Integral Equations - Lecture 1

1 Introduction

Physics 6303 discussed integral equations in the form of integral transforms and the calculus of variations. An integral equation contains an unknown function within the integral. The case of the Fourier cosine transformation is an example.

$$F(k) \,=\, \sqrt{2/\pi} \int_0^\infty dx \, cos(kx) \, f(x)$$

It is assumed that $F(k)$ is known and $f(x)$ is to be determined. The function $f(x)$ can be found by the Fourier inversion theorem.

$$f(x) \,=\, \sqrt{2/\pi} \int_0^\infty dk \, cos(kx) \, F(k)$$

It then becomes useful to classify integral equations so that standard methods may be applied for their solution. Thus consider the following forms.

Voltera Equation

The Voltera equation takes the form;

$$\phi(x) \,-\, \lambda \int_a^x ds \, K(x,s) \, \phi(s) \,=\, f(x)$$

In the above, ϕ is an unknown function and λ is a parameter. Thus λ might be considered an eigenvalue. The function $f(x)$ is known, and $K(x,s)$ is the kernel.

Fredholm Equation

The Fredholm equation takes the form;

$$\phi(x) \,-\, \lambda \int_a^b ds \, K(x,s) \, \phi(s) \,=\, f(x)$$

This equation is the same as the Voltera equation except that the integral is over fixed limits, $a \leq s \leq b$.

If the unknown function ϕ appears only under the integral then the equation is of the first kind, otherwise it is of the second kind. Thus a Fredholm equation of the first kind has the form;

$$f(x) \;=\; \lambda \int_a^b ds\, K(x,s)\, \phi(s)$$

One immediately recognizes this as a solution to Poisson's pde when $f(x)$ is the potential and ϕ the charge density. Assume that;

$$\int_a^b dx \int_a^b ds\, |K(x,s)|^2 \;<\infty$$

In the above example, the solution to Poisson's equation uses the 3-D Green's function as the kernel. We look further into the relationship of an integral equation to a differential form.

$$\frac{d^2y}{dt^2} \;+\; A(t)\,\frac{dy}{dt} \;+\; B(t)\,y \;=\; g(t)$$

We seek a solution with boundary conditions, $y(a) \;=\; y_0$ and $y'(a) \;=\; y_0'$. Integration of the derivatives gives;

$$\int_a^x dt\, \frac{d^2y}{dt^2} \;=\; \frac{dy}{dt}\,|_a^x \;=\; \frac{dy}{dt} - y_0'$$

$$\int_a^x dt\, A(t)\,\frac{dy}{dt} \;=\; Ay\,|_a^x \;-\; \int_a^x dt\, y\, A'$$

The last equation above was obtained through integration by parts. Now substitute into the pde to obtain;

$$y' \;=\; -Ay\; \int_a^x\; dt\,(B - A')y \;+\; \int_a^x dt\, g \;+\; A(a)y_0 + y_0'$$

Integrate a second time to obtain;

$$y \;=\; -\int_a^x dt\, Ay \;-\; \int_a^x du \int_a^u dt\,(B - A')y \;+\; \int_a^x du \int_a^u dt\, g \;+\; [A(a)y_0 + y_0'](x-a) \;+\; y_0$$

Use the identity;

$$\int_a^x du \int_a^u dt\, f(t) \;=\; \int_a^x dt\,(x-t) f(t)$$

to obtain the form;

$$y(x) \;=\; -\int_a^x dt\,[A(t) + (x-t)[B(t) - A'(t)]]\, y(t) \;+\; \int_a^x dt\,(x-t)\, g(t) \;+\; [A(a)y_0 + y_0'](x- a) \;+\; y_0$$

Identify;

$$\lambda K(x,t) \;=\; (t-x)[B(t)-A'(t)] \;-\; A(t)$$

$$f(x) \;=\; \int\limits_a^x dt\,(x-t)\,g(t) \;+\; [A(a)y_0 + y_0'](x-a)\;y_0$$

Substitution produces a Voltera integral equation of the 2^{nd} kind. Note that the boundary conditions are included in the integral equation. This is not the case for the pde where the boundary conditions are added after the pde is solved.

Note that if the kernel is symmetric, $K(x,t) = K(t,x)$. Most physical equations have symmetric kernels. Also if the kernel is separable, $K(x,t) = f(x)g(t)$. Finally if the kernel has the form;

$$K(x,t) \;=\; \begin{bmatrix} K(x,t) & t < x \\ 0 & t > x \end{bmatrix}$$

then the Voltera equation is a limiting case of the Fredholm equation.

2 Example

Suppose the moment problem;

$$M_n \;=\; \int\limits_{-\infty}^{\infty} dx\,x^n\,\phi(x)\rho(x)$$

The weighting function is $\rho(x)$, and we are asked to find the function $\phi(x)$ for a given value of M_n. For this example let $\rho = e^{-x^2}$ and multiply both sides of the equation by $\frac{(2x)^n}{n!}e^{-x^2}$. The first term is the normalization for the Hermite polynomials.

$$\frac{(2x)^n}{n!}e^{-x^2}\,M_n \;=\; \int\limits_{-\infty}^{\infty} dx'\,x'^n\frac{(2x)^n}{n!}e^{-(x^2+x'^2)}\,\phi(x')$$

Then sum over all n;

$$W \;=\; e^{-x^2}[\sum_{n=0}^{\infty}(\frac{(2x)^n}{n!}\,M_n)] \;=\;$$
$$\int\limits_{-\infty}^{\infty} dx'\,x'^n\sum\frac{(2x)^n}{n!}e^{-(x^2+x'^2)}\,\phi(x')$$
$$W \;=\; \int\limits_{-\infty}^{\infty} dx'\,e^{-(x-x')^2}\,\phi(x')$$

The above equation is obtained by expanding the cross product term, $e^{2xx'}$, obtained from $e^{-(x-x')^2}$, in a power series. If the series converges the moment problem is converted into an integral equation which may be solved by Hermite polynomial expansion, $\phi(x) \;=\; \sum c_n\,H_n(x)$.

