Lecture 3: The Electric Field

1 Coulomb’s Law

Coulomb’s law states that the force between two charges is inversely proportional to the square of the distance between the charges, and proportional to the product of the charges. However, any measurement of this inverse square law has some error, and while the result of a good measurement is consistent with a law, any law cannot be exactly proved. In this case, the inverse square behavior is an inductive postulate, which in fact has many consequences, and these can be (and have been) deductively tested. The mathematical form of Coulomb’s law states that for charges at rest, the force on \( q_1 \) due to \( q_2 \) when separated by a distance \( r_{12} \) is described by the vector equation;

\[
\vec{F} = \kappa \frac{q_1 q_2}{r_{12}^2} \hat{r}_{12}
\]

In the above, \( \hat{r}_{12} \) is a unit vector defined by;

\[
\hat{r}_{12} = \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}
\]

A unit vector has unit magnitude and points, as shown in the above equation, from the tip of the vector \( \vec{r}_2 \) to the tip of vector \( \vec{r}_1 \). Coulomb’ law can be written in terms of a field equation for \( \vec{E} \) as;

\[
\vec{F} = q_2 \vec{E}_{12}
\]

\[
\vec{E} = \kappa \frac{q_1}{r_{12}^2} \hat{r}_{12}
\]

In the above, \( \vec{E} \) is the electric field which describes the effect of the charge, \( q_1 \), on the spatial geometry, and the force, \( \vec{F} \), is obtained when \( q_2 \) interacts with the field at a point in space which is a distance \( r_{12} \) from \( q_1 \). The law has consequences which are discussed below. For the moment note that if the units of charge, distance, and force are previously defined, the constant \( \kappa \) must be adjusted to balance this equation. Thus, if charge is defined as a Coulomb (defined by the definition of current i.e. a Coulomb per second), length measured in meters, and force in Newtons, then \( \kappa \) is set equal to \( \frac{1}{4\pi \epsilon_0} \). In this expression, \( \epsilon_0 \) is the permittivity of free space, and has the value \( 8.85 \times 10^{-12} \) farad/m. These units are the rationalized MKS system, which are standard SI units and are used in this book. However, Coulomb’s law could also be used to define the unit of charge. This later procedure forms the Gaussian unit system. In the Gaussian set of units, length is measured in cm, force in dynes = \( g cm/s^2 \), and the charge, \( q \), is used to balance the force equation. In this system,
the unit of charge is the stata-coulomb, and $\kappa = 1$.

Charge is quantized in units equal to the charge of an electron. The elemental electronic charge has an absolute value of approximately $1.6 \times 10^{-19}$ Coulomb. Positive and negative charge have equal elemental units to the extent that this can be measured. However in classical electromagnetism, we often think of continuous charge distributions, and these will be discussed below.

The equality of the elemental positive and negative charges is an assumed axiom. Thus, suppose an approximate gram-mol of matter. This contains Avagadro’s number, $\approx 6 \times 10^{23}$, atoms which are neutrally charged. So if the positive and negative charges of an atom did not cancel, the forces between atoms could be large, entirely changing chemical or even atomic binding.

Now return to the $1/r^2$ behavior of the force. Is the power of $r$ exactly 2, or is this just approximate? Although, the behavior of the force can be experimentally tested, there is always some uncertainty in any measured number. It is believed, however, that the force decreases exactly as $r^2$. One reason for this axiom, is that Coulomb’s law represents the interaction of charges through the exchange of electromagnetic field quanta, the photons. Photons in free space have zero mass, and for (and only for) zero mass particles does the force decrease as $1/r^2$.

2 Fields

A field can be defined as a mathematical construction which connects a space-time geometry to a physical process. It is obtained by assigning one or more functions to each point in a coordinate system. These functions are used to form a continuous description of an interaction at various space-time points so that the symmetries imposed on the geometric and mathematical description of the field are then impressed on the interaction.

As one example, a physical process must be independent of the type and orientation of a coordinate system, so the field functions must be independent of the spatial operations of rotation, reflection, and translation of the coordinates. Thus a scalar field in Cartesian coordinates has a form, $f(x, y, z)$, and gives a numerical value at the coordinate position, $(x, y, z)$, independent of the orientation of the coordinate system. A vector field has one function for each coordinate direction at each point in space, and the coordinate functions transform like a vector under coordinate transformations. Thus in 3 Cartesian dimensions, the vector field has functions; $f_x(x, y, z)$, $f_y(x, y, z)$, $f_z(x, y, z)$, representing the magnitudes of the field vector along each of the Cartesian axes when evaluated at $(x, y, z)$. The properties of an interaction are also independent of the coordinate representation. Figure 1 illustrates an example of a two-dimensional vector field of wind velocity superimposed on the geometry of a section of the United States. The wind velocity and direction are are overlayed on
the underlying map to form a field. A connection between the coordinates and the velocity vectors is required to apply the field information to an interaction, in this case the velocity of the wind which would act on some object at a given point in space.

