Waves

1 1-D scalar wave equation

The 1-D, homogeneous, scalar wave equation. has the form;

$$\frac{\partial^2 \psi}{\partial x^2} = 1/V^2 \, \frac{\partial^2 \psi}{\partial t^2}$$

This is a 1-D example of a hyperbolic 2^{nd} order pde. In this equation, V is the phase velocity of the wave. Representative solutions can be harmonic;

$$\psi = A \, e^{i[kx \pm \omega t]}$$

with $\omega/k = \pm V$, as can be demonstrated by substitution. However any function, F, of the form, $\psi = F(x - Vt)$ is also a solution. As noted in the last lecture, we choose the complex harmonic form as a convenient solution, and note that by a weighted inverse Fourier transformation, any functional form can be represented as a superposition of the harmonic solutions. For the harmonic form, ω is the (angular) frequency of oscillation ($\omega = 2\pi\nu$), and k is the wave number ($k = 2\pi/\lambda$). Here ν and λ are the frequency and wavelength, respectively.

2 3-D scalar wave equation

Now extend the wave equation to 3 spatial dimensions. In this case the wave number becomes a vector, $\vec{k}$, and we find the harmonic solution;

$$\psi = A \, e^{i[\vec{k} \cdot \vec{x} - \omega t]}$$

The harmonic phase written in Cartesian coordinates, is required to be;

$$\omega^2/k^2 = V^2 = \frac{\omega^2}{k_x^2 + k_y^2 + k_z^2}$$

In spherical coordinates, the wave propagates outward from (or inward to) a point. The wave equation still has the form;

$$\nabla^2 \psi = 1/V^2 \, \frac{\partial^2 \psi}{\partial t^2}$$

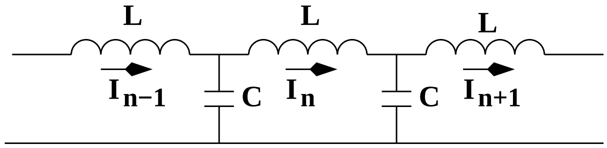

Figure 1: An infinitesmal element of a delay line

However in this case the Laplacian operator is (spherical coordinates);

$$\nabla^2 = (1/r^2)\frac{\partial}{\partial r}[r^2\frac{\partial}{\partial r}] + \frac{1}{r^2\, sin(\theta)}\,\frac{\partial}{\partial\theta}[sin(\theta)\,\frac{\partial}{\partial\theta}] + \frac{1}{r^2\, sin^2(\theta)}\,\frac{\partial^2}{\partial\phi^2}$$

We take the simple case when the solution is independent of angle;

$$(1/r^2)\frac{\partial}{\partial r}[r^2\frac{\partial}{\partial r}]f(r,t) = (1/V^2)\frac{\partial^2 f(r,t)}{\partial t^2}$$

The solution to this equation is;

$$f(r,t) = G(r - Vt)/r$$

As previously, any functional form of $G[r - Vt]$ will be a solution when that function is divided by the radial distance, r.

3 Transmission line

As an example of the 1-D wave equation, consider a simple transmission line that conducts a high frequency current signal, Figure 1, and look for a harmonic solution. The line is divided into discrete elements where the n represents the n^{th} element. The voltage across the inductor in the n^{th} element is $V_I = V_{n+1} - V_n$. The charge flowing in this circuit is Q_n. Use the fact that the voltage across the capacitor is its charge divided by the capacitance, and the voltage across the inductor is the negative of the inductance times the time change of the current flow;

$$-L\frac{\partial I_n}{\partial t} = -\frac{Q_{n+1} - Q_n}{C} + \frac{Q_n - Q_{n-1}}{C}$$

The above is re-written as ;

$$L\frac{\partial^2 Q}{\partial t^2} = (1/C)[\frac{[Q_{n+1} - Q_n]/\Delta x - [Q_n - Q_{n-1}]/\Delta x}{\Delta x}\,\Delta x^2$$

