1 Maxwell’s equations in potential form

Maxwell’s equations consist of 4 1st order pde in terms of fields. As we noted previously, the potentials turn out to be more fundamental that the fields. Thus we write these equations in terms of the potentials. The magnetic vector potential is defined by \((\vec{\nabla} \cdot B = 0)\)

\[
\vec{B} = \vec{\nabla} \times \vec{A}
\]

Now we have from Faraday’s law

\[
\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \frac{\partial \vec{A}}{\partial t}
\]

\[
\vec{\nabla} \times (\vec{E} + \frac{\partial \vec{A}}{\partial t}) = 0
\]

Then we write \(\vec{E}\) in terms of a scalar function (potential) and the vector potential as ;

\[
\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}
\]

This definition satisfies 2 of the Maxwell equations; \(\vec{\nabla} \cdot \vec{B} = 0\) and \(\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}\).

The remaining 2 equations are;

\[
\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} = -\nabla^2 V - \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A})
\]

\[
\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{V}}{\partial t}
\]

Substitute for the potentials in the equation directly above.

\[
\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 (-\vec{\nabla} \frac{\partial V}{\partial t} - \frac{\partial^2 \vec{A}}{\partial t^2})
\]

Then write \(\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}\). This finally gives

\[
\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla} (\vec{\nabla} \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}) = -\mu_0 \vec{J}
\]

We must be careful here. The above notation uses the VECTOR Laplacian which is NOT the scalar operator. It is ONLY the same in Cartesian coordinates. Note that the scalar operator has the form \(\nabla \cdot (\nabla)\). In a general curvilinear coordinate system the differential
operators would operate on the unit vectors that define the coordinate directions. Only in the case of a Cartesian system are these unit vectors constant for all positions in space.

2 Gauge transformations

We have already seen that we may always add the gradient of a scalar function to the vector potential without changing the magnetic field. \( \mathbf{A} \rightarrow \mathbf{A} + \nabla \Lambda \). However if we do this then we need to change the scalar potential

\[
\vec{E} + \frac{\partial \vec{A}}{\partial t} + \vec{\nabla} \frac{\partial \Lambda}{\partial t} = -\vec{\nabla} V - \vec{\nabla} V'
\]

\[V \rightarrow V - \frac{\partial \Lambda}{\partial t}\]

Suppose we choose \( \vec{A}, V \) to satisfy \( \vec{\nabla} \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} = 0 \) The potential equations decouple to obtain;

\[
\nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0}
\]

\[
\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}
\]

This particular choice of potentials is called the Lorentz gauge. The freedom of choice of the potentials is called a gauge transformation. The Lorentz gauge is important because it is the only choice of potentials which are Lorentz invariant. This means that a solution in one reference frame is the same in all other relativistic frames of reference.

Another choice of gauge is of course possible, and solutions would be valid in that frame of reference. For example one could choose \( \vec{\nabla} \cdot \vec{A} = 0 \). This is the Coulomb gauge. In this gauge;

\[
\nabla^2 V = -\frac{\rho}{\epsilon_0}
\]

with solution

\[
V = \frac{1}{4\pi \epsilon_0} \int \frac{\rho}{|\vec{r} - \vec{r}'|} d\tau'
\]

The scalar potential is the instantaneous Coulomb potential. Obvious this cannot be a relativistic invariant. We discuss the full meaning of gauge invariance later.
3 Multipole expansions

Consider the solution for the static potential in spherical coordinates.

\[ V = \frac{1}{4\pi \varepsilon_0} \int d\tau' \frac{\rho}{|\mathbf{r} - \mathbf{r}'|} \]

The geometric representation of this solution is shown in the figure 1. If \( \mathbf{r} > \mathbf{r}' \) we can make a power series expansion of the factor \( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \)

\[ \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_l \frac{r'^l}{r^{l+1}} P_l(\cos(\alpha)) \]

Here \( P_l \) are the Legendre polynomials. By the angular addition theorem we expand the Legendre polynomials in terms of the spherical harmonics

\[ \frac{1}{|\mathbf{r} - \mathbf{r}'|} = 4\pi \sum_{lm} \frac{1}{2l + 1} \frac{r'^l}{r^{l+1}} Y^*(m) Y(l) \int d\tau' r'^l Y_l^m(|\hat{r}'|) \]

This gives the expansion;

\[ V(r) = \frac{4\pi}{4\pi \varepsilon_0} \sum_{lm} \frac{1}{2l + 1} \frac{1}{r^{l+1}} Y_l^m(|\hat{r}|) \int d\tau' r'^l Y_l^m(|\hat{r}'|) \]