Since a vector field is perhaps easiest to visualize, suppose the 3 functions defined above provide the 3 numbers for the magnitudes of the vector interactions at each spatial location. Then require that these functions transform like a vector under coordinate rotations, and inversions. For example, given the form for the vector field;

\[ [f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)] \]

apply a right handed rotation about the z axis to obtain the functions \( f'_{x'} \), \( f'_{y'} \), \( f'_{z'} \) in the new frame \( x', y', z' \). The symmetry relation is then;

\[
\begin{align*}
    f'_{x'}(x', y', z') &= f_x(x, y, z) \cos(\theta) + f_y(x, y, z) \sin(\theta) \\
    f'_{y'}(x', y', z') &= -f_x(x, y, z) \sin(\theta) + f_y(x, y, z) \cos(\theta) \\
    f'_{z'}(x', y', z') &= f_z(x, y, z)
\end{align*}
\]

The magnitude of the vector field is obtained from the dot (scalar) product with itself.

\[
|M(x, y, z)| = |f_x^2 + f_y^2 + f_z^2|^{1/2}
\]

Figure 1: An example of a 2-dimensional vector field superimposed on a geometry described by a geographic location in the United States

The magnitude of the vector is a number which remains constant under rotations and re-
Reflections, since the vector length is independent of the rotation and the type of coordinates use to represent the vector. Thus the vector magnitude at each point forms a scalar field. If the coordinate frame is inverted (the parity symmetry), $x' = -x$, $y' = -y$, $z' = -z$ and the vector field has the properties:

$$f_x(x', y', z') = -f_x(x, y, z)$$
$$f_y(x', y', z') = -f_y(x, y, z)$$
$$f_z(x', y', z') = -f_z(z, y, z).$$

Although the vector field changes sign under reflection, a scalar field (magnitude) does not (and should not) change sign. Generalizing these operations, a field is characterized by its symmetry transformations as given in Table 1 and these symmetries are then impressed on the mathematical description of an interaction.

<table>
<thead>
<tr>
<th>Rotation(Field Type)</th>
<th>Reflection(Parity)</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scalar</td>
<td>Positive</td>
<td>Potential Energy</td>
</tr>
<tr>
<td>Pseudo-scalar</td>
<td>Negative</td>
<td>Nuclear Force Field (Pion Field)</td>
</tr>
<tr>
<td>Vector(rank 1 tensor)</td>
<td>Negative</td>
<td>Electric Field</td>
</tr>
<tr>
<td>Pseudo-vector (rank 1 tensor)</td>
<td>Positive</td>
<td>Magnetic Field</td>
</tr>
<tr>
<td>Tensor (rank 2 tensor)</td>
<td>Positive</td>
<td>Gravitational Field</td>
</tr>
<tr>
<td>Pseudo-tensor (rank 2 tensor)</td>
<td>Negative</td>
<td></td>
</tr>
</tbody>
</table>

3 Superposition

In addition to Coulomb’s law, the electric field also obeys the law of superposition. This law states that the total force on a charge is the linear, **VECTOR SUM**, of all the electrostatic forces acting on the charge.

$$\vec{F}_T = \kappa q \sum_i \frac{q_i}{r_{1i}^2} \hat{r}_{1i}$$

The accumulation of force due to many charges does not have to be linear (superposition), but experimentally electric forces are found (within error) to add vectorially and linearly. Thus superposition is a separate law, in addition to Coulomb’s law, which must be applied to describe the interaction of fields.
4 The Electric Field

Now return to the electric field. Electric charges create an electric field in space-time such that if another charge is placed at a specific point, the charge experiences a force given by Coulomb’s law. In the case of static charges, the interaction is mutual and symmetric. The force on a charge, $q_1$, due to a number of other charges, $q_i$, using superposition has the form.

$$
\vec{F}_1 = q_1 [\kappa \sum_i \frac{q_i}{r_{1i}^2} \hat{r}_{1i}]
$$

Here $\kappa = \frac{1}{4\pi\varepsilon_0}$. The electric field is;

$$
\vec{E}_1 = [\kappa \sum_i \frac{q_i}{r_{1i}^2} \hat{r}_{1i}]
$$

From our previous definition of the electric field and force;

$$
\vec{F}_1 = q_1 \vec{E}_1
$$

The electric field is a vector quantity, which is a function of position. If the field is known at ALL points in space then the charge distribution which caused the field can be determined.