Then write the capaciatnce and inductance per unit length as $c = C/\Delta x$ and $l = L/\Delta x$ and

allow the element length $\Delta x \to 0$. The wave equation results with $V = \sqrt{1/lc}$
Now for 2 thin parallel wires of diameter, a, separated by a distance s with a highly developed skin effect (surface current)

$$c = 2\pi\epsilon_0/(ln[(s-a)/a/2])$$

$$l = \mu_0(ln[(s-a)/a/2])/(2\pi)$$

Thus $V = \sqrt{1/\epsilon_0\mu_0}$ and substitution for ϵ_0 and μ_0 gives $V = c$, the velocity of light in vacuum. The current travels with light velocity down the wires. Thus we find that the solution for the sources (charges) travels as a wave, but the sources produce fields, which also describe the flow of energy.

4 Wave equation for the fields

Maxwell's equations in a source free region of space are;

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

To separate these coupled equations, take the curl of Faraday's law.

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{\nabla} \times \vec{B}$$

Then use the identity $[\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla}(\vec{\nabla} \cdot \vec{E} - \nabla^2 \vec{E})]$, Ampere's law, and Gauss' law to write;

$$\nabla^2 \vec{E} = \mu_0\epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

In the above, the Laplacian operator must operate on a vector, so we consider this equation **ONLY** in Cartesian coordinates, where we can consider each component separately, ignoring changes of the unit vector directions. Later we may apply a vector Laplacian operator, but this can be quite complicated depending on the coordinate system. The above equation has 3 spatial coordinates and operates on 3 field variables. The same wave equation results when separating Maxwell's equations for the magnetic field, $\vec{B}$. Suppose we then choose to remove the time dependence by using a representation of the form;

$$\vec{E}(\vec{x},t) \rightarrow \vec{E}(\vec{x})e^{-i\omega t}$$

$$\vec{B}(\vec{x},t) \rightarrow \vec{B}(\vec{x})e^{-i\omega t}$$

The solution to the wave equation takes the form;

$$\vec{E} = \vec{E}_0 e^{i[\vec{k}\cdot\vec{x}-\omega t]}$$

with the same form for $\vec{B}$. Then put these into Faraday's law;

$$\vec{k} \times \vec{E} = -\omega\vec{B}$$

In the above $\vec{k}$ is in the direction of the wave motion, which means $\vec{E}$ and $\vec{B}$ are transverse to the direction of the wave, and the magnitude of B is a factor of $\omega/k = c$ less than the magnitude of E. The electromagnetic wave is transverse in free space.

5 Power flow

The Poynting vector calculates power flow in the fields. Suppose an electromagentic wave in free space, moving in the $\hat{z}$ direction with transverse fields E_x and B_y. The time averaged Poynting vector is

$$\langle\vec{S}\rangle = 1/2Re(\vec{E} \times \vec{H}^*) = (1/2\mu_0)Re(\vec{E} \times \vec{B}^*)$$

Insertion of the fields shows that $\vec{S}$ points in the $\hat{z}$ direction. Insertion of $B = E/c$ gives;

$$\langle S\rangle = \frac{E^2}{2\mu_0 c} = \frac{c\epsilon E^2}{2}$$

The time average energy density in the fields is;

$$\langle\mathcal{W}\rangle = (1/4)[\epsilon_0 E^2 + (1/\mu_0)B^2] = \epsilon_0 E^2/2$$

The energy flowing through a 1m^2 cross section in (x,y) is $\epsilon_0 E^2(ct/2)$ which has power flow $c\epsilon_0 E^2/2$ as obtained from the Poynting vector. Now the momentum in the field can be obtained from the relativistic relation $E^2 = (pc)^2 + (mc^2)^2$ with zero rest mass;

$$E = pc$$

The time average momentum per unit time crossing the 1m^2 area is;

$$\langle \mathcal{P} \rangle = \epsilon E/2$$

As an example, sunlight crossing a 1 m^2 area above the Earth's atmosphere is 1.3-1.4 kW. Attenuation occurs as the light pennetrates the atmosphere, and the surface of the earth is not perpendicualr to the direction of the radiation, so the effective solar constant is smaller. The solar constant contains all electromagnetic frequencies radiated by the sun.

