at $r = a$ is;

$$\sigma = \vec{P} \cdot \hat{r} = \epsilon_0(\epsilon_r - 1)\vec{E} = \epsilon_0(\epsilon_r - 1)\frac{1}{4\pi\epsilon} \frac{q}{a^2}$$

Inside there is an induced charge symmetrically placed about q at a finite radius so that the total induced charge sums to zero. Outside the water tank the field is the same as the field from a point charge q in the vacuum.

8 Dipole placed at the center of a Spherical Tank of Water

This problem is similar to the problem in the last section, but the point charge is replaced by a dipole aligned along the $\hat{z}$ axis. The field of the dipole in vacuum is;

$$\vec{E}_d \,=\, \kappa \frac{p}{r^3}[2\,cos(\theta)\hat{r} \,+\, sin(\theta)\,\hat{\theta}]$$

Put this dipole inside a small spherical volume of radius, b, in the center of the tank. This keeps the solution appropriately bounded as $r \to 0$. Thus the boundry conditions at $r = b$ are;

$$\epsilon E_r(water) \,=\, \epsilon_0 E_r(vacuum)$$

$$E_{\|}(water) \,=\, E_{\|}(vacuum)$$

Therefore inside the water;

$$\vec{E}(water) \,=\, \frac{\kappa p}{r^3}[2(\epsilon_0/epsilon)\,cos(\theta)\,\hat{r} \,+\, sin(\theta)\,\hat{\theta}]$$

Solve for the potential in the water using separation of variables. The solution has the forms;

$$r > a$$

$$V \,=\, \sum A_l\, r^{-(l+1)}\, P_l(x)$$

$$b < r < a$$

$$V \,=\, \sum [B_l\, r^{-(l+1)} \,+\, C_l\, r^l] P_l(x)$$

Now match the boundary conditions at $r = b$.

$$\epsilon[2B_1/b^3 \,-\, C_1] cos(\theta) \,=\, 2\epsilon_0 \kappa p/b^3\, cos(\theta)$$

$$(1/b)[B_1/b^2 \,+\, C_1 b] sin(\theta) \,=\, \kappa p/b^3\, sin(\theta)$$

All other coefficients vanish. Solve for B_1 and C_1.

$$B_1 \;=\; \frac{\kappa p}{3\epsilon_r}[\epsilon_r + 2]$$

$$C_1 \;=\; \frac{2\kappa p}{3\epsilon_r b^3}[\epsilon_r - 1]$$

This gives the potential;

$$V \;=\; \frac{\kappa p}{3\epsilon_r}[\frac{\epsilon_r + 2}{r^3} \;+\; \frac{2(\epsilon_r - 1)}{b^3} r] cos(\theta)$$

From this one gets the field;

$$\vec{E} \;=\; \frac{\kappa p}{3\epsilon_r}[\,[\frac{\epsilon_r + 2}{r^2} \;+\; \frac{2(\epsilon_r - 1)r}{b_3}]\, cos(\theta)\, \hat{r} +$$
$$[\frac{\epsilon_r + 2}{r^3} + \frac{2(\epsilon_r - 1)}{b^3}]\, sin(\theta)\, \hat{\theta}]$$

The polarization is $\vec{P} \;=\; \epsilon_0(\epsilon_r - 1)\vec{E}$. So that the volume charge density is;

$$\rho \;=\; -\vec{\nabla} \cdot (\epsilon_0(\epsilon_r - 1)\vec{E})$$

$$\rho \;=\; \frac{\epsilon_0(\epsilon_r - 1)\,\kappa p}{3\epsilon_r r^2}[[\frac{2(\epsilon_r + 2)}{r^2} \;-\; \frac{2(\epsilon_r - 1)}{b^3}]\, cos(\theta) +$$
$$[\frac{\epsilon_r + 2}{r^3} \;+\; \frac{2(\epsilon_r - 1)}{b^3}] cos(\theta)]$$

$$\rho \;=\; \frac{\epsilon_0(\epsilon_r - 1)\,\kappa p}{3\epsilon_r r^2}[3(\epsilon_r + 2) \;+\; 6(\epsilon_r - 1)(r/b)^3]\, cos(\theta)$$