5 Uniform Charge Distribution

Although charge is quantized, the elemental charge is very small so we can assume that it is continuous for a sufficiently large volume. The function, $\rho(x, y, z)$, is then defined as a continuum charge density so that;

$$
Q = \sum_i q_i \rightarrow \int d\tau' \rho
$$

Here $Q$ is the total charge contained in the volume $\int d\tau'$.

The electric field is obtained from the differential $\Delta Q = \rho \Delta \tau'$;

$$
\vec{E}_j = \kappa \sum_i \frac{q_i}{|\vec{r}_j - \vec{r}_i|^3} (\vec{r}_j - \vec{r}_i)
$$

$$
\vec{E}_j = \kappa \sum_i \frac{\rho(x_i, y_i, z_i) d\tau_i}{|\vec{r}_j - \vec{r}_i|^3} (\vec{r}_j - \vec{r}_i) \rightarrow \kappa \int d\tau_i \frac{\rho(\vec{r}_j - \vec{r}_i)}{|\vec{r}_j - \vec{r}_i|^3}
$$

Pictorially this is shown in Figure 2, where we use $\vec{r}_{ij} = (\vec{r}_j - \vec{r}_i)$ in future notation.
Figure 2: The electric field obtained from a uniform charge distribution

6 Gauss’ Law

Gauss’ law assumes both Coulomb’s law and the law of superposition. Recall that superposition was used to obtain the expression for the electric field in the last section. Now find the field intensity which penetrates a surface enclosing a net charge. Without loss of generality, assume a positive charge density, \( \rho \). The field intensity (flux) through a surface is illustrated in Figure 3.

The elemental charge creating a quantity of elemental flux is \( \rho \, d\tau' \). The flux through the surface area, \( d\sigma \), is then (define \( \vec{r} = \vec{R} - \vec{r}' \));

\[
d(\text{flux}) = \vec{E} \cdot d\vec{\sigma} = \kappa \frac{\rho \, d\tau}{\rho^2} \hat{r} \cdot d\vec{\sigma}
\]

The solid angle subtended by \( d\sigma \) about the origin \( O \) is \( d\Omega = \sin(\theta) \, d\theta \, d\phi \). Then write \( d\vec{\sigma} = R^2 \, d\Omega \, \hat{r} \);

\[
d(\text{flux}) = \vec{E} \cdot d\vec{\sigma} = \kappa \, dQ \frac{(r' - R\cos(\theta))r^2 \, d\Omega}{[R^2 + r'^2 - 2Rr'\cos(\theta)]^{3/2}}
\]

Keep \( R \) constant for a spherical surface, and integrate over angles.

\[
d(\text{flux}) = (dQ/2\varepsilon_0) \int d\cos(\theta) \frac{1 - (r'/R)\cos(\theta)}{[1 + (r'/R)^2 - 2(r'/R)\cos(\theta)]^{3/2}}
\]
The above integrates to $dQ/\varepsilon_0$ which should not be surprising, since one can always draw a sphere centered on the charge $dQ$. The total flux through this sphere is given by Coulomb’s law and equals $dQ/\varepsilon_0$. However, the flux out of this sphere equals the flux out of any surface which encloses the sphere and thus the charge, $dQ$. Then use superposition of all the elements of enclosed charge to obtain the total flux.

$$\text{Flux} = \oint \vec{E} \cdot d\vec{\sigma} = \frac{Q_{\text{Total}}}{\varepsilon_0}$$

This is Gauss’ law. Use the divergence theorem (Gauss’ theorem) to write;

$$\int \text{div} \vec{E} \, d\tau = \frac{Q}{\varepsilon_0} = \int \frac{\rho}{\varepsilon_0} \, d\tau$$

Because the volume is arbitrary, the above can be written in the differential form, ;

$$\text{div} \vec{E} = \frac{\rho}{\varepsilon_0}$$

Obviously the result remains the same for a combination of both positive and negative charges. By superposition the charge $Q$ is the net enclosed charge within the volume. Gauss’ law as written, is always true for free space but can be modified to include the effects of a medium by changing the value of $\varepsilon_0$ as will be seen in a later chapter.
The Application of Symmetry to Gauss’ Law

In some cases, symmetry allows the flux integral to be easily evaluated to obtain the electric field. As an example, consider a uniform, spherical charge distribution with maximum radius, \( r_0 \). Draw a spherical shell centered on the distribution having a radius, \( r > r_0 \). This is called a Gaussian surface. By Gauss’ law, the flux out of this shell is \( \frac{Q}{\varepsilon_0} \) where \( Q \) is the total charge enclosed by the surface. Because the problem is completely symmetrical, \( \vec{E} \) cannot depend on angle, and is only a function of the radial distance, \( r \). The integration over the Gaussian surface of the shell keeps \( r \) constant, and \( \vec{E} \) is directed along \( d\vec{\sigma} \). Thus \( E = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \).

\[
\oint \vec{E} \cdot d\vec{\sigma} = E \oint d\sigma = E 4\pi r^2 = \frac{Q}{\varepsilon_0}
\]

To complete the evaluation, the direction of \( \vec{E} \) is obtained from the direction of the dot product with the area vector.

