Solution to Laplace’s Equation in Spherical Coordinates

Lecture 7

1 Introduction

First write the potential for a charge distribution \( \rho \) given by;

\[
V = \kappa \int d\tau' \frac{\rho(r')}{|\vec{r} - \vec{r}'|}
\]

Now suppose \( r > r' \) and look at the term;

\[
\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r} \frac{1}{\sqrt{1 + (r'/r)^2 - 2(r'/r)\cos(\theta)}}
\]

Make a power expansion of the fraction;

\[
\frac{1}{\sqrt{1 + (r'/r)^2 - 2(r'/r)\cos(\theta)}} = [1 + (r'/r)\cos(\theta) - (r'/r)^2 + (3/2)(r'/r)^2\cos^2(\theta) + \cdots]
\]

Here introduce the Legendre polynomials;

\[
P_n(x) = \sum_{k=0}^{n/2} (-1)^k \frac{(2n - 2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}
\]

For even \( n \), \( P_n \) has even powers of \( x \) and for odd \( n \), the polynomial has odd powers. Some examples are;

\[
P_0(x) = 1
\]

\[
P_1(x) = x
\]

\[
P_2(x) = 1/2(3x^2 - 1)
\]

These polynomials can be generated by the Rodrigues formula;

\[
P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [x^2 - 1]^n
\]

Expansion then takes the form;
\[
\frac{1}{|r - r'|} = (1/r) \left[ \sum_{n=0}^{\infty} (r'/r)^n P_n(\cos(\theta)) \right]
\]

which forms an orthogonal set of functions;

\[
\int_{-1}^{1} dx P_n(x) P_m(x) = \frac{2}{2n + 1}
\]

Where \( n = m \) otherwise the integral vanishes since these polynomials are orthogonal. The functions are also complete, since a power series is complete converging for \(-1 < x < 1\). The integral also satisfies the recursion relations;

\[
(n + 1)P_{n+1} - (2n + 1)xP_n + nP_{n-1} = 0
\]

\[
\frac{dP_{n+1}}{dx} - x\frac{dP_n}{dx} - (n + 1)P_n = 0
\]

\[
(x^2 - 1)\frac{dP_n}{dx} - nxP_n + nP_{n-1} = 0
\]

From the above the value of the polynomials can be obtained numerically and various integrals evaluated. For example

\[
\int_{-1}^{1} dx x P_n P_m = \begin{cases} 
\frac{2(n + 1)}{(2n + 1)(2n + 3)} & n = m + 1 \\
\frac{2n}{(2n - 1)(2n + 1)} & n = m - 1
\end{cases}
\]

2 Separation of Variables for Laplace’s equation in Spherical Coordinates

In spherical coordinates Laplace’s equation is obtained by taking the divergence of the gradient of the potential. As previously developed, this is;

\[
\nabla^2 V = (1/r^2)\left[ \frac{\partial}{\partial r}(r^2 \frac{\partial V}{\partial r}) \right] + \left( \frac{1}{r \sin(\theta)} \right)\left[ \frac{\partial}{\partial \theta}(\sin(\theta) \frac{\partial V}{\partial \theta}) \right] + \left( \frac{1}{r^2 \sin^2(\theta)} \right)\left[ \frac{\partial^2 V}{\partial \phi^2} \right] = 0
\]

Then attempt to obtain a solution by separation of variables. Thus assume that \( V = R(r)\Theta(\theta)\Psi(\phi) \). Substitution into Laplace’s equation and division by \( V \) gives;

\[
\frac{\sin^2(\theta)}{R} \frac{d}{dr}\left( r^2 \frac{dR}{dr} \right) + \frac{\sin(\theta)}{\Theta} \left( \frac{\sin(\theta)d\Theta}{d\theta} \right) + \frac{1}{\Psi} \frac{d^2\Psi}{d\phi^2} = 0
\]

As with separation in Cartesian coordinates, isolate terms which depend on only one variable. Because the terms can take arbitrary values, they must equal a constant. In the above note that;
\[ \frac{1}{\Psi} \frac{d^2 \Psi}{d\phi^2} = -m^2 \]

\[ \frac{d^2 \Psi}{d\phi^2} + m^2 \Psi = 0 \]

Here \( m^2 \) is a separation constant. The equation takes the form of an eigenvalue equation with the boundary condition that the function \( \Psi \) must repeat as \( \phi \) circles beyond \( 2\pi \). Because the solution is harmonic \( m \) must be an integer forming harmonic eigenvalues and eigenfunctions.