However in this case, we solve the integral equation by Fourier transform. Let;

$$\Phi(k) \;=\; \sqrt{1/2\pi} \int_{-\infty}^{\infty} dx\, e^{-ikx}\, \phi(x)$$

Then multiply both sides of the integral equation by e^{ikx} and integrate over k.

$$\sum_{n=0} \int_{-\infty}^{\infty} dx\, \frac{(2x)^n}{n!}\, e^{-x^2}\, e^{ikx} \;=$$
$$\sqrt{1/2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dk'\, e^{-(x-x')^2}\, e^{ikx}\, e^{-ik'x}\, \Phi(k')$$

Then consider the integral;

$$W \;=\; \sqrt{1/2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx'\, e^{-(x-x')^2}\, e^{ikx}\, e^{-ik'x}$$

Change variable $u = x - x'$ to obtain

$$W \;=\; \sqrt{1/2\pi} \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dx'\, e^{-u^2}\, e^{iku}\, e^{-i(k-k')x'}$$

The integral over x' yields $2\pi\delta(k-k')$. Thus the right side of the integral equation becomes, $\sqrt{\pi}e^{-k^2/4}\,\Phi(k)$. The left side is the integral;

$$V \;=\; \sum_{n=0} M_n\, \frac{2^n}{n!} \int_{-\infty}^{\infty} dx\, x^n e^{-x^2+ikx}$$

This integral can be integrated by parts for odd and even integer values of n. This results in the form;

$$V \;=\; -\sqrt{\pi} \sum M_n \frac{2^n}{n!}\, (2i)^n 2^n\, \frac{d^n}{dk^n} e^{-k^2/16}$$

Equating both sides we find the solution for the Fourier transform of $\phi(x)$.

$$\Phi(k) \;=\; -e^{k^2/4} \sum_{n=0} M_n\, \frac{4^n\,(2i)^n}{n!}\, \frac{d^n}{dk^n} e^{-k^2/16}$$

3 Equations of the 2^{nd} kind

Consider the Fredholm equation;

$$\phi(x) \;-\; \lambda \int_a^b ds\, K(x,s)\, \phi(s) \;=\; f(x)$$

In the case where $f(x) = 0$ the homogeneous equation results.

$$\phi(x) = \lambda \int_a^b ds\, K(x,s)\, \phi(s)$$

The solution to this problem occurs only for certain values of $\lambda \rightarrow \lambda_n$, and thus the functions, $\phi \rightarrow \phi_n$ are eigenfunctions. If the kernel is symmetric and non-singular, the eigenvalues are real and the eigenvalue spectrum is discrete. It the kernel is symmetric but singular, then part of the spectrum may be continuous. If it is not symmetric, the eigenfunctions are not necessarily real. To initiate a solution, expand the kernel in a complete set of functions, ψ_n, to obtain a separable kernel. This technique works only if the functions, ψ_n satisfy the boundary conditions of the problem.

$$K(x,s) = \sum \psi_n(x)\, g_n(s)$$

Also expand $\phi(x)$ using the function set.

$$\phi(x) = \sum a_b\, \psi_n(x)$$

Substitute into the integral equation.

$$\sum_n a_n\, \psi_n(x) = \lambda \int_a^b ds \sum_{m,l} \psi(x)\, g(s)\, a_l\, \psi(s)$$

Use linear independence of the function set to write.

$$a_n = \lambda \sum_l a_l \int_a^b ds\, g_n(s) \psi_l(s)$$

Define;

$$\alpha_{nl} = \int_a^b ds\, g_n(s)\, \psi_l(s)$$

$$a_n = \lambda \sum_l \alpha_{nl}\, a_l$$

$$\sum_l [\lambda \alpha_{nl} - \delta_{nl}] a_l = 0$$

This is a set of linear, homogeneous, algebraic equations which have solutions only if the determinate equals zero. This is an eigenvalue problem in determinant form requiring diagonalization.

$$|\lambda\, \alpha_{nl} - \delta_{nl}| = 0$$

4 Perturbations

The Fredholm equation may be solved by perturbation techniques. Define;

$$
\begin{aligned}
A(x) \;&=\; \int_a^b ds\, K(x,s)\, \phi(s) \\
\phi(x) \;&=\; f(x) \;+\; \lambda\, A(x)
\end{aligned}
$$

This equation is solved by successive approximations. Let the first approximation be $\phi_1 = f(x)$. Then;

$$
\begin{aligned}
\phi_2 \;&=\; f(x) + \int_a^b ds\, K(x,s)\, \phi_1(s) \\
\phi_3 \;&=\; f(x) + \int_a^b ds\, K(x,s)\, \phi_2(s) \\
&\qquad\quad \cdots \\
\phi_n \;&=\; f(x) + \int_a^b ds\, K(x,s)\, \phi_{n-1}(s)
\end{aligned}
$$

Assume the kernel is quadratically summable and $\lambda < 1/B$. The series converges if.

$$
B^2 \;=\; \int_a^b dx \int_a^b ds\, |K(x,s)|
$$

5 Fourier transformation solutions

Again consider the equation of the form;

$$
\phi(x) \;-\; \lambda \int_{-\infty}^{\infty} ds\, g(x-s)\, \phi(s) \;=\; f(x)
$$

Since the kernel has the form $g(x-s)$ one can successfully apply a Fourier transform.

$$
\begin{aligned}
\Phi(k) \;&=\; \sqrt{1/2\pi} \int_{-\infty}^{\infty} dx\, e^{ikx}\, \phi(x) \\
F(k) \;&=\; \sqrt{1/2\pi} \int_{-\infty}^{\infty} dx\, e^{ikx}\, f(x) \\
G(k) \;&=\; \sqrt{1/2\pi} \int_{-\infty}^{\infty} dx\, e^{ikx}\, g(x)
\end{aligned}
$$

Apply the Fourier transform to the integral equation.

$$\Phi k \,-\, \lambda\sqrt{1/2\pi}\int_{-\infty}^{\infty} ds\,\phi(s)\int_{-\infty}^{\infty} dx e^{ikx}\,g(x-s) \,=\, F(k)$$

Make a change of variable, $u = x - s$.

$$\int_{-\infty}^{\infty} dx\,e^{ikx}\,g(x-s) \,=\, e^{iks}\int_{-\infty}^{\infty} du\,e^{iku}\,g(u)$$