Suppose a uniform, spherical charge distribution has a spherical void centered within the charge distribution. Apply symmetry to find \( \vec{E} \) within this void. Thus, \( \vec{E} \) can only have a radial dependence, but in this case there is no enclosed charge, and no field is present in the void. However, if \( \rho \) is not uniform all the flux which enters the void also exits it as well, so Gauss’ law is still valid, \( \int \vec{E} \cdot d\vec{\sigma} = 0 \), but in this case there is an electric field in the void. In general, just because Gauss’ law vanishes one cannot assume the electric field vanishes. However by the application of symmetry, there is no electric field inside a void of a uniformly distributed, spherical charge distribution.

In the case that a spherical shell lies within a uniformly distributed charge, \( Q \), only the enclosed charge contributes to the field penetrating the shell. This results in \( Q' = (4/3)\pi r'^3 \rho \), so that;

\[
E' = \frac{1}{4\pi\varepsilon_0} \frac{Q'}{r'^2}
\]

Here \( Q' < Q \) is the enclosed charge and \( r' < r \). Finally, consider an infinitely long wire which is uniformly charged with a charge per unit length, \( \lambda \). The Gaussian surface in this case is cylindrical with the cylindrical axis centered on the wire. Because of symmetry (there is no unique origin along the length of the wire) the electric field vector must then be perpendicular to the wire and independent of rotation angle. It can only depend on the radial distance from the wire. Thus evaluate the flux out of a Gaussian cylindrical surface of radius, \( r \), and length, \( L \). In this case, \( \vec{E} \) only penetrates the cylindrical surface and not the end caps, and \( \vec{E} \) is perpendicular to the differential area, \( r d\theta \, dz \).

\[
\oint \vec{E} \cdot \hat{r} (r \, d\theta \, dz) = E 2\pi r L = \frac{L\lambda}{\varepsilon_0}
\]
This equals the enclosed charge \( L\lambda \) so that:

\[
E = \frac{\lambda}{2\pi\varepsilon_0 r}
\]

The field is obviously directed perpendicular to the wire.

8 Example

Find the electric potential due two infinite, uniformly-charged sheets placed symmetrically about the origin of the \( x \) axis, and an infinite plate of thickness, \( 2b \), which has a uniform charge density, \( \rho \) is placed symmetrically between the sheets. The geometry with dimensions is shown in Figure 4.

![Figure 4: An example used to find the field between two uniformly charged sheets and a uniformly charged plate](image)

Solution

This problem is solved using symmetry and the law of superposition by applying Gauss’ law. The direction of the electric field above an uniformly charged sheet of infinite extent must be perpendicular to the sheet, and independent of the distance above the sheet, \( \sigma/2\varepsilon_0 \). This result can also be obtained using symmetry and Gauss’ law. Note in the figure, the small cylinder around the surface of the sheet carrying the surface charge, \( \sigma \), in Figure 4. This is a Gaussian surface over which Gauss’ law is applied.

\[
\oint d\vec{A} \cdot \vec{E} = \frac{Q_{enc}}{\varepsilon_0}
\]

The above integral is over the surface with the differential surface area element, \( d\vec{A} \), with the direction being the outward normal to the surface. By symmetry, the integral can be evaluated since \( \vec{E} \) is in the direction of \( d\vec{A} \) on the end caps of the cylinder and is parallel to
\[ d\vec{A} \] on the cylindrical surface. Thus the result is:

\[ \oint d\vec{A} \cdot \vec{E} = 2E \int dA_{end-cap} = 2EA_{end-cap} = \frac{Q_{enc}}{\epsilon_0} \]

\[ E = \frac{Q/A_{end-cap}}{2\epsilon_0} = \frac{\sigma}{2\epsilon_0} \]

Superimpose the fields from each sheet to obtain the total field. The field from each sheet is obtained by Gauss' law. Thus for an infinitesimal component of the plate width, the effective charge enclosed by a cylindrical Gaussian surface of radius \( A \) is \( Q = \rho A dx \). This is integrated from \(-b\) to \( b\) and the magnitude of the resulting field obtained from the above relation for Gauss' law. There are several regions where the fields from these components are combined, after carefully noting the directions of each of the components, the result is shown in Table 2.

