SET 11

10.1)

Given;

$$\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}\vec{A}) = -\rho/\epsilon$$

$$\nabla^2 \vec{A} - \mu\epsilon\frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}(\vec{\nabla}\cdot\vec{A} + \mu\epsilon\frac{\partial V}{\partial t}) = -\mu\vec{J}$$

The equation for the vector potential is automatically satisfied is we choose;

$$L = \vec{\nabla}\cdot\vec{A} + \mu\epsilon\frac{\partial V}{\partial t}$$

Substitute for $\vec{\nabla}\vec{A} = L - \mu\epsilon\frac{\partial V}{\partial t}$ into the equation for the scalar potential to obtain;

$$\nabla^2 V - \mu\epsilon\frac{\partial^2 V}{\partial t^2} + \frac{\partial L}{\partial t} = -\rho/\epsilon$$

10.4)

Given;

$$V = 0 \text{ and } \vec{A} = A_0\, sin(kx - \omega t)\,\hat{y}$$

Then find the fields;

$$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} = \omega A_0\, cos(kx - \omega t)$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = kA_0\, cos(kx - \omega t)$$

Substitute into Maxwell's equations to find that;

$$\vec{\nabla}\cdot\vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu\epsilon \frac{\partial \vec{e}}{\partial t}$$

To satisfy the equations we need the dispersion relation;

$$k = \omega\sqrt{\mu\epsilon}$$

10.7)

For the Lorentz gauge we need

$$\vec{\nabla} \cdot \vec{A} + \mu\epsilon \frac{\partial V}{\partial t} = 0$$

Choose a gauge transformation;

$$\vec{A} \to \vec{A} + \vec{\nabla}\Lambda$$

$$V \to V - \frac{\partial \Lambda}{\partial t}$$

Thus obtain the Lorentz gauge;

$$\vec{\nabla}\vec{A} + \mu\epsilon \frac{\partial V}{\partial t} = \nabla^2\Lambda - \mu\epsilon \frac{\partial^2 \Lambda}{\partial t^2} = 0$$

We find a solution to a specific gauge which makes the above equation vanish. We could choose $V = 0$ and obtain an electric field from $\frac{\partial \Lambda}{\partial t}$. We cannot choose $\vec{A} = 0$ because $\vec{\nabla} \times \vec{\nabla}\Lambda = 0$ which makes the magnetic field always vanish.

10.10)

The vector potential has the form;

$$\vec{A} = \frac{\mu}{4\pi} \int d\tau' \, \frac{\vec{J}(\vec{r'}, t')}{|\vec{r'} - \vec{r}|}$$

In the above we use the expression for the retarded time, $t' = t - (|\vec{r'} - \vec{r}|)/c$. We also note that $\int d\vec{a} \cdot \vec{J} = I = kt' = kt - kr/c$, and that $d\tau' = d\vec{a} \cdot d\vec{l}$ where $\vec{l}$ points in the direction of the current. The integral around the current loop can be divided into 4 components and each integral evaluated.

$$\int_0^{\pi} I/r \, b \, d\phi$$

$$-\int_0^{\pi} I/r \, a \, d\phi$$

$$\int_{-a}^{-b} I/r \, b \, dx$$

$$\int_b^{a} I/r \, b \, dx$$

Substitute for I as above and complete the integrals. Collecting terms we find at the origin;

$$\vec{A} = \frac{\mu k t}{2\pi} \, ln(b/a) \, \hat{x}$$

In this case $V = 0$ so that;

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -\frac{\mu k}{2\pi} \, ln(b/a) \, \hat{x}$$

However $\vec{B}$ cannot be obtained unless we know more about the spatial dependence of the vector potential.