This then results in;

$$\Phi(k) \,-\, \lambda\sqrt{2\pi}\,\Phi(k)\,G(k) \,=\, F(k)$$

$$\Phi(k) \,=\, \frac{F(k)}{1-\sqrt{2\pi}\lambda G(k)}$$

The solution is the Fourier inverse;

$$\phi(x) \,=\, \sqrt{1/2\pi}\int_{-\infty}^{\infty} dk\,e^{ikx}\,\frac{F(k)}{1-\sqrt{2\pi}\lambda G(k)}$$

6 Example

The Schrodinger equation for a velocity dependent potential must be written in integral form. Suppose the potential is;

$$V \,=\, V(\vec{r},\vec{p})$$

In the above, $\vec{p}$ is the momentum of a particle in a potential well, $\vec{p} \,=\, -i\hbar\vec{\nabla}$. The Schrodinger equation then takes the form;

$$\nabla^2\psi \,+\, \frac{2m}{\hbar^2}[E - V(\vec{r},\vec{p})] \,=\, 0$$

Apply a Fourier transform. This changes the equation from coordinate space to momentum space with;

$$\Psi(\vec{p}) \,=\, (1/2\pi)^{3/2}\int_a^b d\vec{r}\,e^{-i\vec{k}\cdot\vec{r}}\,\psi(\vec{r})$$

$$\psi(\vec{r}) \,=\, (1/2\pi)^{3/2}\int_a^b d\vec{r}\,e^{i\vec{k}\cdot\vec{r}}\,\Psi(\vec{p})$$

Multiply the Schrodinger equation by $(1/2\pi)^{3/2}\,e^{-\vec{k}\cdot\vec{r}}$ and integrate. Use the convolution theorem to rewrite the right hand side to obtain;

$$\frac{k^2}{2m}\Psi(\vec{k}) + \int_{-\infty}^{\infty} d\vec{k}\,\Psi(\vec{k}')\,V((\vec{k}'-\vec{k}),\vec{k}) = E\Phi(\vec{k})$$

$$V(\vec{k}-\vec{k}',\vec{k}') = (\frac{1}{2\pi})^{3/2} \int_{-\infty}^{\infty} d\vec{r}e^{i(\vec{k}'-\vec{k})\cdot\vec{r}}\,V(\vec{r},\vec{k}')$$

This is a Fredholm equation of the 2^{nd} kind.

7 Solution to the Fredholm equation of the 1^{st} kind

Consider the equation;

$$f(x) = \int_a^b dt\,K(x,t)\,\phi(t)$$

This appears to be an integral transformation. Thus assume that the function $\phi(t)$ can be expressed in a set of complete, orthogonal functions, $\psi_n(t)$. In the above, $K(x,t)$ would be the expansion coefficients. Thus;

$$\phi(t) = \sum_n a_n\,\psi_n(t)$$

$$f(x) = \sum_n a_n \int_a^b dt\,K(x,t)\psi_n(t)$$

Use orthogonality to write;

$$\int_a^b dt\,\psi_n^*(t)\,\psi_m(t)\,\rho = \delta_{nm}$$

In the above, ρ is the weighting function. Using this define;

$$g_n(x) = \int_a^b dt\,K(x,t)\psi_n(t)\,\rho$$

$$g_n = \sum_m b_{mn}\psi_m(x)$$

$$b_{mn} = \int_a^b dx\,\psi_m(x) \int_a^b dt\,K(x,t)\,\psi_n(t)$$

$$b_{mn} = \int_a^b dx \int_a^b dt\,\psi_m(x)\psi_n(t)\,K(x,t)$$

$$f(x) = \sum_{nm} a_n\,b_{mn}\,\psi_m$$

Then let $f(x) = \sum_m c_m \psi_m(x)$

$$c_m = \int_a^b dx\, f(x)\, \psi_m(x)\, \rho(x)$$

This results in $c_m = \sum_n a_n\, b_{mn}$

We need a_n to find ϕ;

$$\phi(t) = \sum_n a_n\, \psi_n(t).$$

This forms a matrix equation which can be inverted. The formulation is useful if there are a small number of coefficients, *ie* if the matrix has small dimensions.

8 Integral Transformations

Again consider the equation;

$$f(x) = \int_a^b dt\, K(x,t)\, \phi(t)$$

Note that $f(x)$ is the integral transformation of $\phi(t)$. We need the inverse transformation to find $\phi(t)$ from $f(x)$. For special kernels use otho-normality of the kernels to write;

$$\int dx\, K^*(x,t)\, K(x,t') = \delta(t-t')$$

This occurs if the kernel is of harmonic form, e^{ixt}, however a kernel in the form of a Bessel function, $K(x,t) = \sqrt{t}\, J_m(kt)$ is equally valid. The requirement that the kernel take this form is quite restrictive. However if this form occurs, then the solution is straight forward.

9 Example

Consider a Fredholm equation of the form;

$$\phi(x) = x^2 + 3 + 2 \int_0^1 dy\, (3x^2 + 3y^2 - 2y)\, \phi(y)$$

The solution must have the form;

$$\phi(x) = a + bx^2$$

Substitute into the equation to obtain;

$$\phi(x) \;=\; x^2+3+6x^2\int_0^1 dy\,(a+by^2) \;+\; 2\int_0^1 dy\,(3y^2-2y)(a+by^2)$$

$$\phi(x) \;=\; a+bx^2 \;=\; x^2+3+6x^2a+2x^2b+2(a-2+3b/4-2b/3)$$

Use linear independence;

$$a = b/8$$

$$b = (1+6a+6b/3)$$
$$a = -1/14 \quad b = -4/7$$

10 Example

Consider the Voltera equation;

$$\phi(k) \;=\; c\int_{-\infty}^{\infty} dx\, e^{ikx}\,\phi(x)$$

This has the form of a Fourier transform. Let $u = cx$ and $s = k/c$. Then;

$$\phi(s) \;=\; \int_{-\infty}^{\infty} du\, e^{ius}\,\phi(u)$$

Write this so that odd or even symmetry can be applied.

$$\phi(s) \;=\; \int_{-\infty}^{\infty} du\, cos(us)\,\phi(u) \;+\; i\int_{-\infty}^{\infty} du\, sin(us)\,\phi(u)$$

$$\phi(s) \;=\; \int_0^{\infty} du\, cos(us)\,(\phi(u)+\phi(-u)) +$$
$$i\int_{-\infty}^{\infty} du\, sin(us)\,(\phi(u)-\phi(-u))$$

The solution is obtained by expansion of ϕ in the harmonic forms of $sin(us)$ and $cos(us)$ which match the boundary conditions of the problem.