<table>
<thead>
<tr>
<th>Region</th>
<th>Sheet 1 Plate</th>
<th>Sheet 2 Plate</th>
<th>Total Field</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x &lt; -a )</td>
<td>( +\sigma/2\epsilon_0 )</td>
<td>( -\rho b/\epsilon_0 )</td>
<td>( -\sigma/2\epsilon_0 )</td>
</tr>
<tr>
<td>(-a &lt; x &lt; -b)</td>
<td>( -\sigma/2\epsilon_0 )</td>
<td>( -\rho b/\epsilon_0 )</td>
<td>( -\sigma/2\epsilon_0 )</td>
</tr>
<tr>
<td>(-b &lt; x &lt; 0)</td>
<td>( -\sigma/2\epsilon_0 )</td>
<td>( -\rho x/\epsilon_0 )</td>
<td>( -\sigma/2\epsilon_0 )</td>
</tr>
<tr>
<td>( 0 &lt; x &lt; b )</td>
<td>( -\sigma/2\epsilon_0 )</td>
<td>( \rho x/\epsilon_0 )</td>
<td>( -\sigma/2\epsilon_0 )</td>
</tr>
<tr>
<td>( b &lt; x &lt; a )</td>
<td>( -\sigma/2\epsilon_0 )</td>
<td>( \rho b/\epsilon_0 )</td>
<td>( -\sigma/2\epsilon_0 )</td>
</tr>
<tr>
<td>( x &gt; a )</td>
<td>( -\sigma/2\epsilon_0 )</td>
<td>( \rho b/\epsilon_0 )</td>
<td>( \sigma/2\epsilon_0 )</td>
</tr>
</tbody>
</table>

### 9 Line Integrals and the Electric Potential

From mechanics, the differential energy, \( dW \), in moving a charge, \( q \), a distance \( d \) under the influence of a static force field is, \( dW = \vec{F} \cdot d\vec{l} \). The total energy to move a charge in a static field, \( \vec{E} \), between positions \( a \) and \( b \) along a path is obtained from:

\[ W = \int_{a, path}^{b} d\vec{l} \cdot \vec{F} = Q \int d\vec{l} \cdot \vec{E} \]

The differential energy at a particular point on the path is shown in Figure 5. First choose the field to be created by a point charge, \( Q \), so that:

\[ d\vec{l} \cdot \vec{F} = dl F \cos(\theta) = -F \, dr \]
By observation of Figure 5;

\[ W = - \int_{\text{Path}} F \, dr = \frac{qQ}{4\pi\epsilon} \int_{a}^{b} \frac{1}{r^2} \, dr = \frac{qQ}{4\pi\epsilon} \left[ \frac{1}{r} \right]_{a}^{b} \]

Thus the energy is independent of the path, and depends only on the end points. By superposition, add contributions for each charge of the distribution, showing that the energy obtained by moving a charge in any static electric field, is independent of the path. The energy obtained from an integral over a closed path must then vanish.

\[ \oint \vec{F} \cdot d\vec{l} = 0 = \int (\nabla \times \vec{F}) \cdot d\sigma \]

The above result is just Stokes theorem where the integral on the right hand side is over the area enclosed by the path integral on the left hand side. The path and the area are arbitrary so that the integrand must vanish;

\[ \nabla \times \vec{F} = 0 \]

This result is a consequence of Coulomb’s law, superposition, and the fact that the charge is static.

Energy is a **numerical value** for a specific charge distribution. Replace the force with the electric field times a charge \( q \), and write the energy when this charge is placed in the electric field. From the above equations;

\[ dW = q \, dV = -q \, \vec{E} \cdot d\vec{s} \]

This results in the definition of the electric potential, \( V \), which is a **scalar field**, and can be obtained from the vector electric field, \( \vec{E} \). If a unit charge is placed at a point in an electric potential field, the potential energy of the charge distribution including this charge is obtained. Thus the field is a potential energy per unit charge at that point in
space. Potential energy is measured relative to another point in space, \( \text{i.e.} \) in this case, the potential at \( a \) is measured relative to that at \( b \) which are the two limits of integration. From the definition of the gradient, the electric field may be obtained from the potential field using;

\[
\vec{E} = -\vec{\nabla}V
\]

To emphasize, the above results are for a static charge distribution.

### 10 Using the Electric Potential to find the Electric Field

Obviously one can use Coulomb’s law to obtain the electric field, for a continuous charge distribution, \( \rho \). This results in an integral of the form;

\[
\vec{E}(\vec{r}) = \kappa \int d\tau' \frac{\rho}{|\vec{r} - \vec{r}'|} (\vec{r} - \vec{r}')
\]

While correct, the above integral is usually used in only a few simple cases. Note that in general the notation expresses what is really 3 separate integrals, one for each of the vector directions. On the other hand, the expression for the electric potential is;

\[
V(\vec{r}) = \kappa \int d\tau' \frac{\rho}{|\vec{r} - \vec{r}'|}
\]

This is a simpler expression, if for no other reason than it only represents one integration. However, integration over the factor \( \frac{\rho}{|\vec{r} - \vec{r}'|} \) can sometimes lead to difficulties and divergences if not properly handled. Once \( V \) is known, the field can be obtained by straightforward, partial differentiation of the potential.