The expansion coefficients of the multipole series are;

\[ q_l^m = \int r'^l Y_l^{*m}(|\hat{r}'|) d\tau' \]
and:

\[ V(r) = \frac{1}{\epsilon_0} \sum_{lm} \left( \frac{1}{2l+1} \right) \left( \frac{1}{r^{2l+1}} \right) q_l^m Y_l^m(|\hat{r}|) \]

This expression makes sense if it converges in a reasonable number of terms. The first few terms are:

- \( q_0^0 \rightarrow \text{charge } q \)
- \( q_1^m \rightarrow \text{dipole moment } p_1^m \)
- \( q_2^m \rightarrow \text{quadrupole moment } Q_{ij} \)

We usually only use the 1\textsuperscript{st} non-zero term as the representative distribution of the charge. We note that these definitions are obtained for a spherical coordinate system. The expansion can also be written in Cartesian coordinates.

\[ V(r) = \frac{1}{4\pi\epsilon_0} \left[ \frac{q}{r} + \frac{\vec{p} \cdot \vec{r}}{r^3} + \frac{1}{2} \sum_{ij} Q_{ij} \frac{r_i r_j}{r^5} + \cdots \right] \]

Here \( \vec{p} \) is the dipole moment and \( Q_{ij} \) is the quadrupole moment.

\[ \vec{p} = \int d\tau' \vec{r}' \rho \]

- \( q_0^0 = \sqrt{3/4\pi} p_z \)
- \( q_1^1 = -\sqrt{3/8\pi} (p_x - ip_y) \)
- \( q_1^{-1} = \sqrt{3/8\pi} (p_x + ip_y) \)

The quadrupole tensor can be represented as a symmetric matrix. We obtain

- \( Q_{ij} = \int d\tau [3x_i x_j - r^2 \delta_{ij}] \rho \)
- \( q_2^2 = (1/12) \sqrt{15/2\pi} [Q_{11} - 2iQ_{12} - Q_{22}] \)
- \( q_1^2 = -(1/3) \sqrt{15/8\pi} [Q_{13} - 2iQ_{23}] \)
- \( q_0^2 = (1/2) \sqrt{5/4\pi} [Q_{33}] \)

The vector potential can be expanded in a similar way. Thus:

\[ \vec{A} = \frac{\mu}{4\pi} \int d\tau' \frac{\vec{J}}{||\vec{r} - \vec{r}'||} \]
In Cartesian coordinates each coordinate has the form:

\[ A_i = (\mu/4\pi) \left[ \int \frac{J_i \, d\tau'}{r} + \frac{\vec{r} \cdot \int J_i \vec{r}' \, d\tau'}{r^3} \cdot \ldots \right] \]

The first term vanishes because there is no magnetic charge. The second term can be re-written in the form:

\[ \int (\vec{r} \cdot \vec{r}') \vec{J} \, d\tau' = -(1/2) \int \vec{r} \times \vec{r}' \times \vec{J} \, d\tau' \]

The magnetic moment is

\[ \vec{m} = 1/2 \int d\tau' (\vec{r}' \times \vec{J}) \]

Then to second order in the multipole expansion;

\[ \vec{A} = (\mu/4\pi) \frac{\vec{m} \times \vec{r}}{r^3} \]

If the current is confined to a loop in a plane, the dipole moment is evaluated as;

\[ \vec{m} = I/2 \int \vec{r}' \times d\vec{l} = I(\vec{a} \vec{e} \alpha) \]

Note that the moment is independent of the shape of the loop.

4 Fields in media

First consider an electric field due to charges in a material. We have a geometry as shown in figure 2. In the figure, \( P \), is the field point, \( O \) is the lab coordinate origin, \( \vec{R}_j \) is the vector to the \( j^{th} \) charge that has a distribution \( \rho_j \), and \( \vec{r}_p \) is the vector to the field point.

