Dispersion Relations Review Wave Guides
Lecture 4

1 Dispersion relations

Dispersion relations were introduced into physics with the work of Kronig and Kramers in the field of optics. Waves in materials may have an imaginary wave vector as we found in the last lecture. An imaginary wave vector will produce an imaginary index of refraction and this can be related to the absorption of the EM wave. We find that the imaginary part is related to the real component due to the analytic properties of the optical functions and the requirement of causality.

This concept can be generalized to a number of applications in physics, in particular scattering where the total scattering cross section can be related to the forward scattering amplitude by the optical theorem. We develop below dispersion relations for an EM wave in a medium.

1.1 Mathematical basis

Suppose a function, $f(z)$, analytic with a complex variable, z, in some region of space. If we apply the Cauchy integral formula;

$$\frac{1}{2\pi i} \oint_C dz \, \frac{f(z)}{z - z_0} \; = \; \begin{bmatrix} f(z_0) & z_0 \, \text{within the intergal space C} \\ 0 & \text{otherwise} \end{bmatrix}$$

If the point is on the contour, then;

$$\frac{1}{\pi i} \, P \oint_C dz \, \frac{f(z)}{z - z_0} \; = \; f(z_0)$$

Now write for $z = x + iy$ and put the pole on the x axis;

$$f(x_0) \; = \; (1/\pi i) \, P \int_{-\infty}^{\infty} dx \frac{f(x)}{x - x_0}$$

Let;

f(x) = u(x) + i v(x)

Substitution gives;

$$u(x) \, + \, iv(x) \; = \; (1/\pi) \int_{-\infty}^{\infty} dx \frac{v(x)}{x - x_0} \; - \; (i/\pi) \int_{-\infty}^{\infty} dx \frac{u(x)}{x - x_0}$$

This gives;

$$u(x_0) = (1/\pi)\, P \int_{-\infty}^{\infty} dx \frac{v(x)}{x - x_0}$$
$$v(x_0) = (-1/\pi)\, P \int_{-\infty}^{\infty} dx \frac{u(x)}{x - x_0}$$

These are dispersion relations. If they have a symmetry such that $f(-x) = f^*(x)$ then the crossing relations are obtained;

$$u(-x) = u(x) \text{ and } v(-x) = -v(x)$$

These relations then allow the following forms for the integrals;

$$u(x_0) = (2/\pi)\, P \int_0^{\infty} dx \frac{x v(x)}{x^2 - x_0^2}$$
$$v(x_0) = (-2/\pi)\, P \int_0^{\infty} dx \frac{x_0 u(x)}{x^2 - x_0^2}$$

1.2 Examples

Consider a plane wave in a conducting medium. We have the dispersion relation;

$$k^2 = \mu\epsilon\omega^2(1 + i(\sigma/\omega\epsilon))$$

The index of refraction has the form $n^2 = (ck/\omega)^2$ therefore;

$$n^2 = 1 + i(\sigma/\epsilon\omega)$$

Now n^2 does not approach 0 as $\omega \to \infty$ so it does not have appropriate properties to directly use in a dispersion relation, however $(n^2 - 1) \to 0$ as $\omega \to \infty$, has a simple pole on the $Re\,\omega$ axis, and is otherwise analytic. Therefore $n^2 - 1$ is an appropriate function to use in a dispersion relation. We also note that $f(\omega) = n^2 - 1 = f^*(-\omega)$ so crossing symmetry holds. The dispersion relations are then

$$Re[n^2(\omega_0 - 1] = (2/\pi)\, P \int_0^{\infty} d\omega \, \frac{\omega\, Im[n^2(\omega) - 1]}{\omega^2 - \omega_0^2}$$
$$Im[n^2(\omega_0 - 1] = (-2/\pi)\, P \int_0^{\infty} d\omega_0 \, \frac{\omega\, Im[n^2(\omega) - 1]}{\omega^2 - \omega_0^2}$$

The analytic properties of the function are a direct consequence of causality. Look at the definition of electric susceptibility, χ. The polarization of a uniform, isotropic medium is $\vec{P} = \epsilon_0 \chi \vec{E}$ Then the electric displacement is