6 Polarization

A linear polarized wave occurs when the electric vector lies along one direction perpendicular to the direction of motion of the wave. A circularly polarized wave has electric field projections along the two axes perpendicular to the direction of motion of the wave which are out of time phase by $\pi/2$. Elliptical polarization occurs when the time phases of the projections are not equal 0 or $\pi/2$. Examples;

Linear Polarization

$$E_x = E_{0\,x} e^{i[kz-\omega t]}$$

$$E_y = E_{0\,y} e^{i[kz-\omega t]}$$

Circular Polarization

$$E_x = E_0 e^{i[kz-\omega t]}$$

$$E_y = E_0 e^{i[kz-\omega t]+\pi/2}$$

Elliptic Polarization

$$E_x = E_0 e^{i[kz-\omega t]}$$

$$E_y = E_0 e^{i[kz-\omega t]+\pi/2+\phi}$$

7 Transmission and reflection

We know that light (electromagnetic radiation) travels with a velocity lower than c when in materials. This is easily seen as the speed of the wave is given by $V^2 = \frac{1}{\epsilon\mu}$. Insertion of the values for the dielectric constant and magnetic permeability gives the square of the velocity, which in vacuum is c^2. Most optical materials have $\mu = \mu_0$ but $\epsilon > \epsilon_0$. Other materials can have differing values of μ. Therefore in a medium the velocity is $V = 1/\sqrt{\epsilon\mu} = c/n$ with n the index of refraction, and for optical materials $n = \epsilon/\epsilon_0 > 1$. This means that at an interface between different materials, the electromagnetic wave divides so that some of the wave is transmitted and some reflected. The reason for this can be understood in terms of conservation of energy. Energy carried by the fields propagates with the velocity of the wave as can be seen from the Poynting vector. For energy conservation, the power incident on an interface must equal the power out of the interface, so that if the velocities of the energy transfer differ across the interface, then some energy must be reflected in order to conserve energy flow (power). This is true for all wave propagation across boundaries.

Below we specifically look at electromagnetic waves. Consider a plane wave incident on an interface as shown in Figure 2. There is an incident wave, and both reflected and transmitted (refracted) waves are given by the following equations.

Incident

$$\vec{E}_I = \vec{E}_{0\,I}\, e^{i[\vec{k}\cdot\vec{x}-\omega t]}$$

Refracted

$$\vec{E}_T = \vec{E}_{0\,T}\, e^{i[\vec{k}''\cdot\vec{x}-\omega'' t]}$$

Reflected

$$\vec{E}_R = \vec{E}_{0\,R}\, e^{i[\vec{k}'\cdot\vec{x}-\omega' t]}$$

The wave must satisfy a continuity requirement at all times. For any time and value of $\vec{x}$ this requires $\omega = \omega'' = \omega'$. Since the reflected and incident wave are in the same medium, $\vec{k} = \vec{k}'$. Also we must have that the phase factors are equal at $z = 0$ (the amplitude of the transmitted wave cannot depend on position). Apply these relations to the phase when $t = 0$.

$$(\vec{k}\cdot\vec{x})_{z=0} = (\vec{k}''\cdot\vec{x})_{z=0} = (\vec{k}'\cdot\vec{x})_{z=0}$$

The above relations represent conservation of momentum and require that all 3 vectors lie in the same plane. From the Figure 2;
$k\, sin(\theta_I) = k''\, sin(\theta_R) = k'\, sin(\theta_T)$

