Solutions to Laplace's Equation in Cylindrical Coordinates and General Numerical solutions

Lecture 8

1 Introduction

We obtained general solutions for Laplace's equation by separtaion of variables in Cartesian and spherical coordinate systems. The last system we study is cylindrical coordinates, but Laplaces's equation is also separable in a few (up to 22) other coordinate systems as previously tabulated. Thus one chooses the system in which the appropriate boundry conditions can be applied. It is only by the application of the boundry conditions (Dirichlet of Neumann on a closed surface) that one finds a unique solution to a problem.

In cylindrical coordinates the divergence of the gradient of the potential yields;

$$\nabla^2 V(\rho, \phi, z) \,=\, \rho\,\frac{\partial^2 V}{\partial \rho^2} \,+\, \frac{\partial V}{\partial \rho} \,+\, (1/\rho)\,\frac{\partial^2 V}{\partial \phi^2} \,+\, \frac{\partial^2 V}{\partial z^2} \,=\, 0$$

Look for solutions by separation of variables;

$$V \,=\, \mathcal{R}(\rho)\Psi(\phi)\mathcal{Z}(z)$$

As previously, one finds 2 separation constants, k and ν, which result in 2 eigenfunction ode equations. The three separated ode equations are;

$$\frac{d^2\mathcal{Z}}{dz^2} - k^2\mathcal{Z} \,=\, 0$$

$$\frac{d^2\Psi}{d\phi^2} \,+\, \nu^2\Psi \,=\, 0$$

$$\frac{d^2\mathcal{R}}{d\rho^2} \,+\, (1/\rho)\,\frac{\mathcal{R}}{d\rho} \,+\, (k^2 - (\nu/\rho)^2)\mathcal{R} \,=\, 0$$

The later 2 equations can be set up as eigenfunction equations. Solutions are;

$$\mathcal{Z} \,\propto\, e^{\pm kz}$$

$$\Psi \,\propto\, e^{\pm i\nu\phi}$$

$$\mathcal{R} \propto\, J_\nu(k\rho), \text{ or/and } N_\nu(k\rho)$$

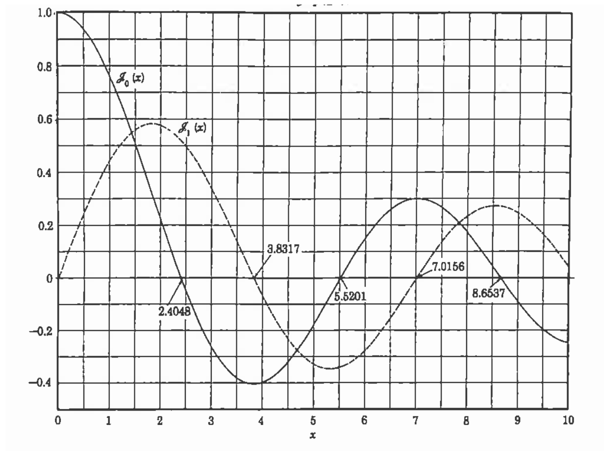

Figure 1: An example of the Cylindrical Bessel function $J(x)$ as a function of x showing the oscillaltory behavior

2 Bessel Functions

In the last section, $J_\nu(k\rho)$, $N_\nu(k\rho)$ are the 2 linearly independent solutions to the last separated ode above - Bessel's equation. Bessel functions oscillate, but are not periodic like harmonic functions, see Figure 1. Thus we expect that the harmonic function solution for Ψ and the Bessel function solution for $\mathcal{R}$ are the eigenfunctions when the boundry conditions are imposed. The Bessel functions, $J_\nu(x)$, are regular at $x = 0$, while the Bessel functions, $N_\nu(x)$, are singular at $x = 0$.