### 11 Example

**Example A**

Consider a line charge of length, \( L \), having a variable charge per unit length, \( \lambda(x) \), see Figure 6. Then complete the following exercises.

1. Write an integral expression for the electric field vector, \( \vec{E}(x, y, z) \), at all points in space.

2. When \( \lambda(x) = 2\lambda_0 x/L \), find the electric field at the point, \((0, 0, z)\)
Solution A

1) Because the charge density is a function of position, first find the electric field for each component of \( \vec{E} \).

\[
\vec{E}(x, y, z) = \frac{1}{4\pi \varepsilon} \int_{-L/2}^{L/2} dx' \lambda(x') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}
\]

In the above, \( \vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \) and \( \vec{r}' = x'\hat{x} \). Upon substitution this yields;

\[
\vec{E}(x, y, z) = \frac{1}{4\pi \varepsilon} \int_{-L/2}^{L/2} dx' \lambda(x') \frac{[(x - x')\hat{x} + y\hat{y} + z\hat{z}]}{|(x - x')^2 + y^2 + z^2|^{3/2}}
\]

Evaluation of the above integral gives the electric field at an arbitrary point in space, \((x, y, z)\). If we wish the field at the point \((0, 0, z)\), substitute this position into \((x, y, z)\) to obtain;

\[
\vec{E}(0, 0, z) = \frac{1}{4\pi \varepsilon} \int_{-L/2}^{L/2} dx' \lambda(x') \frac{[-x'\hat{x} + z\hat{z}]}{|x'|^{3/2}}
\]

To proceed further, the function, \( \lambda(x) \), must be specified. Thus;

2) Let \( \lambda(x) = 2\lambda_0 x/L \). The integral becomes;

\[
\vec{E}(0, 0, z) = \frac{2\lambda_0}{2\pi \varepsilon_0 L} \int_{-L/2}^{L/2} dx' \frac{[-x'^2\hat{x} + x'z\hat{z}]}{|x'^2 + z^2|^{3/2}}
\]

Evaluating the integral, one obtains;

\[
\vec{E}(0, 0, z) = \frac{\lambda_0}{2\pi \varepsilon_0 L} \left[ \frac{2L}{|4z^2 + L^2|^{1/2}} - ln\left(\frac{(4z^2 + L^2)^{1/2} + L}{4z^2}\right) \right] \hat{x}
\]

Figure 6: The field components of a line charge along the \( x \) axis centered on the coordinate origin
The charge is an odd function of $x$, so there is the same amount of positive charge for $0 \leq x \leq L/2$ as negative charge for $-L/2 \leq x \leq 0$. Obviously, the value of field, $|E_x|$, created by the positive charge equals the value created by the negative charge, but the fields for each $z$ component point in opposite directions. Thus the value of the field along the $z$ axis vanishes. Along the $x$ axis the fields point in the same direction so they add.

This problem also may be solved by computing the components of the electric field in polar coordinates. Note that the field must be symmetric in the rotation angle about the $x$ axis, so the solution is independent of the $y$ coordinate, as is clear in the above development. The symmetry is most easily applied in polar coordinates;

$$E_x = E \cos(\theta)$$
$$E_z = E \sin(\theta)$$

$$\cos(\theta) = \frac{x}{|\vec{r} - \vec{r}'|}$$
$$\sin(\theta) = \frac{z}{|\vec{r} - \vec{r}'|}$$

This result, of course, is identical to the above. Now notice that the solution points in the $x$ direction. The integral evaluating $E_x$ is;

$$\vec{E}(0,0,z) = \frac{\lambda_0}{2\pi \epsilon_0 L} \int_{-L/2}^{L/2} dx' \left[ \frac{x^2}{x'^2 + z^2} \right] \hat{x}$$

Example B

As an example, find the electric field at a distance from the edge of a wire which has a uniform charge per unit length, $\lambda$, This is shown in Figure 7.

Solution B

A differential element of the field can be written;

$$d\vec{E} = \kappa \frac{\lambda \ dy}{[y'^2 + z^2_0]} \hat{r}$$

Note in this case, the field has 2 components which are to be calculated. This is because the geometry is symmetric with respect to a rotation in angle about the wire axis, which reduces the independent number of coordinates from 3 to 2. Therefore the differential components take the form;

$$dE_\perp = \kappa \frac{\lambda \ dy}{[y'^2 + z^2_0]} \sin(\theta)$$
Figure 7: An example of a geometry to find the electric field at the edge of a charged wire

\[ dE_\parallel = \kappa \frac{\lambda \, dy \, \cos(\theta)}{[y^2 + z_0^2]} \]