Since $\vec{k} = \vec{k}''$ and $k = \omega\, n/c$, where n is the index of refraction, $n = \sqrt{\epsilon/\epsilon_0}$, Snell's law results.

$$\theta_I = \theta_R$$

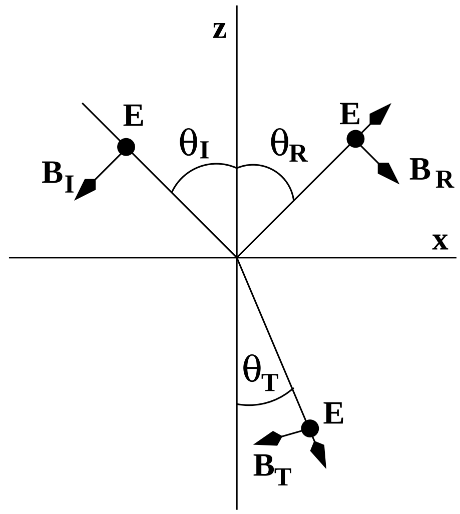

Figure 2: Reflection and refraction at an interface. In this figure $\vec{E}$ is perpendicular to the plane of incidence

$$n_I \, sin(\theta_I) \, = \, n_T \, sin(\theta_T)$$

Now let $\vec{E_I}$ be perpendicular to the plane of incidence as shown in Figure 2. The boundary conditions at a dielectric interface require that;

The tangenial component of E is continuous;

$$E_{0I} \, + \, E_{0R} \, = \, E_{0T}$$

The tangential component of H is continuous;

$$-(B_{0I}/\mu_I) \, cos(\theta_I \, + \, (B_{0R}/\mu_I) \, cos(\theta_I) \, =$$

$$-(B_{0T}/\mu_T) \, cos(\theta_T)$$

The solution is found using $B = \sqrt{\epsilon\mu}E$;

$$\frac{E_{0T}}{E_{0I}} \, = \, \frac{2\sqrt{\epsilon_I/\mu_I} \, cos(\theta_I)}{\sqrt{\epsilon_I/\mu_I} \, cos(\theta_I) \, + \, \sqrt{\epsilon_T/\mu_T} \, cos(\theta_T)}$$

$$\frac{E_{0R}}{E_{0I}} \, = \, \frac{\sqrt{\epsilon_I/\mu_I} \, cos(\theta_I) \, - \, \sqrt{\epsilon_T/\mu_T} \, cos(\theta_T)}{\sqrt{\epsilon_I/\mu_I} \, cos(\theta_I) \, + \, \sqrt{\epsilon_T/\mu_T} \, cos(\theta_T)}$$

The figure when $\vec{E_I}$ lies in the plane of incidence is similar to Figure 2 but with E and B

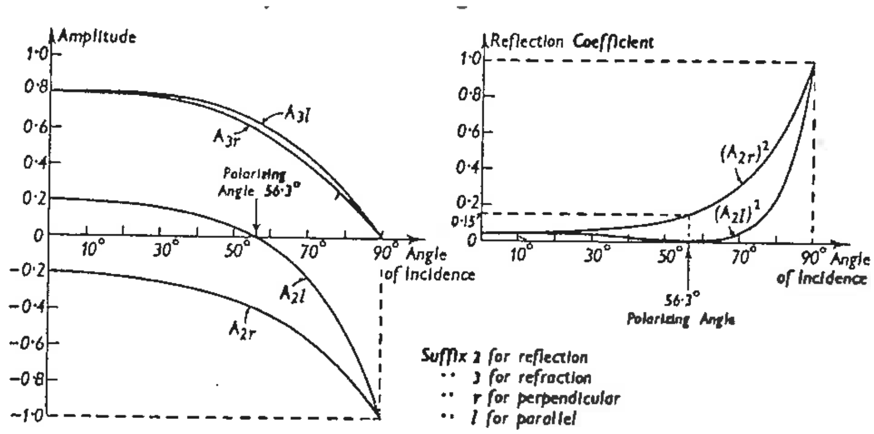

Fig. 21–3. Reflected and refracted amplitudes for $n'/n = 1{\cdot}5$.