\[ \Psi = A e^{\pm im\phi} \]

The geometry is 3 dimensional so there will be 2 eigenvalue equations. In the above analysis one eigenfunction and one separation constant \( m^2 \) was obtained. Now separate the terms in \( r \) and \( \theta \). This results in the two equations;

\[ \frac{1}{\mathcal{R}} \frac{d}{dr} \left( r^2 \frac{d \mathcal{R}}{dr} \right) = l(l+1) \]

\[ \frac{1}{\Theta \sin(\theta)} \frac{d}{d\theta} \left( \sin(\theta) \frac{d \Theta}{d\theta} \right) - \frac{m^2}{\sin^2(\theta)} = -l(l+1) \]

In the above, \( l(l+1) \) is a separation constant chosen to have this particular form for convenience, as will be seen later. As will be explored below, the equation for \( \Theta \) becomes an eigenvalue equation when the boundary condition \( 0 \leq \theta \leq \pi \) is applied requiring \( l \) to integral. The radial equation for \( \mathcal{R} \) cannot be an eigenvalue equation, so \( l \) and \( m \) are specified by the other two equations. The radial equation has the following form when \( U = r \mathcal{R} \);

\[ \frac{d^2 U}{dr^2} - \frac{l(l+1)}{r^2} U = 0 \]

This has solutions;

\[ U = Ar^{l+1} + Br^{-l} \]

\[ \mathcal{R} = Ar^l + Br^{-(l+1)} \]

Note that every 2\(^{nd}\) order differential equation has 2 linearly independent solutions. Now return to the equation for \( \Theta \). To simplify the equation, make the substitution \( x = \cos(\theta) \). This results in the equation;

\[ \frac{d^2 \Theta}{dx^2} - \frac{2\Theta}{1-x^2} \frac{d\Theta}{dx} + \left[ \frac{l(l+1)}{1-x^2} - \frac{m^2}{(1-x^2)^2} \right] \Theta = 0 \]

The above is the associated Legendre equation which has associated Legendre polynomials as solutions. In problems with axial symmetry, the solutions are independent of \( \phi \) so that \( m = 0 \). In this case, solutions to the above equation, are the Legendre polynomials as de-
veloped in the first section.

3 The Method of Frobenius

To explore the above solution further, return to Legendre’s equation with $m = 0$;

$$\frac{d^2 \Theta}{dx^2} - \frac{2x}{1-x^2} \frac{d\Theta}{dx} + \left[ \frac{l(l+1)}{1-x^2} \right] \Theta = 0$$

Look for a series solution of the form;

$$\Theta = \sum_{n=0}^{\infty} a_n x^{n+s}$$

The coefficients $a_n$ are to be chosen to satisfy the ode, and the series is assumed to begin with the $s^{th}$ power of $x$. Put the above series into the ode above to obtain;

$$(1 - x^2) \sum a_n (n+s)(n+s-1) x^{n+s-2} - 2x \sum a_n (n+s) x^{n+s-1} + l(l+1) \sum a_n x^{n+s} = 0$$

Then re-order the sums to contain the same power of $x$. All terms of the same power of $x$ are linearly independent, which means the coefficients of each term must vanish. For the lowest two orders of $x$ there are 2 possibilities;

$$s(s-1) = 0 \text{ if } a_0 \neq 0 \text{ here set } n = 0$$

$$s(s+1) = 0 \text{ if } a_1 \neq 0 \text{ here set } n = 1$$

Thus choose either $a_0 = 0$ or $a_1 = 0$, but of course not both. From the above, if $a_1 = 0$ then $s = 1$ and only even powers of $x$ are generated. If $a_0 = 0$ then $s = -1$ and only odd powers of $x$ are allowed. This is shown in the recursion equation obtained from the re-ordered sum of powers.

$$a_{n+2} = \frac{(n+s)(n+s+1) - l(l+1)}{(n+s+2)(n+s+1)} a_n$$

There is one unknown constant, $a_0$ or $a_1$, remaining in the 2 different series solutions. Inspection of the terms in each series shows that it converges for $|x| < 1$. However when $x = \pm 1$ the series does not converge, but of course a solution for $\theta = 0$ and $\pi$ is also possible. This can happen if the series does not sum to infinity, but terminates at some value of $n$, say $N$. Then note that terms for $n > N$ vanish as can be verified by applying the recursion equation. For odd powers, $s = 1$ the series can be terminated if $n = (l-1)$.
\[ a_{n+2} = \frac{(n+1)(n+2) - l(l+1)}{(n+3)(n+2)} a_n \]

For even powers of \( x \) choose \( s = -1 \) and terminate the series using \( n = l + 1 \)

\[ a_{n+2} = \frac{n(n-1) - l(l+1)}{(n+1)n} a_n \]

This generates the Legendre polynomials as given in the first section of this lecture. Note that \( l \) is an integer of an eigenvalue equation.