From the requirement that the solution is single valued as $\phi \to 2\pi$, *ie* the solution must not change when ϕ is replaced by $\phi + 2\pi$, the values of ν are integral and this produces eigenfuntions in Ψ. The series solution for the Bessel function J_ν can be found by the method of Frobenius. However, the second linearly independent equation is not easily obtained when n is an integer, and another technique is required. The series representation of the Bessel function takes the form;

$$J_\nu(x) \;=\; \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(\nu+s)!}\,(x/2)^{\nu+2s} \;=\; \frac{x^\nu}{2^n n!} \;-\; \frac{x^{\nu+2}}{2^{\nu+2}(\nu+1)!} \;+\; \cdots$$

For integral values of ν one can show $J_{-\nu} \;=\; (-1)^\nu J_\nu$. Bessel functions also satisfy the recurrence relations;

$$J_{\nu-1}(x) \;+\; J_{\nu+1}(x) \;=\; \frac{\nu}{x}\,J_\nu(x)$$

$$J_{\nu-1}(x) \;=\; \frac{x}{\nu}\,J_\nu(x) \;+\; \frac{dJ_\nu(x)}{dx}$$

$$\frac{d}{dx}[x^\nu J_\nu(x)] \;=\; x^\nu J_{\nu-1}(x)$$

At times an integral representation is useful.

$$J_\nu(x) \;=\; (1/\pi) \int_0^\pi d\theta \, e^{ix\,cos(\theta)}\, cos(\nu\theta)$$

Bessel functions are orthogonal, but not normalized.

$$\int_0^\infty kdk\, J_\nu(k\rho)\, J_\nu(k\rho') \;=\; \delta(\rho-\rho')/r$$

$$\int_0^a \rho d\rho\, J_\nu(\alpha_{\nu k}\rho/a)\, J_\nu(\alpha'_{\nu k}\rho/a) \;=\; (a^2/2)[J_{\nu+1}(\alpha_{\nu k})]^2\, \delta_{kk'}$$

In the above $\alpha_{\nu k}$ are the zeros of the Bessel function of order ν where k orders these zeros. As the Bessel functions form a complete set, any function can be expanded in a Bessel series or Bessel integral if the space is infinite.

$$F(\rho) \;=\; \int kdk\, A(k)\, J_\nu(k\rho)$$

3 Integral Representation

Let;

$$f_n \;=\; (1/\pi) \int_0^\pi d\omega \, cos(x\, sin(\omega) - n\omega)$$

Then look at the relations for f_{n+1} and f_{n-1}. Subtracting these we obtain;

$$f_{n+1} - f_{n-1} \;=\; (1/\pi) \int_0^\pi d\omega\, [cos(x\, sin(\omega) - n\omega)\, cos(\omega) + sin(x\, sin(\omega) - n\omega)\, sin(\omega) -$$
$$cos(x\, sin(\omega) - n\omega)\, cos(\omega) + sin(x\, sin(\omega) - n\omega)\, sin(\omega)$$

$$f_{n+1} - f_{n-1} \;=\; (2/\pi) \int_0^\pi d\omega\, sin[x(\omega) - n\omega)]\, sin(\omega)$$

The term on the right is twice f'_n. Thus the recursion relation for the Bessel function is reproduced, so identify $f_n \to J_n$. The integral representation is;

$$J_n(x) \;=\; (1/\pi) \int_0^\pi d\omega\, cos(x\, sin(\omega) - n\omega) \;=$$
$$(1/2\pi) \int_{-\pi}^{\pi} d\omega\, e^{i(x\, sin(\omega) - n\omega)}$$

This implies that the Bessel function, J_n, is the n^{th} Fourier coefficient of the expansion; $e^{i2\,sin(\omega)} = \sum_{n=-\infty}^{\infty} J_n e^{in\omega}$. This allows the following expressions for the generating function;

$$cos(z\,sin(\omega)) = \sum_{n=-\infty}^{\infty} J_n\,cos(n\omega)$$
$$sin(z\,sin(\omega)) = \sum_{n=-\infty}^{\infty} J_n\,sin(n\omega)$$