Fig. 21–4. Reflected intensities.

Figure 3: Reflection and refraction amplitude and energy coefficients for glass, $n - 1.5$ as a function of angle

interchanged and E rotated by 180° so that $\vec{S}$ points along the propagation direction. The boundary conditions in this case are used to produce the following set of coupled equations.

The tangenial component of E is continuous;

$$E_{0I}\,cos(\theta_I) \;+\; E_{0R}\,cos(\theta_I) \;=\; E_{0T}\,cos(\theta_T)$$

The tangential component of H is continuous and $B = \sqrt{\epsilon\mu}E$;

$$\sqrt{\epsilon_I/\mu_I}E_{0I} \;-\; \sqrt{\epsilon_I/\mu_I}E_{0R} \;=\; \sqrt{\epsilon_T/\mu_T}$$

The solution is;

$$\frac{E_{0T}}{E_{0I}} = \frac{2\sqrt{\epsilon_I/\mu_I}\,cos(\theta_I)}{\sqrt{\epsilon_I/\mu_I}\,cos(\theta_T) \;+\; \sqrt{\epsilon_T/\mu_T}\,cos(\theta_I)}$$

$$\frac{E_{0R}}{E_{0I}} = \frac{\sqrt{\epsilon_I/\mu_I}\,cos(\theta_T) \;-\; \sqrt{\epsilon_T/\mu_T}\,cos(\theta_I)}{\sqrt{\epsilon_I/\mu_I}\,cos(\theta_T) \;+\; \sqrt{\epsilon_T/\mu_T}\,cos(\theta_I)}$$

These solutions are ploted in Figure 3 for glass with $n = 1.5$. The figure shows that the amplitude of the reflected wave vanishes at an angle of 56.3°. Thus all reflected light is polarized with the E vector perpendicular to the plane of incidence. Assuming $\mu_I = \mu_T = \mu_0$ which holds for almost all optical materials, this results in;

$$\frac{sin(2\theta_I) - sin(2\theta_T)}{sin(2\theta_T) + (\mu_I/\mu_T)\,sin(2\theta_I)} = 0$$

The solution occurs when $\theta_I + \theta_T = \pi/2$ or when $tan(\theta_I) = n_T/n_I$. In this case the transmitted E vector is parallel to the reflected propagation direction. Also from Snell's law;

$$sin(\theta_T) = (n_I/n_T)\, sin(\theta_I)$$

For values of $n_I > n_T$ $sin(\theta_T) > 1$, which of course is not possible, and there is no transmitted radiation through the boundary. The critical angle θ_C occurs when θ_C is complex as obtained in the relation above when $sin(\theta_T > 1$.

$$cos(\theta_T) = [1 - sin^2(\theta_T)]^{1/2} = i[sin^2(\theta_T) - 1]^{1/2}$$

The refracted wave amplitude is multiplied by the exponential ;

$$e^{i\vec{k}\cdot\vec{r}} = e^{ik[x\, sin(\theta_T) + z\, cos(\theta_T)]} = e^{ik\, x(n_I/n_T) sin(\theta_I)}\, e^{-k\, z[(n_I/n_T)^2\, sin^2(\theta_I) - 1]^{1/2}}$$

The refracted wave propagates parallel to the surface but attenuates into the medium. Thus, there is no energy flow through the boundary. This is obvious from $\vec{S} = Re(\vec{E} \times \vec{H}) = 0$ obtained by substitution of the above values into the expression for $\vec{S}$.