However, be careful. There are problems where boundaries exclude the \( z \) axis. Thus the Legendre polynomials are not always sufficient to expand an arbitrary function which does not include \( x = \pm 1 \). A second order ode has 2 linearly independent solutions, so there is another set of solutions which are infinite at \( x = \pm 1 \) and these solutions must also be used in some problems. However, this case is not considered here. Examples of the first few Legendre functions of both the 1\(^{st} \) and 2\(^{nd} \) kind are shown in Figure 1.

4 Associated Legendre Polynomials

In the case on non-axial symmetry, \( m \neq 0 \) in the Legendre ode.

\[
\frac{d^2\Theta}{dx^2} - \frac{2x}{1-x^2} \frac{d\Theta}{dx} + \left[ \frac{l(l+1)}{1-x^2} - \frac{m^2}{(1-x^2)^2} \right] \Theta = 0
\]

The solutions are the associated Legendre polynomials, written \( P_l^m(x) \) with \( -m \leq l \leq m \) as shown by the generating equations below. They can also be obtained by differentiating the \( m = 0 \) Legendre equation \( m \) times.

\[ P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x) \]

The Rodrigues formula given below also produces the associated Legendre polynomials:

\[ P_l^m(x) = \frac{(-1)^m (1-x^2)^{m/2}}{2^l l!} \frac{d^{l+m}}{dx^{l+m}} (x^2 - 1)^l \]

It can be shown that:

\[ P_l^{-m}(x) = \frac{(-1)^m (l-m)!}{(l+m)!} \]

The orthogonality condition is ;
Figure 1: Examples of Legendre functions
\[ \int dx P^m_l(x) P^m_{l'}(x) = \frac{2(l + m)!}{(2l + 1)(l - m)!} \delta_{l,l'} \]

5 Azimuthal examples symmetric in spherical coordinates

In problems with azimuthal symmetry \((m = 0)\) the separated solution in spherical coordinates takes the form;

\[ V = \sum_{l=0}^{\infty} A_l r^l P_l(\cos(\theta)) + \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos(\theta)) \]

As a simple problem, consider a conducting sphere, separated at the mid-plane so that the upper hemisphere is charged to potential \(V_0\) and the lower hemisphere charged to potential \(-V_0\). Then find the potential outside a sphere of radius \(R\). The problem is axially symmetric so from the above general solution, choose \(A_l = 0\) in order that \(V \to 0\) as \(r \to \infty\). Then;

\[ V = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos(\theta)) \]

Set \(r = R\) and apply the boundary condition that;

\[ V(R, \theta) = \begin{cases} V_0 & 0 \leq \theta < \pi/2 \\ -V_0 & \pi/2 < \theta \leq \pi \end{cases} \]

Multiply by \(P_m(x)\) integrate over the solid angle \(d\Omega\), and use the orthogonality of the Legendre polynomials.

\[ B_l = \frac{2l + 1}{4\pi R^{l+1}} \int_{-1}^{1} dx V(x) P_l(x) \]

Thus only odd values of \(l\) contribute and integrate;

\[ \int_{0}^{1} dx P_n(x) = 1/(n + 1)[P_{n+1}(1) - P_n(1)] \]

This is obtained using;

\[ \frac{dP_{n+1}}{dx} - x \frac{dP_n}{dx} - (n + 1)P_n = 0 \]

Substituting for \(P_l\), integrating and combining terms.

\[ B_l = \frac{2l + 1}{4\pi R^{l+1}} (2V_0/(n + 1))[P_{l+1}(1) - P_l(1)] \]
\[ B_l = \left(-1/2\right)^{(l-1)/2} \frac{(l - 2)! \cdot 2!}{2[(l + 1)!!]} \]

then:

\[ V = V_0 [\left(3/2\right)(R/r)^2 P_1 - (7/8)(R/r)^4 P_3 + (11/16)(R/r)^6 P_5 \cdots ] \]

As will be seen this forms what is called the multipole series.

