SET 10

9.20)

a) The energy density is; $\mathcal{W} = (1/2)[\epsilon E^2 + b^2/\mu]$. This is integrated over time to get the time average.

$$E = E_0 e^{-\kappa x} e^{i[kx - \omega t + \delta]}$$

$$B = B_0 e^{-\kappa x} e^{i[kx - \omega t + \delta']}$$

In the above, κ is the imaginary component and k is the real component of the wave vector. The phase differences in the fields are represented by δ and δ'. The time average is obtained by $(1/2)EE^*$ for example. Thus

$$\langle \mathcal{W} \rangle == (e^{-2kz}/4)[\epsilon |E|^2 + |B|^2/\mu]$$

$$B_0/E_0 = \sqrt{\epsilon\mu \sqrt{1 + (\sigma/\epsilon\mu)^2}}$$

$$\sqrt{1 + (\sigma/\epsilon\mu)^2} = (2k^2/\omega^2 \epsilon\mu)$$

Then ;

$$\langle \mathcal{W}_B \rangle / \langle \mathcal{W}_E \rangle = \sqrt{1 + (\sigma/\epsilon\mu)^2} = (2k^2/\omega^2 \epsilon\mu)$$

b) Evaluate the Poynting vector $\vec{S} = \vec{E} \times \vec{H}$ or take the energy density above and multiply by the speed of propagation ω/k

9.25)

The group velocity is $V_g = \frac{d\omega}{dk}$. Then the dispersion relation is $(\gamma = 0)$;

$$k = \omega/c[1 + (Nq^2/2m\epsilon) \sum_i \frac{f_i}{\omega_i - \omega_0}]$$

Taking the derivative;

$$V_g \;=\; c[(Nq^2/2m\epsilon)\,\sum_i\, f_i\, \frac{\omega_i^2+\omega^2}{\omega_i-\omega_0}^2\,]^{-1}$$

The result is always less than c but the phase velocity, $V_P = \omega/k$ can be greater than c.

9.29)

Apply the fields to the time averaged Poynting vector $\vec{S} \;=\; (1/2)\vec{E}\times\vec{H}^*$. For the TE fields, $E_z = 0$ and

$$B_z \;=\; B_0\, cos(\tfrac{m\pi x}{a})\, cos(\tfrac{n\pi y}{a})$$

$$E_x \;=\; B_0\frac{-i\omega}{\gamma^2}(n\pi/b)\, cos(\tfrac{m\pi x}{a})\, sin(\tfrac{n\pi y}{a})$$

$$E_y \;=\; B_0\frac{i\omega}{\gamma^2}(m\pi/a)\, sin(\tfrac{m\pi x}{a})\, cos(\tfrac{n\pi y}{a})$$

$$B_x \;=\; B_0\frac{-ik}{\gamma^2}(m\pi/a)\, sin(\tfrac{m\pi x}{a})\, cos(\tfrac{n\pi y}{a})$$

$$B_y \;=\; B_0\frac{-ik}{\gamma^2}(n\pi/b)\, cos(\tfrac{m\pi x}{a})\, sin(\tfrac{n\pi y}{a})$$

In the above $\gamma^2 \;=\; (\omega/c)^2 - k^2$. Substiture for $\vec{S}$.

$$\langle S\rangle \;=\; \frac{\pi^2\omega k B_0^2}{2\mu\gamma^2}[(m/a)^2\, sin^2(\tfrac{m\pi x}{a})\, cos^2(\tfrac{m\pi y}{k}) \;+\; (n/b)^2\, cos^2(\tfrac{m\pi x}{a})\, sin^2(\tfrac{m\pi y}{k})]$$

Then integrate this over the cross section $dx\,dy$. Use $B = E/c$

$$\int \langle S\rangle\, dx\, dy \;=\; \frac{\pi^2\omega k B_0^2}{\gamma^2}(ab/8\mu)[(m/a)^2 + (n/b)^2]$$

Then obtain the energy density as previously, $\langle\mathcal{W}\rangle \;=\; (1/4)Re[\epsilon|E|^2 + |B|^2/mu]$. Integrate this over the cross sectional area;

$$\int \langle\mathcal{W}\rangle\, dx\, dy \;=\; \frac{\omega^2 ab B_0^2}{8\mu\omega_{mn}}$$

Divide these expressions;

$$\frac{\int S\rangle ds\, dy}{\int \langle \mathcal{W}\rangle ds\, dy} = (c/\omega)\sqrt{\omega^2 - \omega_{mn}^2} = V_g$$

9.38)

To find the resonant fields for the cavity, use the field expressions for the fields in the wave guide and require that the longitudinal components expressed as $e^{i[kz\pm\omega t]}$ are superimposed to form standing waves. For the wave to have nodes at $0, d$, require that the normal component of the TE mode, $B_z = 0$, and the transverse components of E for the TM mode, $E_x = 0$ and $E_y = 0$.

TE mode

$$E_z = B_0\, cos(\tfrac{m\pi}{a}x)\, cos(\tfrac{n\pi}{a}y)\, e^{i[kz\pm\omega t]} = \\ B_0\, cos(\tfrac{m\pi}{a}x)\, cos(\tfrac{n\pi}{a}y)\, sin(\tfrac{l\pi}{d}z)$$

TM mode

$$E_z = E_0 sin(\tfrac{m\pi}{a}x)\, sin(\tfrac{n\pi}{a}y)\, e^{i[kz\pm\omega t]}$$

$$E_x = E_0 cos(\tfrac{m\pi}{a}x)\, sin(\tfrac{n\pi}{a}y)\, e^{i[kz\pm\omega t]}$$

$$E_y = E_0 sin(\tfrac{m\pi}{a}x)\, cos(\tfrac{n\pi}{a}y)\, e^{i[kz\pm\omega t]}$$

So that;

$$E_z = E_0 sin(\tfrac{m\pi}{a}x)\, sin(\tfrac{n\pi}{a}y)\, cos(\tfrac{l\pi}{d}z)$$

From this, the dispersion relation becomes;

$$k^2 = (\omega/c)^2 = (m\pi/a)^2 + (n\pi/b)^2 + (l\pi/d)^2$$