These forms are integrated over the wire length to obtain the field components.

\[
E_\perp = \kappa \int_0^{y_0} dy \, \frac{\lambda}{[y^2 + z_0^2]} \sin(\theta) = \kappa \int_0^{y_0} dy \, \frac{\lambda z_0}{[y^2 + z_0^2]^{3/2}} \\
E_\parallel = \kappa \int_0^{y_0} dy \, \frac{\lambda}{[y^2 + z_0^2]} \cos(\theta) = \kappa \int_0^{y_0} dy \, \frac{\lambda y}{[y^2 + z_0^2]^{3/2}}
\]

Integration yields;

\[
E_\perp = \frac{\kappa \lambda}{z_0} \frac{y_0}{[y_0^2 + z_0^2]^{1/2}} \\
E_\parallel = \frac{\kappa \lambda}{z_0} \left[1 - \frac{z_0}{[y_0^2 + z_0^2]^{1/2}}\right]
\]

On the other hand, we can obtain the electric potential from the integral;

\[
V = \kappa \int_0^{y_0} dy \, \frac{\lambda}{[y^2 + z_0^2]^{1/2}} \\
V = \kappa \ln(y + [y^2 + z^2]^{1/2})
\]

The field is obtained from the partial derivatives evaluated at the point \( y = 0, z = z_0 \), yielding the same result.
\[ E_{||} = -\frac{\partial V}{\partial y} \bigg|_{y=0 \ z=z_0} \]
\[ E_{\perp} = -\frac{\partial V}{\partial z} \bigg|_{y=0 \ z=z_0} \]

**Example C**

An arc of an annular disk subtends an angle \( \theta_0 \), and has a uniform charge density (charge per unit area), \( \sigma \), Figure 8. The inner radius is \( a \) and outer radius \( b \). If the object lies in the \( x/y \) plane, determine the electric field along the \( z \) axis.

![Diagram of an annular disk](image)

Figure 8: The example of the field calculation on the \( z \) axis for a uniformly charged arc in the \( (x, y) \) plane

**Solution C**

For a point on the \( z \) axis, \((0, 0, z)\), the vector between the element of charge, \( \delta q = \sigma \rho d\rho d\theta \), is \( \vec{r} \). This is used in the integral for the electric field in the previous development. For an incremental element of charge,

\[ dq = \sigma \left( \rho d\rho d\theta \right) \]

The field at a point \((0, 0, z)\) is;

\[ \vec{E} = \frac{1}{4\pi \varepsilon_0} \int \rho d\rho \int d\theta \frac{\sigma}{r^2} \hat{r} \]

Integration limits are \( a \leq \rho \leq b \) and \( 0 \leq \theta \leq \theta_0 \). The charge density, \( \sigma \), is constant, and the unit vector, \( \hat{r} \), written in cylindrical coordinates, is obtained from;

\[ r \hat{r} = -\rho \cos(\theta) \hat{x} - \rho \sin(\theta) \hat{y} + z \hat{z} \]
The negative signs indicate that \( \hat{r} \) points into the quadrant in \((x, y)\) plane 180° from the charged arc. Substitution into the integral results in;

\[
\vec{E} = \frac{\sigma}{4\pi\epsilon_0} \int_a^b \rho \, d\rho \int_0^{\theta_0} d\theta \frac{-\rho \cos(\theta) \hat{x} - \rho \sin(\theta) \hat{y} + z \hat{z}}{[\rho^2 + z^2]^{3/2}}
\]

The integral over angle is easily done, and integration over the radial variable, \( \rho \), can also be found analytically. The result below leaves the radial integral for the student to complete.

\[
\vec{E} = \frac{\sigma}{4\pi\epsilon_0} \int_a^b d\rho \left[ \rho^2 \sin(\theta_0) \hat{x} + \frac{\rho^2 (\cos(\theta_0) - 1)}{[\rho^2 + z^2]^{3/2}} \hat{y} + \frac{z \rho \theta_0}{[\rho^2 + z^2]^{3/2}} \hat{z} \right]
\]

Now expand this example by applying several limits. Exploring solutions by applying limits is an important technique to test the correctness of any mathematical representation of a physical process. As an example, apply the value of the field when the angle, \( \theta_0 \to 2\pi \). In this case the charge is symmetrically distributed around the \( z \) axis. By symmetry arguments, it is expected that \( E_x = E_y = 0 \). Using the expression for the field on the \( z \) axis, the result is;

\[
E_z = \frac{z \sigma}{2\epsilon_0} \int_a^b d\rho' \frac{\rho'}{[\rho'^2 + z^2]^{3/2}} = \frac{z \sigma}{2\epsilon_0} \left[ \frac{1}{[\rho'^2 + z^2]^{1/2}} \right]_a^b
\]