The potential is:

\[ V_j = \left( \frac{1}{4\pi \epsilon_0} \right) \int d\tau_j \frac{\rho_j}{|\vec{r}_p - \vec{R}_j - \vec{r}_j|} \]

We must sum over all charges, \( j \). The field is obtained by the gradient with respect to \( r_p \). Now because of the large number of charges we will take a continuum limit, replacing the sum over charge with an integral over a charge density. We also assume that \( |\vec{r}_p - \vec{R}_j| > r_j \).

\[ V = \left( \frac{1}{4\pi \epsilon_0} \right) \sum_j \int d\tau_j \frac{\rho_j}{|\vec{r}_p - \vec{R}_j - \vec{r}_j|} \]

Expand in the multipole series and take only the first 2 terms.
V = ( \frac{1}{4\pi \epsilon_0} ) \sum_j \frac{1}{|\vec{r}_p - \vec{R}_j|} \left[ q_j + \vec{p}_j \cdot (\vec{r}_p - \vec{R}_j) \right] |(\vec{r}_p - \vec{R}_j)^2|

In the continuum limit $q_j \rightarrow \rho d\tau_R$ and $\vec{p}_j \rightarrow \vec{P} d\tau_R$, where $\vec{P}$ is the dipole moment per unit volume which is the polarization of the medium. Converting the above sum to an integral:

$$V = ( \frac{1}{4\pi \epsilon_0} ) \int d\tau_R \frac{\rho}{|\vec{r}_p - \vec{R}_j|} \left[ q_j + \int d\tau_R \frac{\vec{P} \cdot (\vec{r}_p - \vec{R})}{|\vec{r}_p - \vec{R}_j|^3} \right]$$

Then since:

$$\vec{\nabla}_R \left( \frac{1}{|\vec{r}_p - \vec{R}_j|} \right) = \frac{(\vec{r}_p - \vec{R})}{|\vec{r}_p - \vec{R}_j|^3};$$

$$\vec{\nabla}_R \cdot \left( \frac{\vec{P}}{|\vec{r}_p - \vec{R}_j|} \right) = \frac{\vec{\nabla}_R \cdot \vec{P}}{|\vec{r}_p - \vec{R}_j|} + \vec{P} \cdot \vec{\nabla}_R \left( \frac{1}{|\vec{r}_p - \vec{R}_j|} \right)$$

$$V = ( \frac{1}{4\pi \epsilon_0} ) \int d\tau_R \left[ \frac{\rho_{free}}{|\vec{r}_p - \vec{R}_j|} - \vec{\nabla}_R \cdot \vec{P} / |\vec{r}_p - \vec{R}_j| \right] + ( \frac{1}{4\pi \epsilon_0} ) \int dA \cdot \frac{\vec{P}}{|\vec{r}_p - \vec{R}_j|}$$

There is an effective surface charge, $\vec{P} \cdot \hat{n}$ ($\hat{n}$ the outward normal to the surface) and an induced volume charge given by $-\vec{\nabla} \cdot \vec{P}$. This leads to an extension of Gauss' law:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_{free}}{\epsilon_0} - \vec{\nabla} \cdot \vec{P} / \epsilon_0$$

$$\vec{\nabla} \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho_{free}$$

We define the electric displacement $D$ as

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

In the case of a class A dielectric, $\vec{P}$ and $\vec{E}$ are linear and in the same direction so that
$$\vec{D} = \varepsilon \vec{E}$$

In the case of the magnetic field, the development proceeds in a similar way. The current density is described by 2 components, 1) a free current density, and 2) a bound current density due to motion of charges in the material. The magnetic moment is the leading term in the multipole expansion. The free current density results in the same equation for $\vec{A}$ as previously obtained. The bound current generates a magnetization (magnetic moment per unit volume) which when combined with the free current density results in the equation

$$\vec{A} = \frac{\mu_0}{4\pi} \int d\tau_R \frac{\vec{J}_{\text{free}}}{|\vec{r}_p - \vec{R}|} + \frac{\mu_0}{4\pi} \int d\tau_R \frac{\vec{M} \times (\vec{r}_p - \vec{R})}{|\vec{r}_p - \vec{R}|^3}$$

A surface integral is neglected. The above is written;

$$\vec{A} = \frac{\mu_0}{4\pi} \int d\tau_R \frac{\vec{J}_{\text{free}} + \vec{\nabla} \times \vec{M}}{|\vec{r}_p - \vec{R}|}$$

Then we rewrite Ampere’s law to include the bound current distribution

$$\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J}_{\text{free}} + \vec{\nabla} \times \vec{M})$$

$$\vec{B} - \mu_0 \vec{M} = \mu_0 \vec{H}$$

If $\vec{B}$ and $\vec{H}$ proportional and in the same direction then we can write

$$\vec{B} = \mu \vec{H}$$

This is not true for ferromagnetic materials. Maxwell’s equations in media are;

$$\vec{\nabla} \cdot \vec{D} = \rho_{\text{free}}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{E}}{\partial t}$$
5 Sample problems

1.) A magnetic material is in the shape of a right circular cylinder of length, L, and radius, a. The cylinder has a permanent magnetization, \( \mathbf{M} \), which is uniform and parallel to the axis of the cylinder. Determine \( \mathbf{B} \) and \( \mathbf{H} \) everywhere on the cylindrical surface.