8 Normal incidence and impedance

Suppose a wave is incident normally to the boundary. We define a quantity called the impedance of the medium by $Z = \sqrt{\mu/\epsilon}$. In general terms, the impedance presented to any wave is obtained by considering the transmitted power. Recall that;

$$\text{Power} = \text{Force} \times \text{ Velocity}$$

In terms of Ohm's law, Power = VI and Z (or for charge flow R) equals V/I. Thus we identify an impedance by Z = Force/Velocity. For the EM wave, power is obtained from the Poynting vector, $\vec{S} = \vec{E} \times \vec{H}$ so we take $Z = E/H = \sqrt{i\omega\mu/(i\omega\epsilon)}$. The expression for Z and also be obained from Faraday's law. Units are in ohms as expected.

Take $\theta_I = 0$ and solve the coupled equations for the tangential and perpendicular components of the E field. This leads to the coupled equations;

$$E_{0I} + E_{0R} = E_{0T}$$

$$\sqrt{\epsilon_I \mu_I}\, [E_{0I} - E_{0R}] = \sqrt{\epsilon_T/\mu_T}\, E_t$$

These equations are solved to give;

$$\frac{E_{0T}}{E_{0I}} = \frac{2Z_T}{Z_I + Z_T}$$
$$\frac{E_{0R}}{E_{0I}} = \frac{Z_T - Z_I}{Z_I + Z_T}$$

Note if $Z_I = Z_T$ then there is no reflection and the amplitude of the incident wave is completely transmitted. Look at the Poynting vector of the plane wave, $\vec{S} = \vec{E} \times \vec{H}$. Use $B = \sqrt{\mu\epsilon}E = \mu H$ to obtain; $S = \sqrt{\epsilon/\mu}E^2$. Note that energy of the reflected wave moves opposite to the incident wave in front of the boundary while the transmitted wave moves in the same direction as the incident wave behind the boundary. Then $S_I - S_R = S_T$. Check by using the relations below to demonstrate energy conservation.

$$[\frac{E_{0T}}{E_{0I}}]^2 = \frac{4Z_T^2}{(Z_T + Z_I)^2}$$
$$[\frac{E_{0R}}{E_{0I}}]^2 = \frac{(Z_T - Z_I)^2}{(Z_T + Z_I)^2}$$

The impedance of free space is $Z_0 = \sqrt{\mu_0/\epsilon_0} = 377\,\Omega$. It is of interest to ask whether on can terminate free space so that a wave is reflected.

9 Waves in a conductor

When the EM wave travels in a conductor, the E field causes a current to flow. Consider the Maxwell equations;

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t}$$

As previously, uncouple these two equations to obtain;

$$\nabla^2 \vec{E} - \mu\epsilon\sigma \frac{\partial \vec{E}}{\partial t} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t} = 0$$

Choose a solution of harmonic form;

$$\vec{E} = E_0\, e^{i[kz - \omega t]}\, \hat{x}$$

Substitution gives the dispersion relation;

$$-k^2 + i\mu\sigma\omega + \mu\epsilon\omega^2 = 0$$

The solution is

$$k^2 = \mu\epsilon\omega^2[1 + i\sigma/(\epsilon\omega)]$$

The wave vector, $\vec{k}$, is complex indicating that the amplitude will attenuate, and also that the phase velocity depends on the frequency. Thus the wave disperses as the frequency components of the wave travel with different velocities. The above equation is called the dispersion relation. The wave takes the form;

$$\vec{E} = \vec{E}_0\, e^{i[\alpha z - \omega t]}\, e^{-\beta z}$$

where $k = \alpha + i\beta$. We then identify;

$$\alpha = \omega\sqrt{\mu\epsilon}\left[\frac{\sqrt{1 + (\sigma/(\epsilon\omega))^2} + 1}{2}\right]^{1/2}$$

$$\beta = \omega\sqrt{\mu\epsilon}\left[\frac{\sqrt{1 + (\sigma/(\epsilon\omega))^2} - 1}{2}\right]^{1/2}$$