11 Green's Function

Suppose we have a 2^{nd} order ode of the form;

$$a_1\, y''(x) + a_2\, y'(x) + a_3\, y(x) \;=\; \mathcal{L}\, y \;=\; f(x)$$

Attempt to put the homogeneous form of the equation in self adjoint form.

$$\mathcal{L}\, y \;=\; \frac{d}{dx}\,[p(x)\,\frac{dy}{dx}] \;+\; q(x)y(x) \;=\; 0$$

This is done by multiplying by ;

$$\frac{1}{a_1}\, exp[\int^{x} dx'\;\frac{a_2}{a_1}]$$

Note that this equation can be written;

$$\frac{d}{dx}[exp[\int^{x} dx'\;\frac{a_2}{a_1}]\frac{dy}{dx}] \;+\; \frac{a_3}{a_1}\, exp[\int^{x} dx'\;\frac{a_2}{a_1}]\, y \;=\; 0$$

In the self adjoint form the homogeneous operator is;

$$\mathcal{L} \;=\; \frac{d}{dx}[p(x)\frac{d}{dx}] \;+\; q(x)$$

The boundary conditions for $a < x < b$ are set to be;

$$y(a) \;=\; 0 \quad y'(a) \;=\; 0$$

$$y(b) \;=\; 0 \quad y'(b) \;=\; 0$$

Since the equation is a linear, second order ode, there are 2 linearly independent solutions. We wish the solutions to be continuous at a point between a and b, $a < x' < b$, and the derivative of the solution to be discontinuous. Let the 2 solutions be;

$$y \rightarrow G_1 \;=\; u(x) \quad a < x < x'.$$
$$y \rightarrow G_2 \;=\; v(x) \quad x' < x < b.$$

For the solutions to be continuous with normalization constants a and b to be determined;

$$u(x') \;=\; v(x')$$

For the derivatives to be discontinuous

$$\int_{x'-\epsilon}^{x'+\epsilon} dx\,\frac{d}{dx}[p(x)\frac{d}{dx}\,y] \;=\; -1$$

Then as $\epsilon \rightarrow 0$

$$p^+ \frac{dv^+}{dx} - p^- \frac{du^-}{dx} = -1$$

The solution of these two equations is possible if;

$$\begin{vmatrix} u & v \\ u' & v' \end{vmatrix} \neq 0$$

The Wronskian is then normalized such that;

$$W = [uv' - u'v] = A/p$$

By substitution one finally obtains;

$$G(x, x') = \begin{bmatrix} -(1/A)u(x)v(x') & a < x < x' \\ -(1/A)u(x')v(x) & x' < x < b \end{bmatrix}$$

Now this solution is such that;

$$\mathcal{L}G = -\delta(x - x')$$

If the inhomogeneous equation has the form;

$$\mathcal{L}\, y = f.$$

Then the solution is

$$y = \int_b^a dx'\, G(x, x')\, f(x')$$

Operation of $\mathcal{L}$ on G produces a δ function within the integral which when integrated gives the inhomogeneous term f. Thus we have obtained an integral equation where the kernel is a delta function.

12 Some general properties of the Green's function

One can prove that the Green's function is symmetric.

$$G(\vec{r}, \vec{r}') = G(\vec{r}', \vec{r})$$

The Green's function is analytic everywhere except at the δ function singularity. Look at the equation;

$$\mathcal{L}\, G + k_n^2\, G = -4\pi\delta(\vec{r} - \vec{r}')$$

Integration over a small sphere centered on $\vec{r}'$ yields -4π. Volume integrals excluding the singularity yield zero. Therefore the operator $\mathcal{L}$ operating on G gives a pole of the form $1/R = \frac{1}{|\vec{r} - \vec{r}'|}$ as the radius of the volume shrinks to zero around the singular point. In the case of 2-D, the singularity is logarithmic, $ln(r)$

13 Expansion of the Green's function in eigenfunctions

An expansion of the Green's function in eigenfunctions is useful only in separable coordinate systems. This is because we need to apply boundary conditions on the coordinate surfaces. Suppose we have a complete set of eigenfunctions satisfying boundary conditions withing some space. We also assume that these functions are orthonormal. Then because the set is complete, we expand the Green's function in an eigenfunction representation.

$$G(r, r') = \sum_n a_n \, \psi_n(r)$$

Also we have that;

$$\mathcal{L} \, \psi_n + k_n^2 \, \psi_n = 0$$

Then

$$\mathcal{L} \, G + k^2 \, G = -4\pi\delta(\vec{r} - \vec{r}')$$

$$\sum a_n(k^2 - k_n^2) \, \psi_n = 4\pi\delta(\vec{r} - \vec{r}')$$

Use ortho-normality to obtain;

$$a_n = \frac{4\pi\psi_n(r')}{k_n^2 - k^2}$$

$$G(r, r') = 4\pi \sum_n \frac{\psi_n(r)\psi_n(r')}{k_n^2 - k^2}$$

Note that the Green's function must satisfy the same boundary condition as the eigenfunctions and G is singular at $k = \pm k_n$. The singularity represents the natural modes of the system.