A uniform magnetization is equivalent to a surface current density. (A volume current density is \( \nabla \times \mathbf{M} \)).

\[ \mathbf{\sigma} = \mathbf{M} \]

Use the equation for a solenoid (problem 5.3). If \( N \) is the number of wire turns per unit length and \( I \) is the current/turn, then these constants produce a surface current per unit width or \( NI = M \).

We write that;

\[ d\mathbf{B} = \frac{\mu I}{4\pi} \frac{\mathbf{dl} \times \hat{r}}{r^2} = -\hat{\theta} \frac{\mu I a}{4\pi r^2} d\phi \]

Now note that we can change variables so that \( z = Z + \phi/(2\pi N) = a/cot(\theta) \)

\[ dz = -2\pi N a csc^2(\theta) d\theta \]

\[ r = acsc(\theta) \]

Then;

\[ d\mathbf{B} = \hat{\theta} \frac{\mu NI}{2} d\theta \]

\[ dB_z = \hat{\theta} \frac{\mu NI}{2} \sin(\theta) d\theta \]

The final integration gives after replacement of \( NI = M \);

\[ B_z = \frac{\mu_0 M}{2} (\cos(\theta_1) + \cos(\theta_2)) \]

2.) Find the multipole moments for the potential of a uniformly charged disk of radius, a.

The charge distribution for the disk in spherical coordinates is;

\[ \rho = \frac{\sigma}{r} \delta(cos(\theta)) \]
This results in a charge, \( Q \)
\[
Q = \int \rho \, d\tau = \int r^2 \, dr \, d\Omega \, \rho = \pi a^2 \sigma
\]

The multipole moments are:
\[
q_l^m = \sigma \int r^2 \, dr \, d\Omega \, r^l \frac{\delta(cos(\theta))}{r} Y_l^m
\]

Therefore \( m = 0 \) (axially symmetric). Integration over the polar angle and radial component gives:
\[
q_0^0 = 2\pi \sigma \sqrt{(2l + 1)/4\pi} \frac{l+2}{l+2} P_l(0)
\]
\[
P_l(0) = (-1)^{l/2} \frac{(l-1)!!}{l!!} \text{ } l \text{ even}
\]

4.) An ion of charge, \( q \), and mass, \( m \), is initially at rest. The ion is produced within an evacuated, conducting sphere of radius, \( a \), at a distance, \( d \), from the center. Show that the time for the ion to reach the wall of the sphere is:
\[
t = k[4\pi ma^3]^{1/2}(K-E)/q;
\]

Use images to get the force on the charge.
\[
F = \kappa \frac{QQ'}{[b-r]^2}
\]
\[
Q' = -\frac{Qa}{b}
\]
\[
b = \frac{a^2}{r}
\]

Then since \( F = ma \)
\[
m\ddot{r} = -\frac{Q^2(a/r)}{[a^2/r - r]^2} = \frac{Q^2(ar)}{[a^2 - r^2]^2}
\]

Change variables;
\[
m\dot{r} \dot{r} = \frac{Q^2ar}{[a^2 - r^2]^2} \, dr
\]
and integrate;
\[(\dot{r})^2 = \frac{2Q^2a}{m} \left[ \frac{1}{a^2 - r^2} - \frac{1}{a^2 - d^2} \right] \]

Integrate again and use elliptic integrals;

\[K(m) = \int_0^1 [(1 - x^2)(1 - mx^2)]^{1/2} dx\]
\[E(m) = \int_0^1 [(1 - x^2)^{-1/2}(1 - mx^2)]^{1/2} dx\]

To get;

\[t = \frac{k}{Q}(ma^3)^{1/2}(K - E)\]

where \(k^2 = (a^2 - d^2)/a^2\), and \(K\) and \(E\) are elliptic integrals of modulus, \(k\).

5.) Find the Green’s function outside an infinite cylinder of radius, \(a\), for a line source placed parallel to the axis of the cylinder. This can be done by a solution involving Bessel fns. We use images here.