6 Interior potential between two spherical shells

The geometry of the problem is shown in Figure 2. The solution is assumed to be axially symmetric. Thus the general form of the solution is:

\[ V = \sum [A_l r^l P_l(\cos(\theta)) + B_l r^{-(l+1)} P_l(\cos(\theta))] \]

The potential at \( r = a \) is then:

\[ V_a(\theta) = \sum [A_l a^l + B_l a^{-(l+1)}] P_l(\cos(\theta)) \]

Note the shell at \( r = a \) cannot be a conductor as the potential is not constant on its surface. However one can proceed to find a solution by separation of variables. Use orthogonality of the Legendre polynomials to write:

\[ \int_{-1}^{1} dx V_a(x) P_l(x) = \frac{2}{2l + 1} [A_l a^l + B_l a^{-(l+1)}] \]

In a similar way for the potential at \( r = b \)

\[ \int_{-1}^{1} dx V_b(x) P_l(x) = \frac{2}{2l + 1} [A_l b^l + B_l b^{-(l+1)}] \]

Which gives:

\[ B_l = \left[ \frac{a^{2l+1}b^{2l+1}}{b^{2l+1} - a^{2l+1}} \right]^{1/2l+1} \int_{-1}^{1} dx [V_a(x)/a^l - V_b(x)/b^{l+1}] P_l(x) \]

\[ A = \frac{2l + 1}{2} \int_{-1}^{1} dx V_a(x) P_l(x) - B_l a^{l+1} \]
The conducting sphere in a uniform field

The geometry of this problem is illustrated in Figure 3. Because the geometry is axially symmetric, use the general form developed above.

\[
V = \sum_{l=0}^{\infty} A_l r^l P_l(\cos(\theta)) + \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos(\theta))
\]

Require as \( r \to \infty \) then \( \overrightarrow{E} \to E_0 \hat{z} = E_r \cos(\theta) \hat{z} \). Since \( E_r = -\frac{dV}{dr} \), apply this as the boundary condition as \( r \to \infty \)

\[
V = -E_0 r \cos(\theta) + \frac{B_1 \cos(\theta)}{r^2} + A_0
\]

Here the value of \( l \) at \( l = 1, 0 \) have been already been applied and the boundary conditions at \( \infty \) and at \( r = R \) gives;

\[
V = V_0|_{r=R} = -E_0 R \cos(\theta) + \frac{B_1 \cos(\theta)}{R^2} + A_0
\]

Solving;

\[
B_1 = E_0 R^3
\]
$A_0 = V_0$

The solution is therefore;

$$V = E_0 \cos(\theta) \left[ R^3/r^2 - r \right] + V_0$$

Now find the charge distribution on the surface of the sphere. This is obtained by Gauss’ law as shown previously.

$$E_n = \sigma/\epsilon_0$$

$E_n$ is the normal to the surface at the point of evaluation. Thus;

$$\vec{E} = \vec{\nabla}V = -\frac{\partial V}{\partial r} \hat{r} - \left( 1/r \right) \frac{\partial V}{\partial \theta} \hat{\theta} - \left( 1/[r \sin(\theta)] \right) \frac{\partial V}{\partial \phi} \hat{\phi}$$

Note that $\vec{E}$ does point radially ($\hat{r}$) when $r = R$ as it must since the sphere is conducting. Thus for $r = R$;

$$E_n = 3E_0 \cos(\theta)$$

The surface charge density is;

$$\sigma = (3E_0/\epsilon_0) \cos(\theta)$$
8 Example

Consider the problem of a small hemispherical boss on an infinite plane. Both are conductors and held at a potential $V_0$. To help describe the geometry, place another conducting plane a distance $d$ above and at a large distance, $d$ away. Restrict the solution to potentials and fields near the boss. The problem has axial symmetry, but the coordinate systems are mixed; (ie near the boss the boundary is specified in spherical coordinates, but far away it is specified in Cartesian coordinates. The potential at large distances from the origin takes the form;

$$V = (V_0/d)[d - z] = V_0[1 - (r/d) \cos(\theta)]$$

The potential at $r = a$ is $V_0$. The solution should have the form;

$$V = \sum A_l r^l P_l + \sum B_l r^{-(l+1)} P_l$$

Then attempt to apply the boundary condition at large distance. This can only be done approximately.

