SET 1

2.26)

Figure 1 shows the geometry for the integration. The differential of charge is;

$$dq = \sigma \rho \, d\phi \, dr$$

The potential is obtained with da the differential area as;

$$V = \kappa \int da' \frac{\sigma}{|\vec{r} - \vec{r}'|}$$

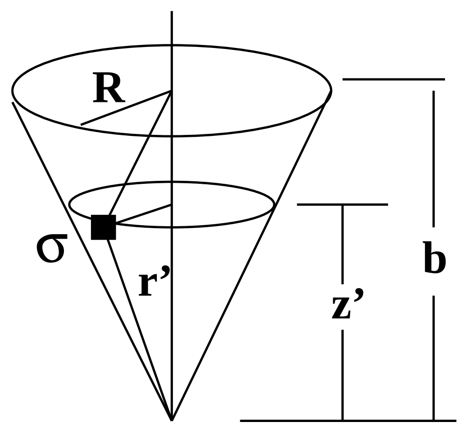

Figure 1: The geometry of the problem

Then $da = \rho d\phi \, dz$ with $z' = rho' \, sin(\theta)$ so the integral becomes;

$$V = 2\pi\sigma\kappa \int \rho' \, drho' \frac{1}{[r^{2\,\prime} + b^2 - 2r'b \, cos(\theta)]^{1/2}}$$

The cone angle is;

$$tan(\theta) = R/b = \rho'/z'$$

$$sin^2(\theta) = \frac{R^2}{R^2 + b^2} = (\rho'/r')^2$$

$$cos^2(\theta) \;=\; \frac{b^2}{R^2 + b^2} \;=\; (z'/r')^2$$

From the above, the integral over $d\phi$ equals 2π. We finally obtain;

$$V \;=\; (2\pi\sigma\kappa)(R/b) \int_0^b dz' \, \frac{z'}{[(R/b)^2 z'^2 + (b - z')^2]^{1/2}}$$

This gives the potential at $z = b$. Then subtract the potential at the apex of the cone due to the charge distribution.

$$V \;=\; (2\pi\sigma\kappa)(R/b) \int_0^b dz' \, \frac{z'}{[(R/b)^2 z'^2 + z'^2]^{1/2}}$$

The result is;

$$\Delta V \;=\; (2\pi\sigma\kappa)(R/b)[1 \;-\; ln(\sqrt{2} + 1)]$$

3.24)

Solution to Laplace's Equation in Polar Coordinates

Lecture 8 Polar Coordinates

1 Introduction

We obtained general solutions for Laplace's equation by separtaion of variables in Cartesian and spherical coordinate systems. The last system we study is cylindrical coordinates, but remember Laplaces's equation is also separable in a few (up to 22) other coordinate systems. As you know, choose the system in which you can apply the appropriate boundry conditions. It is only through application of the boundry conditions (Dirichlet of Neumann on a closed surface) that one finds a unique solution to the problem studied. In cylindrical coordinates apply the divergence of the gradient on the potential to get Laplace's equation.

$$\nabla^2 V(\rho, \phi, z) \;=\; \rho \, \frac{\partial^2 V}{\partial \rho^2} \;+\; \frac{\partial V}{\partial \rho} \;+\; (1/\rho) \, \frac{\partial^2 V}{\partial \phi^2} \;+\; \frac{\partial^2 V}{\partial z^2} \;=\; 0$$

Look for a solution by separation of variables;

$$V \;=\; \mathcal{R}(\rho)\Psi(\phi)\mathcal{Z}(z)$$

As previously, this yields 2 separation constants, k and ν, which will lead to 2 eigenfunction equations. The three separated ode equations are;

$$\frac{d^2\mathcal{Z}}{dz^2} - k^2\mathcal{Z} = 0$$
$$\frac{d^2\Psi}{d\phi^2} + \nu\Psi = 0$$
$$\frac{d^2\mathcal{R}}{d\rho^2} + (1/\rho)\frac{\mathcal{R}}{d\rho} + (k^2 - (\nu/\rho)^2)\mathcal{R} = 0$$