14 Green's function for the scalar wave equation

Suppose the scalar wave equation;

$$\nabla^2\psi \,-\, (1/c^2)\frac{\partial^2\psi}{\partial t^2} \,=\, -4\pi\, q(\vec{r}, t)$$

In the above equation, $q(\vec{r}, t)$ is the space-time source density and c is the wave velocity. The boundary conditions are Cauchy on an open surface. This is the surface defined in 4-D space by the plane in 4-D [volume at a specific time]. The surface provides the initial conditions specifying the value and time derivative of ψ at $t = 0$. This equation is solved through the use of the Green's function. The equation for the Green's function is;

$$\nabla^2 G \,-\, (1/c^2)\,\frac{\partial^2\psi}{\partial t^2} G \,=\, -4\pi\,\delta(\vec{r} - \vec{r}\,')\delta(t - t')$$

Now apply the same boundary conditions on G as those for ψ. We stipulate that the wave propagates in the positive $\vec{r}$ direction and the wave requires that $t > t'$. Thus we have;

$$G\,\frac{\partial G}{\partial t} \,=\, 0 \;\text{ for }\; t < t'$$

Write the equation for time values, t'_1 and t'_2. Then integrate over a spatial volume, V, and time, t after multiplying by $G(\vec{r}, t'_1)$ and $G(\vec{r}, t'_2)$ respectively and subtracting the result.

$$\begin{aligned}
&\int_{-\infty}^{t'} dt \int dV\, [G(\vec{r}, t; \vec{r}\,', t')\, \nabla^2 G(\vec{r}, -t; \vec{r}\,', -t'_2) - \\
&\quad G(\vec{r}, -t; \vec{r}\,'_2, -t'_2)\, \nabla^2 G(\vec{r}, t; \vec{r}\,'_1, t'_1)] + \\
&\quad (1/c^2)[\int_{-\infty}^{t'} dt \int dV\, G(\vec{r}, t; \vec{r}\,', t')\, \frac{\partial^2}{\partial t^2} G(\vec{r}, -t; \vec{r}\,', -t'_2) - \\
&\quad G(\vec{r}, -t; \vec{r}\,'_2, -t'_2)\, \frac{\partial^2}{\partial t^2} G(\vec{r}, t; \vec{r}\,'_1, t'_1)] \,=\, \\
&\quad 4\pi[G(\vec{r}_1, -t_1; \vec{r}_2, t_2) \,-\, G(\vec{r}_2, -t_2; \vec{r}_1, t_1)]
\end{aligned}$$

Apply Green's theorem with the boundary conditions. Assume that $t' > t_1$ and $t' > t_2$, and note that the surface terms vanish. The result is;

$$G(\vec{r}_2, t_2; \vec{r}_1, t_1) \,=\, G(\vec{r}_1, -t_1; \vec{r}_1, -t_1)$$

This shows the symmetry in space-time of the Green's function. Now use the wave equation and Green's function as follows.

$$\nabla^2\psi \,-\, (1/c^2)\frac{\partial^2\psi}{\partial t^2} \,=\, -4\pi\, q(\vec{r}, t)$$

$$\nabla^2 G \,-\, (1/c^2)\,\frac{\partial^2\psi}{\partial t^2} G \,=\, -4\pi\,\delta(\vec{r} - \vec{r}\,')\delta(t - t')$$

Multiply the wave equation by G and the Green's function equation by ψ, subtract the equations from each other, and integrate.

$$\int_0^{t+\epsilon} dt' \int dV' \, [G\nabla^2\psi - \psi\nabla^2 G] + (1/c^2) \, [\int_0^{t+\epsilon} dt' \int dV' \, [\frac{\partial^2 G}{\partial t^2}\psi - G\frac{\partial^2 \psi}{\partial t^2}\psi] \; = \; 4\pi[\psi] - \int_0^{t+\epsilon} dt' \int dV' \, q \, G$$

Use Green's theorem and apply causality so that $G = 0$ when $t > t'$.

$$\psi \; = \; \int_0^{t+\epsilon} dt' \int dV' \, G \, q$$

This is an integral equation which can be easily solved if the kernel (Green's function) is determined. The Green's function is found using a Fourier transform. Thus the delta function and the Green's function are represented in the equations below.

$$\delta(\vec{r} - \vec{r}')\delta(t - t') \; = \; \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d\vec{k} \int_{-\infty}^{\infty} d\omega \, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} \, e^{i\omega(t-t')}$$

$$G \; = \; \int_{-\infty}^{\infty} d\vec{k} \int_{-\infty}^{\infty} d\omega \, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} e^{-i\omega(t-t')} \, g(\vec{k}, \omega)$$

Substitution into the Green's function equation yields;

$$g(\vec{k}, \omega) \; = \; \frac{1}{4\pi^3} \, \frac{1}{(k^2 - \omega^2/c^2)}$$

The Green's function is then;

$$G \; = \; (\frac{1}{4\pi^3}) \int d\vec{k} \int d\omega \, \frac{e^{i\vec{k}\cdot\vec{R}} e^{-i\omega\tau}}{k^2 - (\omega + i\epsilon)^2}$$

In the above $\vec{R} = \vec{r} - \vec{r}'$ and $\tau \; = \; t - t'$, The term $i\epsilon$ is used to insert causality into the integral. Thus for $t' > t$ integrate in the complex ω plane closing the contour in the upper half plane. For $t' < t$ close the contour in the lower half plane. The integral is analytic in the upper half plane so the integral vanishes. The integral in the lower half plane is obtained using the Cauchy residue theorem. Thus for $\tau > 0$

$$G \; = \; \frac{c}{2\pi^2} \int d\vec{k} \, e^{i\vec{k}\cdot\vec{R}} \, \frac{sin(c\tau k)}{k}$$

The final integral then yields;

$$G \; = \; \frac{1}{R} \, \delta(\tau - R/c)$$

Another term $\delta(\tau + R/c)$ has been discarded as it violates causality.

15 Scalar waves

Choose to remove the time dependence in the wave equation (Helmholtz equation) by solving for a particular frequency, ω, where $\omega/k = c$ with $k = |\vec{k}|$ the magnitude of the wave vector. The equation has the form;

$$[\nabla^2 + k^2]\psi(\vec{r}) = q(\vec{r})$$

$$\mathcal{L}\,\psi = q$$

In the above, q is the source of the waves. The source may be dependent on an incident wave, for example a scattering problem. Assume that the wave may be represented by an incident plane wave and an outgoing scattered wave. The form of the solution at a large distance from the scattering center is;

$$\psi = \psi_{in} + \psi_{scat}$$

Each of these wave forms must be a solution of the wave equation. Thus a plane wave incident along the $\hat{z}$ axis has the form;

$$\psi_{in} \propto e^{ikz} = e^{i\vec{k}\cdot\vec{r}}$$

where $\vec{k} = k\hat{z}$. For the scattered wave far removed from the scattering center (*ie* the wave does not act on the scattering center) the solution to the wave equation is an outgoing spherical wave.

$$\psi_{scat}\, \lim_{r\to\infty} \propto e^{ikr}/r$$

In general the wave interacts with the scattering center to act as a source of the outgoing wave. Therefore;

$$q \propto \int d\tau\, V(\vec{r})\psi(\vec{r})$$

This results in an integral equation, which incorporates the boundary conditions of the problem.