$$V \approx A_0 + A_1 r \cos(\theta) + B_0/r + B_1/r^2 \cos(\theta)$$

Apply the conditions as $r \to \infty$

$$V_o - (V_0r/d) \cos(\theta) \approx (A_0 + B_0/r) + (A_1 r + B_1/r^2) \cos(\theta)$$

This gives to first order;

$$A_0 \approx V_0$$

$$A_1 \approx -V_0/d$$

Applying boundary condition at $r = a$ ;

$$B_0 = 0 \text{ and } B_1 \approx V_0(a^3/d) \approx 0$$

9 Spherical Harmonics

In the case where there is no axial symmetry, non-zero values of $m$ must include eigenfunctions of $\phi$ in the solution. This introduces the associated Legendre polynomials as previously discussed. Combine these angular functions into an orthonormal set called the spherical harmonics.
Figure 4: The geometry to find the potential between 2 conduction spherical shells having potentials $V_a(\theta)$ and $V_b(\theta)$

$$Y^m_l(\theta, \phi) = \sqrt{\frac{2l + 1}{4\pi} \frac{(l - m)!}{(l + m)!} P^m_l(\cos(\theta)) e^{im\phi}}$$

Note that;

$$Y^{-m}_l = (-1)^m Y^{m*}_l$$

The functions are orthogonal and normalized to 1;

$$\int d\Omega Y^{m'*}_l Y^{m}_l = \delta_{l,l'} \delta_{m,m'}$$

Figure 5 shows the surface of various spherical harmonics for constant values of $R$ (radial distance). Examples of spherical harmonics are;

$$Y^0_0 = \frac{1}{4\pi}$$
$$Y^1_0 = -\frac{3}{8\pi} \sin(\theta) e^{i\phi}$$
$$Y^0_1 = -\frac{3}{4\pi} \cos(\theta)$$
$$Y^2_0 = -\frac{15}{32\pi} \sin^2(\theta) e^{i2\phi}$$
$$Y^1_1 = -\frac{15}{8\pi} \sin(\theta) \cos(\theta) e^{i\phi}$$
$$Y^0_2 = -\frac{5}{4\pi} ((3/2)\cos^2(\theta) - 1/2)$$

An arbitrary function of the spherical angles $(\theta, \phi)$ can be expanded in spherical harmonics. For example;
Figure 5: Surfaces given by various spherical harmonics for constant amplitude
Figure 6: Expansion of $\frac{1}{|r^2 - r'|}$ in arbitrary coordinates

$$f(\theta, \phi) = \sum_{l,m} A_{lm} Y^m_l(\theta, \phi)$$

The expansion coefficients are:

$$A_{l,m} = \int d\Omega (Y^m_l)^* f$$

The general form of a potential is then:

$$V = \sum_{l,m} [A_{lm} r^l + B_{lm} r^{-(l+1)}] Y^m_l$$

10 The Addition Theorem

Below is given an outline of the development of the relation:

$$P_l(\cos(\gamma)) = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_l^{m*}(\theta', \phi') Y_l^m(\theta, \phi)$$

Using the above relation, write a power expansion when $r_<$ is a radial distance less than $r_>$:

$$\frac{1}{|r^2 - r'|} = 4\pi \sum_{l,m} \frac{1}{2l+1} (r_<^l/r_>^{l+1}) Y_l^{m*}(\theta', \phi') Y_l^m(\theta, \phi)$$

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The geometry is shown in Figure 6. The addition theorem relates the coordinate frame between fixed vectors $\vec{r}_1$ and $\vec{r}_2$ to a general frame of reference. Note that the spherical harmonics form a complete set of functions which span the 3-D angular space. Thus an expansion giving the angular dependence of any function can be obtained using the spherical harmonics. Therefore write;

$$\frac{1}{|\vec{r}_1 - \vec{r}_2|} = \sum_{l_1 l_2 m_1 m_2} f_{l_1 l_2} (r_1, r_2) Y_{l_1}^{m_1} (\theta_1, \phi_1) Y_{l_2}^{m_2} (\theta_2, \phi_2)$$