The surface charge density is;

$$r = b$$

$$\sigma \;=\; -(\epsilon - \epsilon_0)\frac{8\kappa p}{3\epsilon_r b^3}\, cos(\theta)$$

$$r = a$$

$$\sigma \;=\; -(\epsilon - \epsilon_0)\frac{4\kappa p}{3\epsilon_r a^3}[\epsilon_r(1 - \epsilon_r(a/b)^3) \;+\; 2(a/b)^3]\, cos(\theta)$$

Matching the boundry conditions at $r = a$ must now be carefully done. As the field does not $\rightarrow 0$ as $r \rightarrow \infty$ but has a dipole form, with the potential given by;

$$V \;=\; \frac{2\kappa p(\epsilon_r - 1)}{b^3}\, r\, cos(\theta)$$

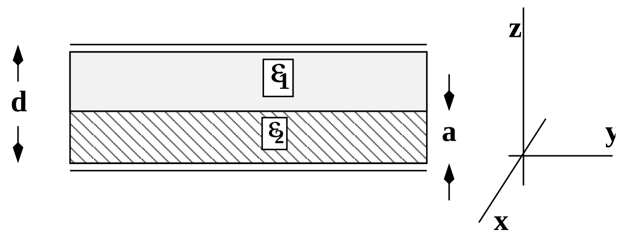

Figure 6: The geometry used to find the capacitance of a parallel palte capacitor filled with 2 dielectrics

This potential should be subtracted from the dipole potential. There still remains a problem in defining a dipole as a point.

9 Examples

9.1 Parallel plate capacior filled with 2 dielectric materials

A parallel plate capacitor is filled with 2 dielectric materials of dielectric constants ϵ_1, and ϵ_2 as shown in figure 6. The plates have area, A, and are separated by a distance, d. The thickness of dielectric, ϵ_2 is a. Find the capacitance.

Place a potential, V_0, between the plates. By symmetry, the potential is assumed to be dependent only on z as the plate lengths are $>> d$. Thus the electric field is perpendicular to the plates. The solution to Laplace's equation, $\nabla^2 V = 0$, must have the form;

$$V = Az + B$$

In the above A and B are constants which are used to satisfy the boundary conditions. Thus we expect;

$$V_2 = A_2 z + B_2 \text{ for } 0 < z < d - a)$$

$$V_1 = A_1 z + B_1 \text{ for } d - a < z < d$$

Then when $Z = 0$

$$V_2 = B_2 = 0$$

and when $z = d$

$$V = A_1 d \, + \, B_1 \, = \, V_0$$

Solving for the constants we obtain for the potentials;

$$V_1 \, = \, [\frac{V_0 - B_1}{d}]z \, + \, B_1$$

$$V_2 \, = \, A_2 z$$

The fields are;

$$E_1 \, = \, -\frac{\partial V_1}{\partial z} \, = \, -[\frac{V_0 - B_1}{d}]$$

$$E_1 \, = \, -\frac{\partial V_2}{\partial z} \, = \, -A_2$$

The remaining constants, A_2 and B_1, are determimed at the dielectric boundary, $z = a$, where;

$$D_1 \, = \, \epsilon_1 E_1 \, = \, D_2 \, = \, \epsilon_2 E_2$$

$$A_2 \, = \, (\epsilon_1/\epsilon_2)[\frac{V_0 - B_1}{d}]$$

We also require; $\int_d^0 d\vec{l} \cdot \vec{E} \, = \, V_0$. Then solve the 2 above equations for the constants.

$$B_1 \, = \, \frac{V_0 a(\epsilon_1 - \epsilon_2)}{[\epsilon_1 a \, + \epsilon_2(d - a)]}$$

$$A_2 \, = \, \frac{\epsilon_1 V_0}{[\epsilon_1 a \, + \epsilon_2(d - a)]}$$

This yields the potentials;

$$V_1 \, = \, \frac{V_0[\epsilon_2 z + (\epsilon_1 - \epsilon_2)a]}{[\epsilon_1 a \, + \, \epsilon_2(d - a)]}$$

$$V_2 \, = \, \frac{\epsilon_1 V_0}{[\epsilon_1 a \, + \, \epsilon_2(d - a)]} \, z$$