The potential about a long wire with charge per unit length, \(\lambda\), is;

\[\phi = \frac{\lambda}{2\pi\epsilon_0} \ln(r);\]

Here, \(r\), is the radial distance from the wire. This potential is obtained from Gauss' law. The Green function is the solution to the equation;

\[\nabla^2 G = -4\pi\delta(\vec{r} - \vec{r}_0);\]

Thus the Green function is the potential for a point charge at \(\vec{r} = \vec{r}_0\), having a strength, \(\frac{1}{4\pi\epsilon_0}\). The boundary conditions are such that \(\phi = 0\) on the surface of the cylinder. Use images to obtain a potential. First try;

\[\phi \to \lambda \frac{1}{4\pi\epsilon_0} [\ln([\vec{R} - \vec{d}]^2) - \ln([\vec{R} - \vec{d}']^2)].\]

This can be written as;

\[\phi \to \lambda \frac{1}{4\pi\epsilon_0} \left[ \ln \left( \frac{d^2(R/d)^2 + 1 - 2(R/d)\cos(\alpha)}{R^2((d'/R)^2 + 1 - 2(d'/R)\cos(\alpha))} \right) \right].\]
Then choose \( R/d = d'/R \). We must subtract a constant term \( \frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 1} \ln(d/R)^2 \) to make the potential zero on the cylindrical surface.

Therefore the Green function is:

\[
G = \ln\left( \frac{(\vec{r} - \vec{d})^2}{(\vec{r} - (R/d)^2 \vec{d})^2} \right) - \ln(d/R)^2
\]

6.) Two long, straight, parallel wires carry a current I in opposite directions, and are a distance, d, apart. Find an expression for the vector potential at an arbitrary point in space.

From the geometry \( r^2 = x^2 + (y - d/2)^2 + z^2 \). Then we find the vector potential for each wire. The currents are in opposite directions which accounts for the difference in signs. Integration from 0 to \( \infty \) and multiply by 2 to get the total integrated value.

\[
\vec{A}_+ = (2\mu_0 I \int_0^\infty dx' \frac{\vec{J}(x')}{|x' - \vec{x}|})
\]

The vector potential is in the \( \hat{x} \) direction. The magnitude is:

\[
A_+ = \left( \frac{\mu_0 I}{2\pi} \right) \int_0^\infty dx' \frac{1}{|x' + \epsilon_+|^{1/2}} \epsilon_+^2 = (y - d/2)^2 + z^2
\]

\[
A_- = -\left( \frac{\mu_0 I}{2\pi} \right) \int_0^\infty dx' \frac{1}{|x' - \epsilon_+|^{1/2}} \epsilon_-^2 = (y + d/2)^2 + z^2
\]

\[
\vec{A} = A_+ + A_- = \frac{\mu_0 I}{2\pi} \hat{x} \left[ \ln\left( \frac{\sqrt{\epsilon_+^2}}{\sqrt{\epsilon_-^2}} \right) \right]
\]

7.) Derive equation 5.39 in the text

Equation 5.36 is:

\[
A_\phi = \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} \cos(\phi) d\phi \left( \frac{1}{a^2 + r^2 - 2ar \sin(\theta) \cos(\phi)} \right)^{1/2}
\]

Factor the term \((a^2 + r^2)\) from the denominator.

\[
A_\phi = \frac{\mu_0 I a}{4\pi(a^2 + r^2)^{1/2}} \int_0^{2\pi} \cos(\phi) d\phi \left( \frac{1}{1 - 2ar \sin(\theta) \cos(\phi)/(a^2 + r^2)} \right)^{1/2}
\]
Then expand in powers;

\[
\frac{1}{(1 - 2ar \sin(\theta) \cos(\theta)/(a^2 + r^2))^{1/2}} = 1 + \frac{1}{2} (\frac{2ar \sin(\theta)}{(a^2 + r^2)} \cos(\theta)) + \\
\frac{1}{2} \frac{(1/2)(3/2)}{2} \left( \frac{2ar \sin(\theta)}{(a^2 + r^2)} \right)^2 \cos^2(\theta) + \\
\frac{1}{2} \frac{(1/2)(3/2)(5/2)}{3 \cdot 2} \left( \frac{2ar \sin(\theta)}{(a^2 + r^2)} \right)^3 \cos^3(\theta)
\]

Substitute into the integral and integrate over powers of \(\cos(\theta)\). Even powers vanish. the result is equation 5.39.

\[
A_\phi = \frac{\mu_0 I a^2 r \sin(\theta)}{4(a^2 + r^2)^{3/2}} [1 + \frac{15a^2 r^2 \sin^2(\theta)}{8(a^2 + r^2)^2} \cdots]
\]