From Figure 6 it is clear that the general function can only depend on the difference in azimuthal angles of the vectors, and this must be single valued as $|\phi_1 - \phi_2|$ turns through $2\pi$. Therefore $m_1 = -m_2 = m$, and $-m \leq l_1, l_2 \leq m$ so $l_1 = l_2 = l$. Now $f$ does not depend on $m$ so that;

$$\frac{1}{|\vec{r}_1 - \vec{r}_2|} = \sum_{l} f_l (r_1, r_2) Y_{l}^{m} (\theta_1, \phi_1) Y_{m}^{-m} (\theta_2, \phi_2)$$

If $\theta_1 = 0$ then $\theta_2 = \gamma$ which is the angle between the vectors. In this case $m = 0$, and the above expression should converge to an expansion in Legendre polynomials.

$$\frac{1}{|\vec{r}_1 - \vec{r}_2|} = \sum_{l} f_l Y_{0}^{l} (0, \phi_1) Y_{0}^{l} (\theta_2, \phi_2)$$

Here;

$$Y_{0}^{l} (0, \phi) = \sqrt{\frac{(2l + 1)}{4\pi}}$$

$$Y_{0}^{l} (\theta, \phi) = \sqrt{\frac{(2l + 1)}{4\pi}} P_l (\cos(\theta))$$

Comparing the above expressions;

$$\frac{(2l + 1)}{4\pi} f_l = \frac{r_l^1}{r_{l+1}}$$

The addition theorem then is obtained after setting $\hat{1}$ and $\hat{2}$ to be the angles of the two radial vectors;

$$P_l (\cos(\theta)) = 4\pi \sum_{l} \frac{1}{2l + 1} Y_{l}^{m} (\hat{1}) Y_{l}^{-m} (\hat{2})$$

Finally using the addition theorem;

$$\frac{1}{|\vec{r}_1 - \vec{r}_2|} = 4\pi \sum_{l m} (r_l^1/r_{l+1}^2) Y_{l}^{m*} (\theta_1, \phi_1) Y_{l}^{m} (\theta, \phi)$$
11 Multipole Expansion

Now find the potential of an arbitrary charge distribution, \( \rho(r, \theta, \phi) \) at a point well outside the distribution. This is illustrated in Figure 7. The vector to the field point, \( \vec{r} \), is larger than the vector to the source point, \( \vec{r}' \). Thus in writing the equation for the potential make a power series expansion in terms of \( (r'/r) \). This takes the form, using the addition theorem above;

\[
V = \kappa \int d\tau' \frac{\rho(r', \theta', \phi')}{|\vec{r} - \vec{r}'|}
\]

\[
V = \left(\frac{1}{\varepsilon_0}\right) \sum_{lm} \frac{Y_m^l(\theta', \phi')}{(2l + 1)^{l+1}} \int d\tau' \frac{Y^*_{m}^{l}(\theta', \phi') r'^l}{r'^{l+1}}
\]

The integral is over the prime variables. In the above, \( q^m_l \) as defined below is called the multipole moment.

\[
q^m_l = \int d\tau' Y^*_{m}^{l}(\theta', \phi') r'^l
\]

Note that the terms in the series decrease in powers of \( (r'/r)^l \) so for very far distances from the charge distribution the dominant term simply becomes \( Q/r = \frac{\int d\tau' \rho}{l} \), as should be obvious. If one deals with a system of point charges the integral is replaced by a sum over these charges. As an example, choose two charges placed on the z axis as shown in Figure 8.

The potential of this configuration is;

\[
V = \frac{q}{4\pi \varepsilon_0} \left[ \frac{1}{r[1 - (r_1/r) \cos(\theta) + (r_1/r)^2]^{1/2}} - \frac{1}{r[1 - (r_2/r) \cos(\theta') + (r_2/r)^2]^{1/2}} \right]
\]
Expansion of the denominators, letting $r_1 = r_2 = d/2$ and taking the lowest order in the power series after combining terms, yields:

$$V = \frac{qd}{4\pi\varepsilon_0 r^2} \cos(\theta)$$

This is the $l = 1$ term in the general multipole expansion above. It is called the dipole term because it is the lowest order potential term for two charges and $\mathbf{p} = qd\mathbf{d}$ is identified as the dipole moment. The moment points from negative to positive charge. As one comes close to the charge distribution higher order terms cannot be neglected.