When the conductivity is large $\sigma/\epsilon\omega \gg 1$ and

$$\beta \approx [\mu\sigma\omega/2]^{1/2}$$

The amplitude of a wave after traveling a distance $\delta = [\frac{2}{\omega\mu\sigma}]^{1/2}$ in a conducting material will be reduced in value by e^{-1}. This distance is the skin depth. For copper $\mu = \mu_0$ and $\sigma = 5.8 \times 10^7$

Table 1: The skin depth as a function of frequency for copper

ω (Hz)	60	10^6	3×10^{10}
δ (m)	9×10^{-3}	6.6×10^{-5}	3.8×10^{-7}

10 Reflection at a conducting surface

Consider a plane, linear polarized wave incident on a conducing medium. As previously, define the wave amplitudes by;

Incident

$$\vec{E}_I = \vec{E}_{0\,I}\, e^{i[\vec{k}\cdot\vec{x}-\omega t]}$$

Refracted

$$\vec{E}_T = \vec{E}_{0\,T}\, e^{i[\vec{k}'\cdot\vec{x}-\omega t]}$$

Reflected

$$\vec{E}_R = \vec{E}_{0\,R}\, e^{i[\vec{k}\cdot\vec{x}-\omega t]}$$

In this case, k' is complex as was obtained in the last section, $k' = \alpha + i\beta$. Assume normal incidence to reduce the complexity of the solution. We apply the boundary conditions at the surface.

Tangential E continuous

$$E_{0I} + E_{0R} = E_{0T}$$

Tangential H continuous

$$\sqrt{\epsilon_I/\mu_I}(E_{0I} - E_{0R}) = \frac{k'}{\omega\mu_T}\, E_{0T}$$

Now set E_{0I} to be real, but both E_{0T} and E_{0R} cannot both be real as k' is complex. The solution is

$$\frac{E_{0R}}{E_{0I}} = \frac{1-(k'/(\omega\mu_T))\sqrt{\mu_I/\epsilon_I}}{1+(k'/(\omega\mu_T))\sqrt{\mu_I/\epsilon_I}}$$

$$\frac{E_{0T}}{E_{0I}} = \frac{2}{1+(k'/(\omega\mu_T))\sqrt{\mu_I/\epsilon_I}}$$

Since k' is complex there will be phase differences not present in the dielectric case. For a good conductor $\sigma/\omega\epsilon \gg 1$ and

$$\frac{E_{0T}}{E_0 I} = (1-i)\delta\sqrt{\epsilon_T/\mu_T}$$

$$\delta = [2/(\mu_T \sigma \omega)]^{1/2}$$

$$\frac{E_{0R}}{E_0 I} \rightarrow -1$$

11 Phase velocity and group velocity

We choose a 1-D wave packet of Gaussian form, composed of a superposition of frequencies.

$$F(x, t = 0) = \frac{1}{\sigma \sqrt{2\pi}} e^{-x^2/2\sigma^2}$$

Apply a Fourier transformation to obtain;

$$\mathcal{F} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x, 0)\, e^{-ikx}$$

$$\mathcal{F} = e^{-\sigma^2 k^2/2}/\sqrt{2\pi}$$

Use this for the inverse transform;

$$f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{i(kx - \omega t)}\, e^{\sigma^2 k^2/2}$$

Now ω is a function of k. Expand $\omega(k)$ in a power series;

$$\omega(k) = \omega_0 + \frac{d\omega}{dk} k + \frac{d^2\omega}{dk^2} k^2/2 + \cdots$$

Keep terms to 2^{rd} order and define $\alpha = \frac{d\omega}{dk} = V_g$ and $\beta^2 = \frac{d^2\omega}{dk^2}$. We use $\omega(k) - \omega(k_0) \rightarrow \omega$ and $k - k_0 \rightarrow k$. The inverse transformation then is;

$$f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, exp[-\sigma^2 k^2/2 + ik(x - \alpha t) - i\beta^2 k^2 t/2]$$