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon \vec{E}$$

Substitute for the polarizarion to obtain $\epsilon/\epsilon_0 = 1 + \chi$. Now the polarization depends on the past history of the fields so that

$$P(t) = \epsilon_o \int_{-\infty}^{\infty} dt'\, E(t') G(t - t')$$

Here $G(t - t')$ is the response function, and because of causality $G(\tau) = 0$ when $\tau = t - t' < 0$. We transform $P = \epsilon_0 \chi E$ to frequency space and because of the faltung theorem

$$\bar{P}(\omega) = \epsilon_0 \chi(\omega) \bar{E}(\omega)$$

$$\chi(\omega) = (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} dt\, G(t)\, e^{i\omega t}$$

$$G(\omega) = (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} d\omega\, \chi(\omega)\, e^{-i\omega t}$$

If $G(t)$ is real then $\chi(\omega) = \chi^*(\omega^*)$. Also χ must be analytic in the upper half of the omplex ω plane in order for $G(\tau)$ to vanish when $\tau < 0$.

Finally consider the following problem.

The dispersion relation for waves in a plasma is given by;

$$0 = 1 + (\omega_p^2/k) \int_{\infty}^{\infty} \frac{\partial f/\partial v}{\omega - kv}\, dv$$

where ω_p is the plasma frequency and

$$f = \sqrt{\frac{m}{2\pi k_b T}}\, exp(-\frac{mv^2}{2k_b T})$$

Landau showed that the integral is along the real v axis and passes under the pole. Show that the integral has the form;

$$\int_{\infty}^{\infty} \frac{\partial f/\partial v}{\omega - kv}\, dv = P \int_{\infty}^{\infty} \frac{\partial f/\partial v}{\omega - kv}\, dv - \frac{i\pi}{k} \frac{\partial f}{\partial v}|_{v=w/k}$$

Justify and indicate all contours over which you integrate.

Solution

The contour is shown in fig. 1

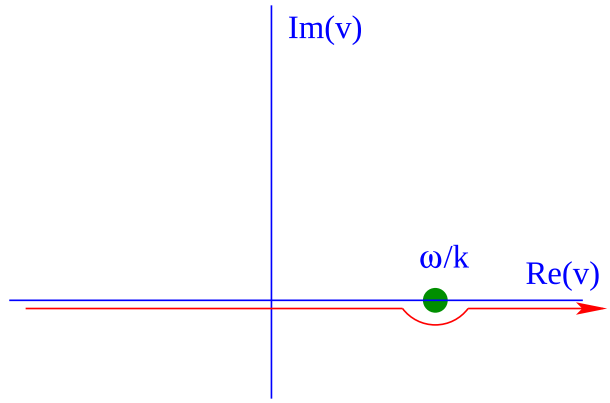

Figure 1: The figure shows the contour of integration

$$f = \sqrt{\frac{m}{2\pi k_b T}}\, exp(-\frac{mv^2}{2k_b T})$$

$$\int_{-\infty}^{\infty} dv\, \frac{\partial f/\partial v}{\omega - kv} = \int_{-\infty}^{\omega/k-\epsilon} dv\, \frac{\partial f/\partial v}{\omega - kv} +$$

$$\int_{\omega/k-\epsilon\ semi-circle}^{\omega/k+\epsilon} dv\, \frac{\partial f/\partial v}{\omega - kv} + \int_{\omega/k+\epsilon}^{\infty} dv\, \frac{\partial f/\partial v}{\omega - kv}$$

the principal value is then;

$$P\int_{-\infty}^{\infty} dv\, \frac{\partial f/\partial v}{\omega - kv} = \int_{-\infty}^{\omega/k-\epsilon} dv\, \frac{\partial f/\partial v}{\omega - kv} + \int_{\omega/k+\epsilon}^{\infty} dv\, \frac{\partial f/\partial v}{\omega - kv}$$