The later 2 equations can be set up as eigenfunction equations. The solutions are;

$$\mathcal{Z} \propto e^{\pm kz}$$

$$\Psi \propto e^{\pm i\nu\phi}$$

$$\mathcal{R} \propto J_\nu(k\rho), \text{ or/and } N_\nu(k\rho)$$

2 Bessel Functions

In the last section, $J_\nu(k\rho)$, $N_\nu(k\rho)$ are the 2 linearly independent solutions to Bessel's equation. Bessel functions oscillate but not harmonically, see Figure 2. Thus we expect that the harmonic function solutions for Ψ and the Bessel function solutions for $\mathcal{R}$ will be the eigenfunctions when the boundry conditions are imposed. The Bessel functions, $J_\nu(x)$, are regular at $x = 0$, while the Bessel functions, $N_\nu(x)$, are singular at $x = 0$.

The limiting values of the Bessel functions are;

$$\lim_{x\to 0} J_\nu(x) \to (\frac{x}{2})^\nu$$
$$\lim_{x\to 0} N_\nu(x) \to \begin{bmatrix} (2/\pi)\, ln(x) & \nu = 0 \\ (2/x)^\nu \frac{\Gamma(\nu)}{\pi} & \text{otherwise} \end{bmatrix}$$
$$\lim_{x\to\infty} J_\nu(x) \to \sqrt{\frac{2}{\pi x}}\, cos(x - \nu\pi/2 - \pi/4)$$
$$\lim_{x\to\infty} N_\nu(x) \to \sqrt{\frac{2}{\pi x}}\, sin(x - \nu\pi/2 - \pi/4)$$

From the requirement that the solution be single valued as $\phi \to 2\pi$, *ie* the solution is must not change when ϕ is replaced by $\phi + 2\pi$, the values of ν are integral and this produces eigenfuntions of Ψ. The series solution for the Bessel function J_ν can be found by the method of Frobenius. However, the second linearly independent equation is not easily obtained when

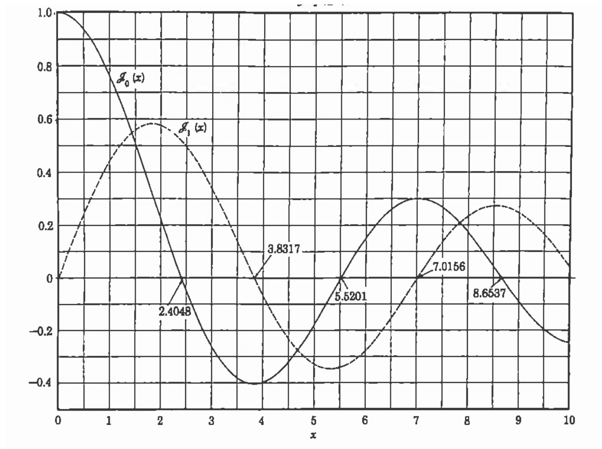

Figure 2: An example of the Cylindrical Bessel function $J(x)$ as a function of x showing the oscillaltory behavior

n is an integer, and another technique is required. The Bessel function takes the form;

$$J_\nu(x) \;=\; \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(\nu+s)!}\,(x/2)^{\nu+2s} \;=\; \frac{x^\nu}{2^n n!} \;-\; \frac{x^{\nu+2}}{2^{\nu+2}(\nu+1)!} \;+\; \cdots$$

For integral values of ν one can show $J_{-\nu} \;=\; (-1)^\nu J_\nu$. The Bessel functions also satisfy the recurrence relations;

$$J_{\nu-1}(x) \;+\; J_{\nu+1}(x) \;=\; \frac{\nu}{x}\,J_\nu(x)$$

$$J_{\nu-1}(x) \;=\; \frac{x}{\nu}\,J_\nu(x) \;+\; \frac{dJ_\nu}{dx}$$

$$\frac{d}{dx}[x^\nu J_\nu(x)] \;=\; x^\nu J_{\nu-1}$$

At times an integral representation is useful.