16 Example - waves in a solid

Waves in a solid can be extremely complicated as the solid transfers forces in all dimensional directions. A fluid for example cannot sustain shear type forces. Consider a solid body under an external force. This force causes a displacement from the original position of body

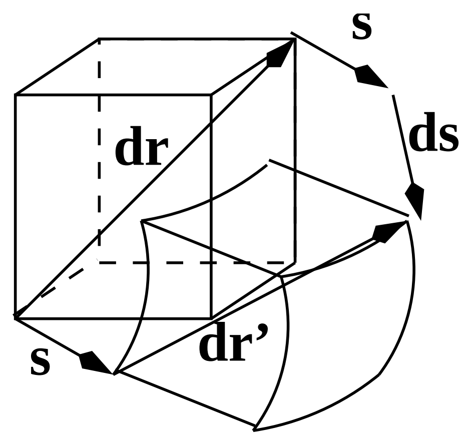

Figure 1: A figure showing how a solid deforms under stress. The stress in one dimension can cause a distortion in all dimensions. Thus i, j, k, l are each 3 dimensional indices.

element. The displacement can be represented by 3 vectors.

$$\vec{\Delta} = \vec{D}(x, y, z) + \vec{P}(x, y, z) + \vec{S}(s, y, z)$$

Here $\vec{D}$ represents the translation of the CM of the element, $\vec{P}$ represents the rotation of the element about the CM, and $\vec{S}$ represents a displacement due to distortion. The vector $\vec{S}$ is of interest for vibrations and waves. We need a measure of the strain of T_{ij} in, and stress, ϵ_{kl} within, and element. Hooke's law is then applied, stress = strain. The strain in a solid can be in any direction to an applied stress. This is illustrated in Figure 1. Stress is defined as the average force per unit area on a surface (dimensions of pressure). The strain is the change in length of a dimension per unit length. We want the change in the displacement $d\vec{S}$ due to a strain.

The elastic strain tensor is $\epsilon_{ij} = (1/2)[\frac{\partial \epsilon_i}{\partial x_j} + \frac{\partial \epsilon_j}{\partial x_i}]$. This tensor is made symmetric so that it depends only on deformations and not rotations. There are $3^4 = 81$ components of the elasticity tensor, c_{ijkl}, however by symmetry, only 21 are independent. The stress tensor, T_{ij}, is the i^{th} component of force acting on an area dA_j. Formally, Hooke's law is commonly written;

$$T_{ij} = -\lambda \, \delta ij \;\; \mathbf{Tr} \, \epsilon - 2\mu \, \epsilon_{ij}$$

The constants λ, μ are the Lame$'$ moduli. Some of the more common moduli are Young's modulus (E), Poisson ratio (σ), and bulk modulus (K). However, the point here is that waves in elastic solids can be quite complicated. In general one expects at least 3 pure wave forms - a longitudinal compression wave, and 2 transverse waves. Reduce the form of Hooke's law above to the wave equation;

$$\rho \frac{\partial s_i}{\partial t^2} \, = \, (1/2) \, c_{ijkl} \, \frac{\partial}{\partial x_j} \, [\frac{\partial s_l}{\partial x_k} \, + \, \frac{\partial s_k}{\partial x_l}]$$

Look for a solution to this equation of the form of harmonic motion.

$$s_i \, = \, S_i \, e^{i\vec{k}\cdot\vec{x}-\omega t}$$

There is a phase velocity, $c \, = \, \omega/|\vec{k}|$, and let $\lambda_{ijkl} = c_{ijlk}/\rho$, and $\Lambda_{il} = \lambda_{ijkl} n_j n_k$ where n_j represents a direction in $\hat{j}$. Substitution into the wave equation yields the matrix equation;

$$\begin{pmatrix} \Lambda_{11} - c^2 & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{12} & \Lambda_{22} - c^2 & \Lambda_{23} \\ \Lambda_{13} & \Lambda_{22} & \Lambda_{33} - c^2 \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} \, = \, 0$$

This is an eigenvalue equation which defines the 3 phase velocities as eigenvalues for the 3 polarization directions. Seismic waves within the earth are discussed in terms of S and P waves. There are other wave forms including surface waves. Primary (P) waves are compressional (longitudinal) waves and travel about 5000 m/s in granite for example. Secondary (S) waves are transverse waves (shear) and have about 10% of the velocity of P waves. The difference in these wave velocities allows an earthquake to be located

We now look at the development of the integral equation involving scattering .

17 Scattering

Wave scattering is defined by an interaction of a wave with a scattering center when the wavelength is large compared to the scattering source. It occurs when some of the incident wave is absorbed and then re-radiated so the scattering amplitude will be a function of the incident wave. In general we keep the lowest terms in a multipole expansion of the scattered fields. Almost always this results in dipole radiation as the dominant term in the series as the incident wave polarizes the scattering center. If the incident wave has large amplitude, then for an EM wave as an example, accelerated charges will not move linearly with the incident E field, and will bend due to the $\vec{V} \times \vec{B}$ force due to the magnetic component in the incident wave. An intense incident wave requires terms of higher order than the Born term in the perturbation expansion for the cross section. Scattering is developed below (we always assume a time dependence of $e^{i\omega t}$).