Therefore we only need to evaluate the integral over the semi-circle contour. Define $v = \omega/k + \epsilon\, e^{i\phi}$. Then this integral becomes;

$$-\int_{\pi}^{2\pi} d\phi\, \frac{i\epsilon\, e^{i\phi}\, \partial f/\partial v}{\omega - kv} = -(1/k)\int_{-\pi}^{0} d\phi\, \frac{i\epsilon\, e^{i\phi} \partial f/\partial v}{\epsilon e^{i\phi}} = -\frac{i\pi}{k}\, \frac{\partial f}{\partial v}|_{v=\omega/k}$$

Note that if $v = x + iy$ then $v^2 = x^2 - y^2 + 2ixy$. The contour cannot be closed by a circle whose radius goes to infinity, so you can't directly use the residue theorem.

2 The vector Laplacian

The vector Laplacian is defined by the expression;

$$\nabla^2 \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla} \times \vec{\nabla} \times \vec{A};$$

This is obtained by defining;

$$\begin{aligned}
\Gamma_1 &= (g_{11}/g^{1/2}) \left[\frac{\partial[(g_{33})^{1/2} A_3]}{\partial u_2} - \frac{\partial[(g_{22})^{1/2} A_2]}{\partial u_3} \right]; \\
\Gamma_2 &= (g_{22}/g^{1/2}) \left[\frac{\partial[(g_{11})^{1/2} A_1]}{\partial u_3} - \frac{\partial[(g_{33})^{1/2} A_3]}{\partial u_1} \right]; \\
\Gamma_3 &= (g_{33}/g^{1/2}) \left[\frac{\partial[(g_{22})^{1/2} A_2]}{\partial u_1} - \frac{\partial[(g_{11})^{1/2} A_1]}{\partial u_2} \right];
\end{aligned}$$

and;

$$\gamma = g^{-1/2} \left[\frac{\partial[(g/g_{11})^{1/2} A_1]}{\partial u_1} + \frac{\partial[(g/g_{22})^{1/2} A_2]}{\partial u_2} + \frac{\partial[(g/g_{33})^{1/2} A_3]}{\partial u_3} \right];$$

Then;

$$\begin{aligned}
\nabla^2 \vec{A} \;=\; & \hat{a}_1 \left[(1/g_{11})^{1/2} \frac{\partial \gamma}{\partial u_1} + (g_{11}/g)^{1/2} \left[\frac{\partial \Gamma_2}{\partial U_3} - \frac{\partial \Gamma_3}{\partial u_2} \right] \right] + \\
& \hat{a}_2 \left[(1/g_{22})^{1/2} \frac{\partial \gamma}{\partial u_2} + (g_{22}/g)^{1/2} \left[\frac{\partial \Gamma_3}{\partial U_1} - \frac{\partial \Gamma_1}{\partial u_3} \right] \right] + && (1) \\
& \hat{a}_3 \left[(1/g_{33})^{1/2} \frac{\partial \gamma}{\partial u_3} + (g_{33}/g)^{1/2} \left[\frac{\partial \Gamma_1}{\partial U_2} - \frac{\partial \Gamma_2}{\partial u_1} \right] \right]. && (2)
\end{aligned}$$

In the above equations g_{ij} are the metric forms for the coordinate system. The metric defines the scale length on the coordinate axes, and for any general orthorgonal curvilinear coordinate system, the length element ds is;

$$ds^2 = dx^2 + dy^2 + dz^2 = g_{11} du_1^2 + g_{22} du_2^2 + g_{33} du_3^2$$

For an orthorgonal system $g_{ij} = 0$ when $i \neq j$. Thus we have defined a system such that;

$$x = x(u_1, u_2, u_3)$$

Circular-cylinder coordinates are employed,

$$\frac{\partial^2 E_r^*}{\partial r^2}+\frac{1}{r}\frac{\partial E_r^*}{\partial r}-\frac{E_r^*}{r^2}+\frac{1}{r^2}\frac{\partial^2 E_r^*}{\partial \psi^2}-\frac{2}{r^2}\frac{\partial E_\psi^*}{\partial \psi}+\frac{\partial^2 E_r^*}{\partial z^2}+\beta^2 E_r^* = 0,$$