$$J_\nu(x) \;=\; (1/\pi)\int_0^\pi d\theta\, e^{ix\,cos(\theta)}\,cos(\nu\theta)$$

The Bessel functions are orthogonal;

$$\int_0^\infty kdk\, J_\nu(k\rho)\, J_\nu(k\rho') \;=\; \delta(\rho-\rho')/r$$

$$\int_0^a \rho d\rho\, J_\nu(\alpha_{\nu k}\rho/a)\, J_\nu(\alpha'_{\nu k}\rho/a) \;=\; (a^2/2)[J_{\nu+1}(\alpha_{\nu k})]^2\,\delta_{kk'}$$

In the above $\alpha_{\nu k}$ are the zeros of the Bessel function of order ν where k orders these zeros. As the Bessel functions form a complete set, any function may be expanded in a Bessel series or integral for an infinite space.

$$F(\rho) \;=\; \int\, kdk\, A(k)\, J_\nu(k\rho)$$

3 Solutions Independent of z

When the solution is independent of the z axis, the coordinates reduce to polar form. Therefore the solution is obtained by setting $k = 0$ in the above development.Laplace's equation for the potential in polar form is ;

$$\nabla^2 V(\rho, \phi, z) \;=\; \rho\, \frac{\partial^2 V}{\partial \rho^2} \;+\; \frac{\partial V}{\partial \rho} \;+\; (1/\rho)\, \frac{\partial^2 V}{\partial \phi^2} \;=\; 0$$

We look for a solution by separation of variables;

$$V \;=\; \mathcal{R}(\rho)\Psi(\phi)$$

This is an equation with 2 independent variables so that we expect 1 eigen value equation with one separation constant. Choose a solution of the form;

$$V \;\propto\; A_m\, \rho^m\, e^{\pm im\phi}$$

Then it is clear that m must be an integer so that the solution remains single valued as ϕ turns past 2π. We can write the harmonic form for the potential as;

$$V \;=\; \sum_m \,[A_m\, \rho^m \;+\; B_m\, \rho^{-m}]\, e^{im\phi}$$

Solve $\nabla^2 V \;=\; 0$ in cylindrical coordinates with a solution is independent of z. Assume a solution of the form;

$$V \;=\; A\rho^n\, e^{im\phi}$$

By substitution this is a solution if $n \;=\; \pm m$. Thus we have a solution in the form;

$$V \;=\; \sum\,[A_m\, \rho^m \;+\; B_m \rho^{-m}]\, cos(m\phi)$$

The boundary conditions $\rho \to \infty$ gives $V \to E_0 \rho\, cos(\theta)$, and $\rho \to a$ gives $V \to 0$. The solution then takes the form;

$$V \;=\; E_0\, \rho\, cos(\theta) \;-\; E_0[a^2/\rho]\, cos(\theta)$$

4.33)

The boundary conditions are with $D = \epsilon E$;

$$D_{1\perp} = D_{2\perp}$$

$$E_{1\parallel} = E_{2\parallel}$$

Then $tan(\theta_1) = \dfrac{E_{1\parallel}}{E_{1\perp}}$
This leads to;

$$\frac{tan(\theta_2)}{tan(\theta_1)} = \epsilon_2/\epsilon_1$$

5.34)

The magnetic moment is;

$$\vec{m} = (1/2) \oint \vec{\rho} \times I d\vec{l}$$

Use $I\, dl = V\, dq = (\omega\rho\sigma)\, \rho\, d\rho\, d\phi$

Substitute and integrate;

$$m = \omega\pi\sigma \int_0^R d\rho\, \rho^3$$