18 Formal mathematical development of scattering

18.1 Lipmann-Schwinger equation

The solution to the inhomogeneous scattering equation can be obtained by the Green's function equation.

$$[\nabla^2 + k^2]\psi = \mathcal{L}\,\psi = -4\pi\delta(\vec{r} - \vec{r}') = e^{ikR}/R$$

with $R = |\vec{r} - \vec{r}'|$. Apply the operator $\mathcal{L}$ to the wave amplitude ψ to get the source. Then if we formally define an operator $\mathcal{L}^{-1}$. The wave far from the scattering center when $S(\vec{r}) \to 0$ has the form;

$$\psi = \psi_{in} + \psi_{out}$$

$$\mathcal{L}\psi_{in} = 0$$

The solution we seek then has the form;

$$\psi_{out} = \mathcal{L}^{-1}\,S.$$

or in terms of the Green function;

$$\psi_{out} = \int d\vec{r}'\,\frac{e^{ikR}}{R}\,S(\vec{r}')\psi$$

This is not a solution but an integral equation because ψ is not known, *ie* the source term in the integrand is unknown. However, formally substitute for $\psi = \psi_{in} + \psi_{out}$ and collect terms. The solution is then written as;

$$\psi_{out} = \frac{1}{1 - \mathcal{L}^{-1}}\,\mathcal{L}^{-1}(S\psi_{in})$$

The above means;

$$\psi_{out} = [1 + \mathcal{L}^{-1}S + \mathcal{L}^{-1}S\mathcal{L}^{-1}S + \cdots]\,\mathcal{L}^{-1}(S\psi_{in})$$

The order of the operators is important and must be preserved. The series converges if the source term is sufficiently small. Generally the scattered EM wave is much less than the incident wave so we take the first term, called the Born term for future calculations.

18.2 Cross section

We consider scattering as the solution to the wave equation, and we use the scalar wave equation recognizing that a vector such as an EM field can be considered as a set of scalar components. The wave equation has the form;

$$[\nabla^2 \,-\, (1/c^2)\frac{\partial^2}{\partial t^2}]\psi \,=\, S(\vec{r},t)$$

In the above, ψ is the wave amplitude and S the scattering center, or in this case the source of the scattered wave. The source of the scattered wave depends on the strength of the incident wave so we rewrite this term as $S \rightarrow S\psi$ The solution, ψ, consists of an incident wave plus an outgoing spherical wave. Scattering assumes that the source term is localized so that sufficiently far away this term $\rightarrow 0$. Thus the incident wave is a solution to the homogeneous equation; *ie* the above wave equation with $S = 0$. Now we also assume that we can write the solution as harmonic in time, $\psi(\vec{r},t) \rightarrow \psi(\vec{r})e^{i\omega t}$ which removes the time dependence of the equation (although this is not really necessary).

$$[\nabla^2 \,+\, k^2]\psi \,=\, S(\vec{r},t)\psi$$

We will take the incident wave to have the solution of a plane wave $\psi_{in} = e^{ikz} = e^{ikr\,cos(\theta)}$ Then

$$\psi \,=\, \psi_{in} \,+\, \psi_{out}$$

The 1^{st} term is a solution to the inhomogeneous equation and the 2^{nd} term to the inhomogeneous equation. As $r \rightarrow \infty$;

$$\psi_{out} \,\rightarrow\, f(\theta,\phi)\,\frac{e^{ikr}}{r}$$

The scattering cross section of EM waves is then the scattered power into a solid angle $(r^2\,d\Omega)$ divided by the incident flux (incident power per area). The flux is equal to the Poynting vector. Therefore suppose we use ψ as a field. Then the differential cross section is;

$$\frac{d\sigma}{d\Omega} \,=\, \frac{(1/c\mu)|f(\theta,\phi)|^2}{(1/c\mu)|e^{ikz}|^2} \,=\, |f(\theta,\phi)|^2$$

19 Expansion of a plane wave in spherical harmonics

The scattering problem is solved naturally in spherical coordinates, as the center of the coordinate system is chosen to lie at the center of the scattering source. Also the scattering solution is to be obtained in a multipole series as the scattering amplitude is observed as the distance from the scattering center $\rightarrow \infty$. However, a plane wave incident on the scattering center is formulated in Cartesian coordinates. Thus we must formulate the incident wave in a spherical system, written in terms of spherical harmonics. We write;

$$e^{ikz} \,=\, e^{ikr\,cos(\theta)} \,=\, \sum C_n\, Y_n^0$$

Use the orthonormal properties of the spherical harmonic to write;

$$C_n \,=\, 2\pi \int d(cos(\theta))\, Y_n^0\, e^{ikr\,cos(\theta)}$$

Note that an integral representation of the spherical Bessel function, $j_l(kr)$, is

$$j_l(kr) \,=\, (-i)^l/2 \int_0^\pi d(cos(\theta))\, e^{ikr\,cos(\theta)}\, P_n\, cos(\theta)$$

We may also use the addition theorem which has the form;

$$P_l(\cos(\theta)) \,=\, \frac{4\pi}{2l+1} \sum_m Y_l^m(\theta_1, \phi_1)\, Y_l^{*\,m}(\theta_2, \phi_2)$$

where the angles (1) and (2) refer to the angles of vectors $\vec{k}$ and $\vec{r}$ in the specified coordinate system. This gives the required form for the plane wave;

$$e^{ikr\,cos(\theta)} \,=\, 4\pi \sum_{l,m} i^l j_l(kr)\, Y_l^m(\hat{k})\, Y_l^{*\,m}(\hat{r})$$

20 Multipole expansion

Separate Maxwell's equations into two sets of equations. Each set separately involves either the electric or the magnetic field. After removal of the time dependence by assuming a harmonic form $e^{-i\omega t}$, these equation sets have the forms;

Set 1

$$(\nabla^2 \,+\, k^2)\vec{B} \,=\, 0$$

$$\vec{\nabla} \cdot \vec{B} \,=\, 0$$

From the solution to these equations, the electric field is obtained by $\vec{E} \,=\, i(z/k)\vec{\nabla} \,\times\, \vec{B}/\mu$. This equation automatically satisfies $\vec{\nabla} \cdot \vec{E} \,=\, 0$

Set 2

$$(\nabla^2 \,+\, k^2)\vec{E} \,=\, 0$$

$$\vec{\nabla} \cdot \vec{E} \,=\, 0$$

From the solution to these equations, the magnetic field is obtained $\vec{E} = -i(1/zk)\vec{\nabla} \times \vec{B}/\mu$. This equation automatically satisfies $\vec{\nabla} \cdot \vec{B} = 0$

In the above, the wave impedance is $z = \sqrt{\mu/\epsilon}$. Now any vector can be written as a sum of a ir-rotational vector (curl vanishes) plus a solenoidal vector (divergence vanishes). In the radiation zone the solution to the above equations is found as $r \to \infty$ and takes the form of an outgoing spherical wave e^{ikr}/r (time dependence suppressed). Applying the divergence to this form, the fields are seen to be transverse, *ie* the vector direction of a field must be perpendicular to the wave vector, k. This is not necessarily true near the source currents. However, we can find any field within the entire space by a superposition of solutions from both of the above equation sets.