$$\frac{\partial^2 E_\psi^*}{\partial r^2}+\frac{1}{r}\frac{\partial E_\psi^*}{\partial r}-\frac{E_\psi^*}{r^2}+\frac{1}{r^2}\frac{\partial^2 E_\psi^*}{\partial \psi^2}+\frac{2}{r^2}\frac{\partial E_r^*}{\partial \psi}+\frac{\partial^2 E_\psi^*}{\partial z^2}+\beta^2 E_\psi^* = 0,$$

$$\frac{\partial^2 E_z^*}{\partial r^2}+\frac{1}{r}\frac{\partial E_z^*}{\partial r}+\frac{1}{r^2}\frac{\partial^2 E_z^*}{\partial \psi^2}+\frac{\partial^2 E_z^*}{\partial z^2}+\beta^2 E_z^* = 0.$$

Figure 2: The equations for $\nabla^2 \vec{E}$ in cylindrical coordinates

$$y = y(u_1, u_2, u_3)$$

$$z = z(u_1, u_2, u_3)$$

and ;

$$g_{ii} = (\frac{\partial x}{\partial u_i})^2 + (\frac{\partial y}{\partial u_i})^2 + \frac{\partial z}{\partial u_i})^2$$

We have defined the determinant for an orthorgonal system, $g = g_{11}g_{22}g_{33}$ as;

$$g = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix}$$

An example of the vector Laplacian operating on the vector electric field in a cylindrical coordinate system is shown in fig. 2. In these equations $\beta = \omega/c$.

Note than only the z component has the same form as for the scalar Laplacian. This is because the unit vector $\hat{z}$ keeps the same direction throught space.

3 Cylindrical wave guide

We propose a hollow metal tube with perfectly conducting walls. We are to determine the fields in the interior of the tube and the flow of energy down the tube. We note that for a perfect condutor the electric field must be perpendicular to the surface, and the magnetic field must be tangential to the surface. These are the boundry conditions and are derived from Maxwell's equations.

$$\vec{E} \times \hat{n} = 0 \text{ and } \vec{B} \cdot \hat{n} = 0$$

In the above, $\hat{n}$ is the normal to the surface. Also suppose there is no free charge or curents within the tube, and we choose a time dependence such that;

$$\vec{E}(\vec{r},t) = \vec{E}(\vec{r})\, e^{-i\omega t}$$

$$\vec{B}(\vec{r},t) = \vec{B}(\vec{r})\, e^{-i\omega t}$$

Substitution in Maxwell's equations gives;

$$\vec{\nabla} \times \vec{E} = i\omega\vec{B} \text{ and } \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = -i\mu\epsilon\omega\vec{E} \text{ and } \vec{\nabla} \cdot \vec{E} = 0$$

These equations are combined to give ($c = (1/\sqrt{\mu\epsilon})$);

$$\nabla^2 + (\omega/c)^2 \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} = 0$$

Note here the use of the vector Laplacian. Because of the geometry we choose to let the z axis be the unique direction along the axis of the tube. Thus write $\nabla^2 = \nabla_z^2 + \nabla_\perp^2$ and search for a wave solution in the $\hat{z}$ direction.

$$\vec{E}(\vec{r},t) = \vec{E}(x,y)e^{i(kz-\omega t)}$$

$$\vec{B}(\vec{r},t) = \vec{B}(x,y)e^{i(kz-\omega t)}$$

Substitution yields;

$$\nabla_\perp^2 + ((\omega/c)^2 - k_z^2) \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} = 0$$

Separate the fields into transverse and longitudinal components.