The fields are;

$$E_2 \, = \, \frac{\epsilon_1}{[\epsilon_1 a + \epsilon_2(d - a)]} \, V_0 z$$

$$E_1 \, = \, \frac{\epsilon_2}{[\epsilon_1 a + \epsilon_2(d - a)]} \, V_0 z$$

These solutons should be checked for the limiting cases when $a = 0$, $a = d$, and $\epsilon_1 = \epsilon_2$

Now the charges on the plates are obtained from Gauss' law, $\int \vec{D} \cdot d\vec{A} = Q_{free}$.

Bottom Plate;

$$\sigma_F = \epsilon_2 E_2 = \frac{\epsilon_1 \epsilon_2}{\epsilon_1 a + \epsilon_2 (d-a)} V_0$$

Top Plate;

$$\sigma_F = \epsilon_1 E_1 = \frac{\epsilon_1 \epsilon_2}{\epsilon_1 a + \epsilon_2 (d-a)} V_0$$

So the free charge on the top and bottom plates are equal as they should be. Then using the free charge, the capacitance per unit area is;

$$C/A = \sigma_F / V_0 = \epsilon_1 E_1 = \frac{\epsilon_1 \epsilon_2}{\epsilon_1 a + \epsilon_2 (d-a)} V_0$$

9.2 The field of a polarized sphere

Suppose a sphere of radius, R, composed of polarized material. The polarization has only and angular dependence given by;

$$\vec{P} = P_0 \, cos(\theta) \, \hat{r}$$

There is a surface charge density;

$$\sigma = \vec{P} \cdot \hat{r} = P_0 \, cos(\theta)$$

The volume charge density is;

$$\rho = -\vec{\nabla} \cdot \vec{P} = -\frac{2 P_0 \, cos(\theta)}{r}$$

We are to find the potential outside the sphere, figure 7. The volume contribution to the potential has the form;

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$$V = \kappa \int d\Omega' \, r'^2 \, \frac{\rho}{|\vec{r}' - \vec{r}|}$$

Substitute for the volume charge density and use the addition theorem to expand $\frac{1}{|\vec{r}' - \vec{r}|}$

$$\frac{1}{|\vec{r}' - \vec{r}|} = \sum_{l,m} \frac{4\pi}{2l+1} \, (r'^l / r^{(l+1)}) \, Y_l^{*\,m}(\theta', \phi') \, Y_l^m(\theta, \phi).$$

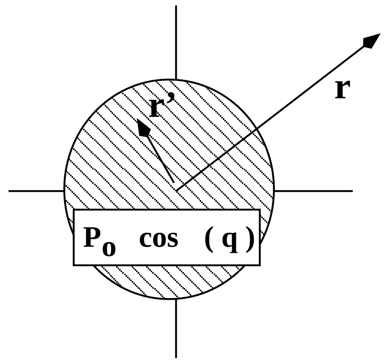

Figure 7: Geometry to obtain the potential of a polarized sphere

Note, $cos(\theta) = \sqrt{4\pi/3}\, Y_l^0$. In the following use the orthogonality of the spherical harmonics.

$$V = 8\pi\kappa(P_0/r) \sum \frac{1}{2l+1}$$
$$[\int d\Omega' \, cos(\theta', \phi') \, Y_l^{*\,m}(\theta', \phi')][\int dr' \, (r'^{\,l+2}/r^l)] \, Y_l^m$$

$$V = 8\pi\kappa(\frac{P_0 R^3}{3r^3}) \, cos(\theta)$$

The integral for the surface proceeds in a similar way.

$$V = -\kappa \int R^2 \, d\Omega' \frac{P_0 \, cos(\theta)}{|\vec{r'} - \vec{r}||}$$
$$V = -4\pi\kappa \, \frac{P_0 R^3 cos(\theta)}{3r^3}$$

The potential is the sum of the potentials from the volume and surface charge densities.