Observe that $J_{-n} = (-1)^n\,J_n$

4 Generating function

Begin with the series in the last section;

$$cos(z\,sin(\omega)) = \sum_{n=-\infty}^{\infty} J_n\,cos(n\omega)$$
$$sin(z\,sin(\omega)) = \sum_{n=-\infty}^{\infty} J_n\,sin(n\omega)$$

Let $\omega = \pi/2$ to obtain;

$$cos(z) = J_0 - 2J_2 + 2J_4 + \cdots$$

$$sin(z) = 2J_1 - 2J_3 + \cdots$$

Let $e^{i\omega} = t$ so that $e^{i\omega} - e^{-i\omega} = 2i\,sin(\omega)$. Then;

$$e^{iz\,sin(\omega)} = e^{z(t-1/t)/2} = \sum_{n=-\infty}^{\infty} J_n\,t^n$$

This is the generating function for J_n.

$$G(z,t) = e^{z(t-1/t)/2} = \sum_{n=-\infty}^{\infty} J_n(z)\,t^n$$

One then obtains the value of $J_n(x)$ by determining the coefficient of the n^{th} power in the series. Thus;

$$\frac{d^n}{dt^n}[G(z,t)]|_{t=0} = n!\,J_n(z)$$

The generating function can be used to establish the Bessel power series, and the recursion relations.

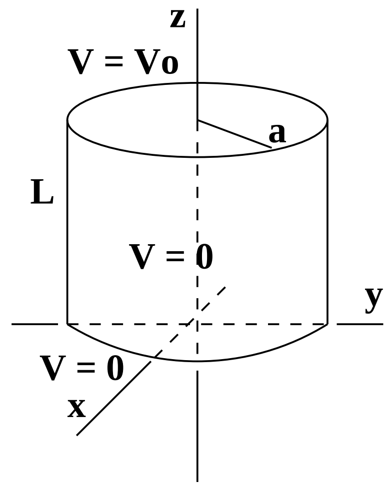

Figure 2: The geometry of a cylinder with one eddcap held at potential $V = V_0(\rho)$ and the other sides grounded

5 Limiting values

Limiting Values for the Bessel fns.

$$J_n(x) \lim_{x \to 0} \to \frac{(x/2)^\nu}{\Gamma(\nu+1)}$$

$$N_0(x) \lim_{x \to 0} \to (2/\pi) ln(x)$$

$$N_\nu(x) \lim_{x \to 0} \to (1/\pi)\Gamma(\nu)(x/2)^{-\nu}$$

$$J_\nu(x) \lim_{x \to \infty} \to \sqrt{2/\pi x}\, cos(x - \nu\pi/2 - \pi/4)$$

$$N_\nu(x) \lim_{x \to \infty} \to \sqrt{2/\pi x}\, sin(x - \nu\pi/2 - \pi/4)$$

$$H^1_\nu(x) = J_\nu + iN_\nu \lim_{x \to \infty} \to \sqrt{2/\pi x}\, e^{i(x - \nu\pi/2 - \pi/4)}$$

$$H^2_\nu(x) = J_\nu - iN_\nu \lim_{x \to \infty} \to \sqrt{2/\pi z}\, e^{-i(x - \nu\pi/2 - \pi/4)}$$

6 Example

Find the solution for the interior of a cylindrical shell with the top end cap held at a potential $V = V_0(\rho)$ and all the other surfaces grounded, Figure 2. The solution we seek has the form;

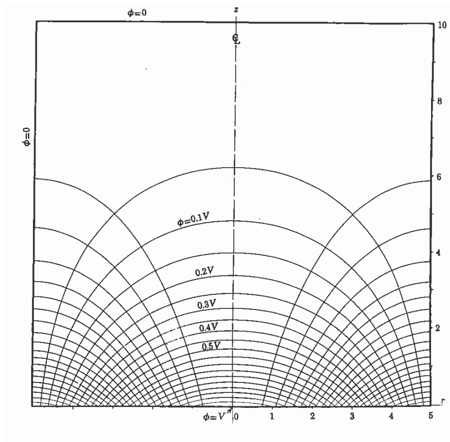

Figure 3: A graphical representation of the above solution

$$V = \sum_{\nu n} A_{\nu k_n} J_\nu(k_n \rho) sinh(k_n z) e^{i\nu\phi}$$

In this case the solution is independent of the angle ϕ so $\nu = 0$. Note that we have excluded N_ν in the solution because we want it to be finite as $\rho = 0$. Also we have chosen $sinh(k_n z)$ to satisfy the boundary condition at $z = 0$. The reduced solution is;