$$\vec{E} = \vec{E}_\perp + \vec{E}_z\hat{z}$$

$$\vec{B} = \vec{B}_\perp + \vec{B}_z\hat{z}$$

Put all this back into the 4 Maxwell's equations above and use the identities;

$$\hat{z} \times \vec{\nabla}_\perp \times \hat{z} \,=\, \vec{\nabla}_\perp$$

$$\hat{z} \cdot (\vec{\nabla}_\perp \times \hat{z}) \,=\, 0$$

$$\hat{z} \times \hat{z} \times \vec{E}_\perp \,=\, -\vec{E}_\perp$$

To obtain the following forms;

$$\vec{E}_\perp \,=\, -\frac{i}{k^2 - (w/c)^2}[k\,\vec{\nabla}_\perp E_z - \omega\,\hat{z} \times \vec{\nabla}_\perp B_z]$$
$$\vec{B}_\perp \,=\, -\frac{i}{k^2 - (w/c)^2}[k\,\vec{\nabla}_\perp B_z - \omega/c^2\,\hat{z} \times \vec{\nabla}_\perp E_z]$$

Therefore $E_\perp$ and $B_\perp$ are defined in terms of the z components. The divergence equations are not yet satisfied.

$$\vec{\nabla}_\perp \cdot \vec{E}_\perp \,=\, -\frac{\partial E_z}{\partial z}$$
$$\vec{\nabla}_\perp \cdot \vec{B}_\perp \,=\, -\frac{\partial B_z}{\partial z}$$

If we choose;

$$\frac{\partial E_z}{\partial z} \,=\, \frac{\partial B_z}{\partial z} \,=\, 0$$

The we have a wave that is totally transverse to $\hat{z}$. This can only occur if

$$k \,=\, \omega/c$$

So in this case the wave would propagate as it would in free space with the free space velocity. Note that this is a 2-D electric potential problem since $\vec{\nabla}_\perp \times \vec{E}_\perp = 0$ and $\vec{\nabla}_\perp \cdot \vec{E}_\perp = 0$. To support this propagation mode there must be another conductor inside the tube, otherwise the potential will be constant through out the interior. Therefore we cannot choose BOTH E_z and B_z to vanish together. We thus consider the two cases, transverse magentic modes (TM where $B_z = 0$) and transverse electric modes (TE where $E_z = 0$)

4 TM modes

The TM mode has $B_z = 0$ and $E_z \neq 0$ except at the surface where $E_z = 0$. In this case we have that;

$$\vec{E}_\perp = \frac{-i\,k\,\vec{\nabla}_\perp\,E_z}{k^2 - (\omega/c)^2}$$

$$\vec{B}_\perp = \frac{-i\,\omega\,\hat{z} \times \vec{\nabla}_\perp\,E_z}{k^2 - (\omega/c)^2}$$

Here we have the wave as $E_z \rightarrow E_z e^{i(k_z z - \omega t)}$

The solution for E_z is obtained from the equation;

$$\nabla_\perp^2 E_z + \gamma^2 E_z = 0$$

with $\gamma^2 = ((\omega/c)^2 - k_z)$ and $(\omega/c)^2 = k^2 = \gamma^2 + k_z^2$. Use cylindrical coordinates (ρ, ϕ, ϕ) assuming a cylindrical boundry at $\rho = a$. The above pde has a solution of the form, $E_z \sim J_\nu(\gamma\rho)\,e^{\pm i\nu\phi}$ This independent of z as we have extracted the Z dependence as indicated above. Now we must have that $E_z(\rho, \phi)|_{\rho=a} = 0$. This will occur by a choice of appropriate zeros of the Bessel function, $\alpha_{n/nu}$.

$$J_\nu(\alpha_{n\nu}/a\,\rho)|_{\rho=a} = 0$$

The final solution can then written;

$$E_z = [\sum_{n,\nu} A_{n\nu}\,J_\nu([\alpha_{n\nu}/a]\rho)\,e^{i\nu\phi}]\,e^{i(k_z z - \omega t)}$$

The dispersion relation is

$$k_z = [(\omega/c)^2 - (\alpha_{n\nu}/a)^2]^{1/2}$$

Then for k_z to be real, $(\omega/c)^2 \geq (\alpha_{n\nu}/a)^2$. Define a cut off frequency $\omega_\lambda = \frac{c\alpha_{n\nu}}{a}$ which placed in the dispersion relation yields;

$$k_z = \sqrt{(\omega/c)^2 - \omega_\lambda^2}$$

When $\omega < \omega_\lambda$ k_z is imaginary and the field decreases exponentially in z. For real k, the wave propagates in the z direction. Modes of the cylindrical guide are shown in fig. 3. It is possible to design a guide to propagate one mode, although propagation of higher frequencies would be allowed. Note that k_z is less than the free space value. Thus the phase velocity $\omega/k_z > c$.