### 12 Shape of the Earth’s gravity field

As another example of the use of spherical harmonics, the gravity field of the Earth, Moon, or planets can be described by a spherical harmonics series.

$$U(r, \theta, \phi) = \frac{GM}{R} \left[ \frac{R}{r} - \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left( \frac{R}{r} \right)^{n+1} (C_{nm} Y_{nm}^c(\theta, \phi) + S_{nm} Y_{nm}^s(\theta, \phi)) \right]$$

In the above $M$ is the mass of body and $R$ the equatorial radius.

$$Y_{nm}^c = P_n^m \cos(m\phi)$$
\[ Y_{mn}^* = P_n^m \sin(m\phi) \]

Here, real trigonometric forms are used instead of the complex exponential form of \( Y_l^m \). Numerical values of the coefficients are given in Table 1. From the coefficients one observes that the gravitational field of the Earth is very slightly pear shaped.

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Earth</th>
<th>Moon</th>
<th>Mars</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_{20} )</td>
<td>1.083 \times 10^{-3}</td>
<td>0.200 \times 10^{-3}</td>
<td>1.96 \times 10^{-3}</td>
</tr>
<tr>
<td>( C_{22} )</td>
<td>0.16 \times 10^{-5}</td>
<td>2.4 \times 10^{-5}</td>
<td>-5 \times 10^{-5}</td>
</tr>
<tr>
<td>( S_{22} )</td>
<td>-0.09 \times 10^{-5}</td>
<td>0.5 \times 10^{-5}</td>
<td>3 \times 10^{-5}</td>
</tr>
</tbody>
</table>

### 13 Angular momentum

The momentum operator in QM has the form \( i\hbar \vec{\nabla} \). The angular momentum operator then has the form:

\[ \vec{L} = -\vec{r} \times i\hbar \vec{\nabla} \rightarrow \vec{r} \times \vec{p} \]

The energy operator in spherical coordinates is;

\[ T = -\frac{\hbar^2}{2M} \nabla^2 = -\frac{\hbar^2}{2M} [\nabla^2_r + \nabla^2_\perp] \]

where the Laplacian operator is divided into operations on the radial variable, \( r \), and the angular variables, \( (\theta, \phi) \). In Cartesian coordinates;

\[ L_x = -i\hbar [y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}] \]
\[ L_y = -i\hbar [x \frac{\partial}{\partial z} - x \frac{\partial}{\partial z}] \]
\[ L_z = -i\hbar [x \frac{\partial}{\partial y} - y \frac{\partial}{\partial z}] \]

The map between Cartesian and spherical coordinates is;

\[ x = r \sin(\theta) \cos(\phi) \]
\[ y = r \sin(\theta) \sin(\phi) \]
\[ z = r \cos(\theta) \]

Make a change of variables to obtain the Cartesian angular momentum operators.

\[
L_x = i\hbar [\sin(\phi) \frac{\partial}{\partial \theta} + \cot(\theta) \cos(\phi) \frac{\partial}{\partial \phi}] \\
L_y = i\hbar [-\cos(\phi) \frac{\partial}{\partial \theta} + \cot(\theta) \sin(\phi) \frac{\partial}{\partial \phi}] \\
L_z = -i\hbar \frac{\partial}{\partial \phi}
\]

Then look at \( L^2 = \vec{L} \cdot \vec{L} \)

\[
L^2 = -\hbar^2 \left[ \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2(\theta)} \frac{\partial^2}{\partial \phi^2} \right]
\]

This can be obtained by using:

\[
\vec{L} = -i\hbar \vec{r} \times \vec{\nabla}
\]

\[
\vec{\nabla} = \hat{r} \frac{\partial}{\partial r} - \vec{r} \times \hat{r} \times \vec{\nabla}
\]

Then an eigenvalue equation results.

\[
L^2 Y_l^m(\theta, \phi) = \hbar^2 l(l+1)Y_l^m(\theta, \phi)
\]

The commutation rules for the angular momentum operator can be worked out as well using the above description of the operators.

\[
[L_i, L_j] = i\epsilon_{ijk} \hbar L_k
\]

\[
L^\pm = L_x \pm iL_y
\]

\[
[L_z, L^\pm] = \pm \hbar L^\pm
\]

\[
[L^+, L^-] = 2L_z
\]

\[
L_z Y_l^m = \hbar m Y_l^m
\]

\[
L_z L^\pm Y_l^m = \hbar (m \pm 1)L^\pm Y_l^m
\]

\( L^2 \) commutes with \( L^\pm \) so from the above, \( L^+(L^-) \) is a raising (lowering) operator on the eigenvalue \( m \) in the eigenfunction \( Y_l^m \)