$$V = \sum_{n} A_n J_0(k_n \rho) sinh(k_n z)$$

Now $V = 0$ for $\rho = a$. This means that;

$$J_0(k_n a) = 0$$

The values of $k_n a$ are the zeros of the bessel function $J_0(k_n a)$. The first few are, $\alpha_{0n} =$ 2.4048, 5.5201, 8.6537, $\cdots$. Then at $Z = L$ we find A_n using the orthogonality of the Bessel functions. The graphic form of the solution is shown in Figure 3.

$$A_n = \frac{2}{a^2 \left[J_1(k_n a)\right]^2 sinh(k_n L)} \int_0^a \rho d\rho \, J_0(k_n \rho) \, V_0(\rho)$$

7 Example

As another example, find the potential inside a cylinder when the potential is specified on the end caps and the cylindrical wall is at zero potential, Figure 4. Note that the figure shows the coordinate origin at the center of the cylinder. Thus the solution is symmetric for $\pm z$ to match the boundary conditions. The boundry conditions are;

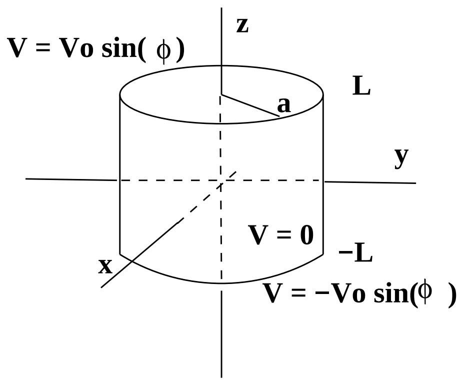

Figure 4: The geometry of the problem with endcaps held at potential $V = \pm V_0\, sin(\phi)$

$V = V_0\, sin(\phi)$ z = L

$V = -V_0\, sin(\phi)$ z = -L

$V = 0$ $\rho = a$

$V = 0$ on the (x, y) plane when $z = 0$

The solution must have the form;

$$V = \sum_{\nu n} A_{\nu n}\, J_\nu(k_n \rho)\, sin(\nu\phi)\, sinh(k_n z)$$

Here solutions cannot contain $N_{\nu n}(k\rho)$ which are infinite at the origin. To match the boundry at $z = \pm L$ we need to have a term $sin(\nu\phi)$ which requires $\nu = 1$. Then require that the Bessel function, $J_{1n}(k_n a) = 0$, which determines the zeros of the Bessel function of order 1. Write these as α_{1n} so that $k_n = \alpha_{1n}/a$. The solution has the form;

$$V = \sum_n A_n\, J_1(\alpha_{1n}\rho/a)\, sinh(\alpha z/a)\, sin(\phi)$$

Finally match the boundry condition at $z = \pm L$ where $V = V_0\, sin(\phi)$. Use orthorgonality to obtain;

$$(L/2)[J_2(\alpha_{1n})]^2 A_n = \frac{1}{sinh(\alpha L/a)} \int_0^a \rho d\rho\, J_1(\alpha_{1n}\rho/a)\, V_0$$

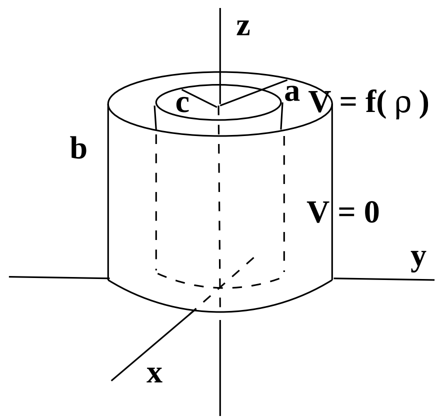

Figure 5: The geometry for the problem of two concentric cylinders

8 Example

As another example, we look at a solution for concentric cylinders with the boundry conditions;