Integrated the result is;

$$f(x, t) = \frac{1}{\sqrt{2\pi}(\sigma^2 + i\beta^2 t)^{1/2}} e^{(x - \alpha t)^2/(2(\sigma^2 + i\beta^2 t))}$$

The wave propagates with velocity $V_g = \alpha = \frac{d\omega}{dk}$. This is the group velocity representing the velocity of the superimposed envelope of all the frequency components of the wave. The phase velocity is $V_p = \omega/k$.In the above example, there is also a dispersion illustrated by the increase in the Gausian width as a function of time. The pulse remains Gaussian but spreads in width as it travels in x. This is due to the fact that the frequency is not a linear function of the wave vector.

12 Index of refraction

The phase velocity of an EM wave in a medium is $V_p = \sqrt{1/\epsilon\mu}$ where almost always $\mu = \mu_0$. In this section we develop a simple model to evaluate the index of refraction. We assume that the E field in a medium takes the form;

$$\langle E \rangle = E_0 + E_p = (1 - \frac{N\alpha}{3\epsilon_0})E_0$$

In the above $E_p = -\vec{P}/\epsilon_0$ is the induced dipole field with P the polarization in the material due to the E_0 vector of the EM wave acting on the electrons in the material. The number of dipoles per unit volume is N, and α is the atomic polarizability (Clausius-Mossotti equation).

The force applied to an electron in the material is $F = eE_0$. These electrons are bound to molecules, and we assume that the binding force as the electron is moved away from equilibrum is linear (small displacements). Thus we have an equation of the form;

$$Force = m\frac{d^2x}{dt} = qE_0 - ax - m\Gamma\frac{dx}{dt}$$

where x is the displacement, xa the restoring force, qE_0 the driving force, and a resistive (dissipative) force $m\Gamma\frac{dx}{dt}$. Collecting terms we obtain;

$$m\frac{d^2x}{dt} = qE_0 - ax - m\Gamma\frac{dx}{dt}$$

Then assume that the driving term is harmonic with a time dependence, $E_0 \rightarrow E_0\,cos(\omega t)$

The solution of the above equation is therefore,

$$x = A\,cos(\omega t) + B\,sin(\omega t)$$

$$A = (qE_0/m)\,\frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \Gamma^2\omega^2}$$

$$B = (qE_0/m)\,\frac{\Gamma\omega}{(\omega_0^2 - \omega^2)^2 + \Gamma^2\omega^2}$$

$$\omega_0^2 = a/(\epsilon m)$$

Then ;

$$D = \epsilon E = \epsilon_0 E + P$$

$$P = \epsilon_0(\epsilon_r - 1)E_0$$

The induced dipole is assumed to be $-ex$ (atoms do not move) and N is the number of atoms per unit volume so that, $P = -Nex$. Collecting terms;

$$\epsilon_r = 1 + NqA/E_0 + (NqB/E_0)\, tan(\omega t)$$

If we neglect Γ then;

$$\epsilon_r = n^2 = 1 + (Nq^2/m)\frac{1}{\omega_0^2 - \omega^2}$$

In glass the resonant frequencies are in the ultraviolet so that $\omega_0 > \omega$. As $\omega \rightarrow \omega_0$ n increases so blue light has a larger index than red light. The other component represents absorption of the EM wave and induces an imaginary component in the index of refraction.

We can obtain the dispersion relation from $n^2 = 1/\sqrt{\epsilon_r} = (c/V_p)^2 = c^2k^2/\omega^2$. Using the above

$$\omega^2 = c^2k^2 - (nq^2)/(m\epsilon_0)\frac{\omega^2}{\omega_0^2 - \omega^2}$$

The phase velocity is;

$$V_p^2 = (\omega/k)^2 = \frac{c^2}{1 + Nq^2/(\epsilon_0 m(\omega_0^2 - \omega^2))}$$

The group velocity can also be found from $\frac{d\omega}{dk}$.