The other field components vanish upon integration, as expected from the symmetry argument above. Now continue to apply limiting conditions, this time let \( a \to 0 \), i.e. create a solid disk, \( 0 \leq \rho \leq b \). Taking this limit in the above equation, one obtains;

\[
E_z = \frac{z \sigma}{2\epsilon_0} \left[ \frac{1}{z} - \frac{1}{(z^2 + b^2)^{1/2}} \right]
\]

Then when \( z \gg b \) expand \( \frac{1}{[z^2 + b^2]^{1/2}} \) in powers of \((b/z)^2\).

\[
E_z = \frac{z \sigma}{2\epsilon_0} \left[ \frac{1}{z} - (1/z)[1 - (1/2)(b/z)^2 + \cdots] \right]
\]

Collect terms to first order in \((b/z)^2\) to obtain;

\[
E_z = \frac{z \sigma}{2\epsilon_0} \left[ \frac{b^2}{2z^3} \right]
\]

The total charge on the disk is;

\[
Q = \sigma \, \text{[Area]} = \pi b^2 \sigma
\]

Using this result, the field on the \( z \) axis for a uniformly charged disk when \( z \gg b \) is finally;
Along the $z$ axis, the field components in the $(x, y)$ directions vanish. Thus, along the $z$ axis the uniformly charged disk appears as a point source of charge, $Q$, placed at the coordinate origin. On the other hand, if $b \gg z$ then:

$$\vec{E} = \frac{\sigma}{2\epsilon_0} \left[ 1 - \frac{z}{(z^2 + b^2)^{1/2}} \right] \hat{z} = \frac{\sigma}{2\epsilon_0} \hat{z}$$

12 Using the Electric Potential in Gauss’ Law

Gauss’ law in differential form is $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon$. Insert the above expression for the electric field in terms of the electric potential. This results in:

$$\vec{\nabla} \cdot \vec{\nabla} V = -\rho/\epsilon_0$$

The above is a partial differential equation, Poisson’s equation, which is solved to obtain the electric potential. In the case where $\rho = 0$ the equation is called Laplace’s equation. The operator, $\vec{\nabla} \cdot \vec{\nabla} = \nabla^2$, is the scalar Laplacian. These equations are $2^{nd}$ order, linear, partial differential equations. Later we find methods to solve these equations.

13 Conductors

When an electric field is applied to a charge, there is a force which causes the charge to move if no constraining force(s) is present. A conducting material is one in which charges are free to move inside the conductor, so if a conducting material is inserted into an electric field, charges move until the fields produced by the new charge distribution conspires to cancel this applied field. After the field is canceled there is then no force, and no further movement, of the charge. Thus for a static charge distribution there is no electric field in a conducting medium. A null electric field implies a constant ($\vec{E} = -\vec{\nabla} V = 0$) potential, so all points of a continuous, conducting medium have the same value of electric potential.

As an illustration, consider the electric field near a sharp edge of a conductor. Approximate this geometry by two conducting spheres connected by a thin, perfectly conducting wire, as shown in Figure 9. The wire allows charge to move between the spheres so that they have the same potential. On the other hand if the wire is thin, its contribution to the field can be ignored.

The field about a conducting sphere must be radial (from the argument above there can be no tangential component of $E$ field) and must have the same magnitude at all points on a
Figure 9: The simplified geometry used to example the field strength near a sharp edge of a conductor

spherical surface. As previously seen from Gauss’ law, the field is:

\[ \vec{E} = \kappa \frac{Q}{R^2} \hat{R} \]

where \( Q \) is the charge on the sphere and \( R \) is its radius. The electric potential of the sphere is:

\[ V = \kappa \frac{Q}{R} \]

When the two spheres are connected by a conducting wire they have the same potential. Write:

\[ V = \kappa \frac{Q_1}{R_1} = \kappa \frac{Q_2}{R_2} \]

From this, the charge on the spheres is related by:

\[ Q_1 = Q_2 \left( \frac{R_1}{R_2} \right) \]

The ratio of the electric field strength is then:

\[ \frac{E_1}{E_2} = \frac{R_2}{R_1} \]

Then as \( R_1 \gg R_2 \) one sees that \( E_2 \gg E_1 \). Applying this simple solution to the more complicated problem of a conductor with a sharp edge, note that the electric field parallel to the conducting surface is zero and the potential is constant. However, the perpendicular component of \( E \) is much larger at an edge than on a smooth, flat surface.

If a pointed conductor is placed in air, the strong field at the tip of the point can strip electrons from the air molecules and attract free charge which might be in its vicinity. These charges are accelerated in the strong field, colliding with other molecules and producing an avalanche of charge which could create a spark of electricity. If the conductor is in vacuum, the strong field can pull electrons from the conducting surface, and this is called field emis-
sion.