$r = a, c$ and $z = 0$ V = 0

$z = b \;\; V = f(\rho)$

This geometry is shown in Figure 5. Choose the solution to have the form;

$$V = \sum_n^\infty A_n \, sinh(k_n z) \, G_0(k_n \rho)$$

Here we choose a superposition of the Bessel and Neumann functions;

$$G_0 = [\frac{J_0(k_n \rho)}{J_0(k_n c)} - \frac{N_0(k_n \rho)}{N_0(k_n c)}]$$

This choice results G_0 when $\rho = c$, *ie* on the cylindrical surface of the inner cylinder. Note we choose $\nu = 0$ because the potential is independent of ϕ, *ie* the problem is aximuthally symmetric. Now choose the values of k_n to make $G_0 = 0$ when $\rho = a$. This selects a set of zeros, $\alpha_{\nu n}$, of the function, G_0. Because G_0) is a solution to the cylindrical Sturm Liouville problem, the functions, G_0, a complete orthogonal set as would any solution to Bessel's equation which satisfies the boundary conditions. This points out that we separated the solutions of the radial ode into a form which was regular at $\rho = 0$ and one which was not as a useful identifier of the 2 lineraly independent solutions of Bessel's equation. But we also could have chosen G_0 as one of the 2 linearly independent solutions. As does and Bessel function, G_0 has oscillating properties. Of course the location of the zeros differs. Finally, use orthogonality to obtain the coefficients in the above equation.

$$H\,A_n \;=\; [1/sinh(k_n b)] \int_c^a \rho d\rho\, V(\rho)\, G(\alpha_n \rho/a)$$

Here;

$$H \;=\; \int_c^a \rho d\rho\, G^2(\alpha_n \rho/a)$$

9 Example

Consider the problem with the cylindrical wall held at potential $V = f(z)$ and the endcaps grounded. This geometry is shown in Figure 6. The boundry conditions are;

$$z = 0, b \;\; V = 0$$

$$\rho = a \;\; V = f(z)$$

In this case we cannot use the hyperbolic function in z to match the boundry conditions. However if we let $k \to ik$ then this function becomes harmonic at the expense of making the argument of the Bessel function imaginary. Note here that the problem is 2-D since the solution it independent of angle. Thus we expect only one eigenfunction, and this now occurs for the z coordinate. An eigenvalue of $k = n\pi/b$ is determined for this harmonic equation. The radial ode with a Bessel function solution of imaginary argument result in an eigenfunction.

$$sin(n\pi z/b) \qquad \text{n integral}$$

The solution has the form;

$$V \;=\; \sum_{n=1}^{\infty} A_n sin(n\pi z/b)\, J_0(in\pi\rho/b)$$

In this case we use the orthogonality of the harmonic functions rather than the Bessel functions to find the expansion coefficients. This results in;

$$A_n \;=\; \frac{2}{b\, J_0(in\pi a/b)} \int_0^b dz\, f(z)\, sin(n\pi z/b)$$

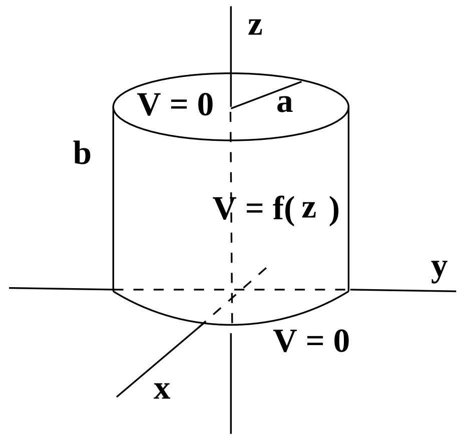

Figure 6: The geometry for the problem where a potential is placed on the cylindrical surface and the end caps grounded