$$V_p = \omega/k_z = \frac{c}{\sqrt{1 - (\omega/\omega_\lambda)^2}}$$

Figure 3: The modes in wave guides of various geometries

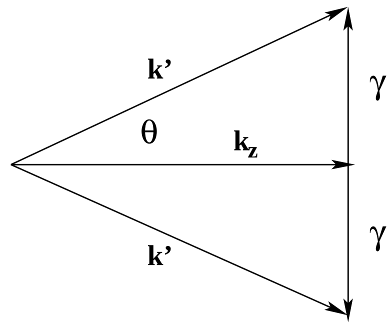

Figure 4: The vector addition for the wave vectors

5 Analysis of group and phase velocities

Choose the lowest mode of excitation. The E field is;

$$E_z = E_0\, J_0([\alpha_{1,0}/a]\rho)\, e^{i(k_z - \omega t)}$$

Now consider a large diameter tube;

$$lim_{x\to\infty}\, J_0(x) \to \sqrt{2/\pi x}\, cos(x - \pi/4) = \sqrt{2/\pi x}[\frac{e^{i(x-\pi/4)} - e^{-i(x-\pi/4)}}{2}]$$

In this mode $x = \gamma\rho$ and;

$$lim_{x\to\infty}\, E_z = \frac{E_0\, e^{ik_z z}}{\sqrt{2\pi}\,\sqrt{\gamma\rho}}\, [e^{i(\gamma\rho - \pi/4 - \omega t)} - e^{-i(\gamma\rho - \pi/4 + \omega t)}]$$

This is the form of a superposition of an outgoing and incoming wave. Define the wave vector as k' as is shown in fig. 4. We also have $\gamma^2 = (\omega/c)^2 - k_z^2$

$$\vec{k}' = \pm\gamma\hat{\rho} + k_z\hat{z}$$

Note that $tan(\theta) = \gamma/k_z$. At the cutoff frequency $k_z = 0$ so the wave propagates perpendicular to the $\hat{z}$ direction. The phase velocity is ;

$$\omega/k' = \frac{\omega}{(\gamma^2 + k_z)^{1/2}} = c$$

The velocity of the wave projected onto the $\hat{z}$ axis is;

$$V_z = V_p\, cos(\theta) = k_z c/k' = \frac{\partial\omega}{\partial k_z} = V_g$$

The traveling wave moves moves at the velocity of a wave in free space. Energy travels

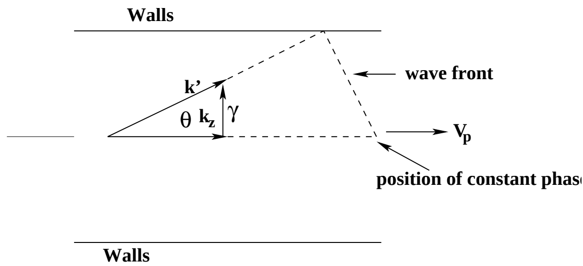

Figure 5: The vector diagram of the wave velocities showing the wave front of an outgoing wave

down the tube with velocity V_g, fig. 5. $V_g = c\,cos(\theta)$, and $V_p = c/cos(\theta)$

6 TE modes

For the TE mode we choose $E_z = 0$ and $B_z \neq 0$. In this case ;

$$\vec{B}_\perp = \frac{ik_z \vec{\nabla}_\perp B_z}{k_z^2 - (\omega/c)^2}$$
$$\vec{E}_\perp = \frac{i\omega \hat{z} \times \vec{\nabla}_\perp B_z}{k_z^2 - (\omega/c)^2}$$