10 Dual Boundary Conditions

The equation for the potential is $\nabla^2 V = 0$. In cylindrical coordinates, Figure 7;

$$(1/\rho)\,\frac{\partial}{\partial \rho}[\rho\,\frac{\partial V}{\partial \rho}] + (1/\rho^2)\,\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Assume a solution of the form;

$$V = \sum_{n=0} V_n(z,\rho) \begin{bmatrix} sin(n\theta) \\ cos(n\theta) \end{bmatrix}$$

In the above, one uses the harmonic functions *sine*, *cosine* or a combination of both to satisfy the angular boundary conditions. Use $sin(\theta)$ as an example.

$$V_n(z,\rho) = 1/\pi \int_0^{2\pi} d\theta\, V(z,\rho,\theta)\, sin(n\theta)$$

This results in the pde for V_n.

$$(1/\rho)\,\frac{\partial}{\partial \rho}[\rho\,\frac{\partial V_n}{\partial \rho}] - (n^2/\rho^2)\,V_n + \frac{\partial^2 V_n}{\partial z^2} = 0$$

Apply a Hankel transform;

$$\int_0^{\infty} \rho\, d\rho\, J_n(\lambda\rho)\,[(1/\rho)\,\frac{\partial}{\partial \rho}[\rho\,\frac{\partial V_n}{\partial \rho}] - (n^2/\rho^2)\,V_n + \frac{\partial^2 V_n}{\partial z^2}] = 0$$

$$\overline{V}_n(\lambda, z) = \int_0^{\infty} \rho\, d\rho\, V_n\, J_n(\lambda\rho)$$

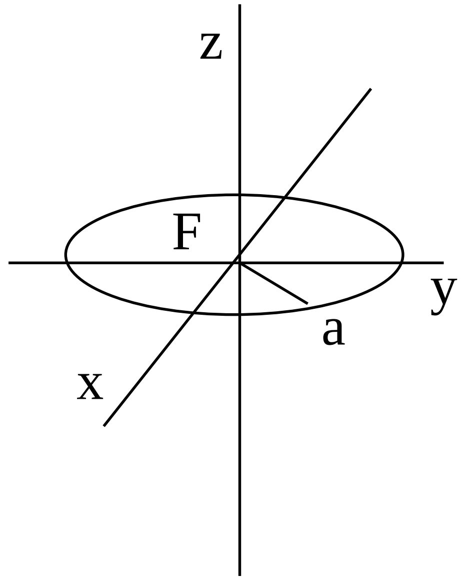

Figure 7: The geometry of a flat, conducting disk of radius a held at a potnetial F

The solution is symmetric for $z \to -z$, so choose $z > 0$. Use eigenfunctions for $J_n(\lambda\rho)$ and $cos(n\theta)$. This means the functional form for z is $e^{-\lambda z}$. Choose $n = 0$ to simplify the exposition below.

$$V_{n=0}(\rho, z) \,=\, \int\limits_0^\infty \lambda\, d\lambda\, A(\lambda)\, e^{-\lambda z}\, J_0(\lambda\rho)$$

The boundary conditions are applied in two steps.

Step 1 - $z = 0$ and $\rho < a$

$$V_0(\rho, 0) \,=\, F(\rho) \,=\, \int\limits_0^\infty \lambda\, d\lambda\, A(\lambda)\, J_0(\lambda\rho)$$

Step 2 - $z = 0$ and $\rho > a$

$$\frac{\partial V_0}{\partial \rho}|_{z=0} \,=\, \int\limits_0^\infty \lambda^2\, d\lambda\, A(\lambda)\, J_0'(\lambda\rho) \,=\, -E_\rho \,=\, 0$$

The above forms a pair of integral equations which must be simultaneously solved.

11 Numerical Solutions

Separation of variables provides an analytic solution when the boundaries of the potential surfaces are the same as those obtained by choosing each variable of the separation geometry as a constant in order to project out a surface in the coordinate system. Of course Laplace's equation also must separate into ode equations each involving only one of these variables. While analytic solutions provide insight into more realistic problems, they are not always useful in obtaining detailed information. Thus we require techniques to obtain accurate numerical solution of Laplace's (and Poisson's) equation.

First consider a result of Gauss' theorem. Integrate Laplace's equation over the volume which bounds the potential surface.

$$\int d\tau \, \nabla^2 V \; = \; \int \vec{\nabla} V \cdot d\vec{\sigma} \; = \; 0$$

In the above, $\vec{\sigma}$ is the surface which encloses the volume τ. In the case of a spherical surface, $d\vec{\sigma} \; = \; R^2 d\Omega \, \hat{r}$. Substitute this in the above to write;

$$R^2 \frac{d}{dR} \int d\Omega \, V \; = \; 0$$

This equation means that $\int d\Omega \, V$ is constant. In Cartesian coordinates divide space into a grid with cells of the dimensions (δx, δy, δz). From the above analysis we know that the potential at the center of a cell will approximately equal the average of the potential over the enclosing surface. In Cartesian coordinates this means that the potential at a point is approximately the average of the sum of the potentials over its nearest neighbors.

$$V_{l,m,n} \; = \; 1/6[V_{l-1,m,n} \; + \; V_{l+1,m,n} \; + V_{l,m-1,n} \; + \\ V_{l,m+1,n} \; + V_{l,m-1,n} \; + V_{l,m,n-1} \; + \; V_{l,m,n+1}]$$

Begin a numerical solution by taking the exact potential values on the surface and assign initial values to the interior potential at all grid points. These initial values can be any guesses. The average values at each point are then obtained, but one keeps the correct potential values on the surface. The process is then iterated to convergence. This proceedure technique is called the "relaxation method". It is stable with respect to iteration and converges rapidly to correct potential inside a volume. The technique (finite element analysis) is generally applied to any process which is described by Laplace's equation. This includes a number of physical problems in addition to electrostatics.

If charge is present, we need a solution to Poisson's equation. For a sphere of radius, r, the potential at the center relative to the surface is;

$$\Delta V \; = \; \rho r^2 / (6 \epsilon_0)$$

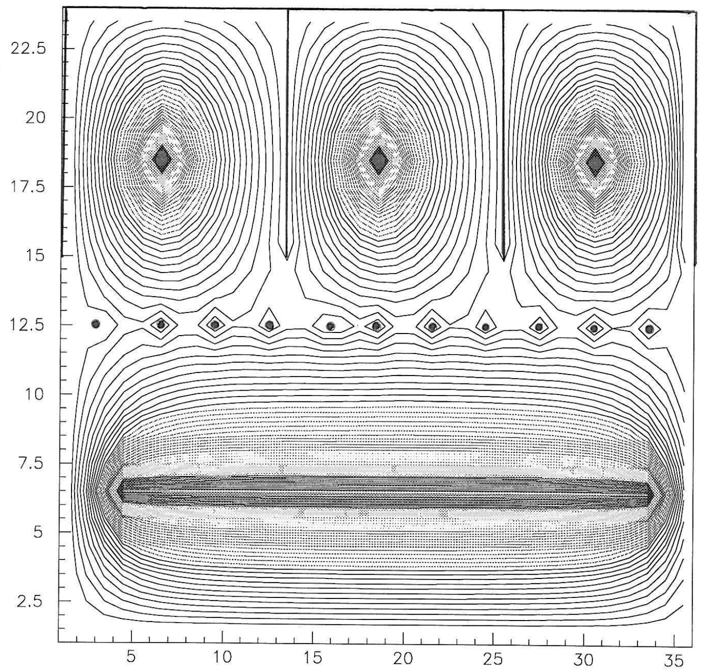

Figure 8: An 3-D numerical example showing contour lines of constant potential of a geometry having grounded metal boundaries and wires held at potential

This is included in the equation above when computing the average. As an example, Figure 8 shows a numerical valuation of the potential of a set of grounded metal boundaries and wires which are held at constant potential.