The equation for the z component is;

$$\vec{\nabla}_\perp^2 B_z + \gamma^2 B_z = 0$$

Where $\gamma^2 = (\omega/c)^2 - k_z^2$. The boundry condition is $\vec{B}_\perp \cdot \hat{\rho} = 0$ and since $\vec{\nabla}_\perp = \hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}(1/r)\frac{\partial}{\partial\phi}$ we force $\hat{\rho} \cdot \vec{\nabla}_\perp B_z = 0$ which results in the derivative of the Bessel function set to zero.;

$$J_\nu'(\gamma a) = 0$$

Then use the notation that $\beta_{n\nu}$ are zeros of the derivative of the Bessel function. By substitution the solution is;

$$B_z = \sum_{n/nu} B_{n\nu}\, J_\nu([\beta_{n\nu}/a]\rho)\, e^{i\nu\phi}$$

The modes are obtained as for the TM case. The lowest mode ;

$$B_z = B_{01}\, J_0([\beta_{01}/a]\rho)$$

$$E_\phi = \frac{i\omega}{k^2(\omega/c)^2}\,\frac{\partial}{\partial\rho}[B_{01}\, J_0([\beta_{01}/a]\rho)]$$

7 TEM mode

For the case when both E_z and B_x equal zero, the fields are completely transverse, and we cannot obtain a representation for these fields in terms of the components in the Z direction. As previously separate out the Z dependence.

$$\vec{E} = \vec{E}(x,y)\, e^{i(k_z z - \omega t)}$$

$$\vec{B} = \vec{B}(x,y)\, e^{i(k_z z - \omega t)}$$

Upon substitution into the separated form of Maxwell's equations

$$[\nabla_\perp^2 + \gamma^2]\begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} = 0$$

Also $\gamma = (\omega/c)^2 - k_z^2$

Note that;

$$\vec{\nabla}_\perp \times (\vec{\nabla}_\perp \times \vec{E}) = \vec{\nabla}_\perp(\vec{\nabla}_\perp \cdot \vec{E}_\perp) - \nabla_\perp^2 \vec{E}_\perp = 0$$

The term $(\vec{\nabla}_\perp \cdot \vec{E}_\perp) = 0$ so $\gamma = 0$ and $\nabla_\perp^2 \vec{E}_\perp = 0$. Then for the 2-D potential $\vec{E}_\perp = -\vec{\nabla}_\perp V$ where V is a scalar potential. This potential must satisfy the equation;

$$\nabla_\perp^2 V = 0$$

Then we solve this scalar equation by separation of variables.

$$V = \sum_\nu C_\nu\, \rho^\nu\, e^{i\nu\phi} + \sum_\nu D_\nu\, \rho^{-\nu}\, e^{i\nu\phi}$$

The constants C_ν and D_ν are determined by the boundry contitions on the cylindrical walls. Suppose $\nu = 0$ then

$$V = C + D\, ln(\rho)$$

The fields become;

$$\vec{E}_\perp = -\vec{\nabla}\phi$$

$$\vec{E}_\perp = -(D/\rho)\hat{\rho}$$

$$\vec{B}_\perp = (1/\omega)\vec{k} \times \vec{E}_\perp$$

The boundry conditions are that $E\phi$ and B_ρ vanish at the walls. When $\nu \neq 0$ ($V = 0$ for $\rho = a, b$);

$$V = \sum_\nu C_\nu[(\frac{\rho}{a})^\nu - (\frac{\rho}{a})^{-\nu}]\, e^{i\nu\phi}$$

The potential vanishes when $\rho = a$. Then

$$\vec{E}_\perp = \sum_\nu C_\nu\, \nu[(\frac{\rho}{a})^{\nu-1} + (\frac{\rho}{a})^{-\nu-1}]\, e^{i\nu\phi}\, \hat{\rho} + \sum_\nu C_\nu\, i\nu[(\frac{\rho}{a})^{\nu-1} - (\frac{\rho}{a})^{-\nu-1}]\, e^{i\nu\phi}\, \hat{\phi}$$

But we must choose the value of $E_\phi = 0$ when $\rho = b$. The only choice is to set $\nu = 0$. There are also possible TE and TM modes.