We now look for multipole solutions of the above equations. Given our past development in vector spherical harmonics we could immediately write a solution in terms of these functions. However, we follow the development in the text. Suppose we were to solve the above equations in Cartesian coordinates. In this case we could apply the scalar Laplacian to each of the 3 field coordinates. For example;

$$B_i = \sum [a_{lm}^{(i,j)} \, h_l^{(j)}(kr)] \, Y_{lm}$$

In the above, i represents one of the three Cartesian coordinates and $j = 1, 2$ represents either an outgoing or incoming boundary condition. The $a^{(i,j)}$ are coefficients that are determined by the source. As there is a solution for each vector component, we choose to write the above solution as;

$$\vec{B} = \sum [\vec{A}^{(j)} h_l^{(1)}] \, Y_{lm}$$

The coefficients $\vec{A}^{(j)}$ are not completely independent as the above function is a solution to the wave equation but not necessarily $\vec{\nabla} \cdot \vec{B} = 0$ This must occur independently for the solution for both equation sets. We then must also have that;

$$\vec{\nabla} \cdot \sum_{lm} h^{(j)}(kr) \vec{A}_{lm}^{(j)} \, Y^{lm} = 0$$

Write;

$$\vec{\nabla} = \hat{r} \frac{\partial}{\partial r} - (i/r^2) \, \vec{r} \times \vec{L}$$

and substitute this into the above equation ;

$$\hat{r} \cdot \sum_l [\frac{\partial h_l^{(j)}}{\partial r} \sum_m \vec{A}_{lm}^{(j)} \, Y_{lm} - (i h_l^{(j)}/r) \, \vec{L} \times \sum_m \vec{A}_{lm}^{(j)} \, Y_{lm}] = 0$$

A particular choice is;

$$\sum_m \vec{A}^{(j)}_{lm} Y_{lm} \;=\; \sum_m a_{lm} \vec{L} Y_{lm}$$

where we have introduced the vector spherical harmonic $\vec{\chi}_{lm} \;=\; \sqrt{\frac{1}{l(l+1)}} \vec{L} Y_{lm}$ which identically satisfies the divergence condition. Note that this makes the wave form transverse to $\hat{r}$ as expected. Now if we do this we need to be sure we have preserved the solution to the wave equation. This occurs because;

$$L^2 \vec{L} \;=\; \vec{L} L^2$$

$$L_i \nabla^2 \;=\; \nabla^2 L_i$$

Thus we obtain special solutions to the above 2 equation sets in terms of multipole fields. A linear combination of these 2 solutions spans the complete space of EM fields. If we choose;

$$\vec{B}_{lm} \;\propto\; f_l(kr) \vec{L} Y_{lm}$$

Then

$$\vec{E} \;=\; (iz/k) \vec{\nabla} \;\times\; \vec{B}_{lm}$$

and we have a solution for magnetic transverse modes (TM) where the magnetic field is transverse to $\hat{r}$. The TE modes have the same form, and a complete solution is superposition of both modes.

21 General solution

We need the vector spherical harmonic previously defined;

$$\vec{\chi}_{lm} \;=\; \sqrt{\frac{1}{l(l+1)}} \vec{L} Y_{lm}$$

It has the ortho-normal properties;

$$\int d\Omega \, \vec{\chi}^*_{lm} \cdot \vec{\chi}_{l'm'} \;=\; \delta_{ll'} \delta_{m'}$$

The radial function, $f_l(kr)$, is a spherical Hankel function that satisfies either spherical outgoing or incoming boundary conditions. To write a completely arbitrary solution, we introduce a similar Hankel function, $g_l(kr)$. Then the general solution takes the form;

$$\vec{B} = \sum_{lm} a_E(l,m)\, f_l(kr)\, \vec{\chi}_{lm} - (i/ck)\, a_M(l,m)\, \vec{\nabla} \times g_l(kr)\vec{\chi}_{lm}$$

$$\vec{E} = \sum_{lm} (ci/k)\, a_E(l,m)\, \vec{\nabla} \times f_l(kr)\vec{\chi}_{lm} +$$
$$a_M(l,m)\, g_l(kr)\, \vec{\chi}_{lm}$$

In the above $a_E(a_M)$ is the amount of electric(magnetic) multipole radiation in the solution. For a wave moving radially outward f_l, g_l are the Hankel functions $h_l^{(1)}$ which have the asymptotic form;

$$h_l^{(1)}(x) \to -e^{i(x-l\pi/2)}/x$$

Magnetic(electric) multipoles are called transverse electric(magnetic) since $\hat{r} \cdot \vec{\chi} = 0$. In the radiation zone $r \to \infty$. We substitute the asymptotic forms for f_l, g_l.

$$\vec{B} = \frac{\mu\, e^{i(kr-\omega t)}}{kr} \sum (-1)^{l+1}[A_E(l,m)\vec{\chi}_{lm} + A_M(l,m)(\hat{r} \times \vec{\chi}lm)]$$

$$\vec{E} = c(\vec{B} \times \hat{r})$$

22 Connection to multipole moments of source distributions

Assume a set of sources with time dependence $e^{-i\omega t}$ that have the forms;

$$\rho(\vec{x})\, e^{-i\omega t}, \;\; \vec{J}(\vec{x})\, e^{-i\omega t}, \;\; \vec{M}(\vec{x})\, e^{-i\omega t}$$

which are the source terms in Maxwell's equations;

$$\vec{\nabla} \cdot \vec{B} = 0 \;\; \vec{\nabla} \cdot \vec{E} = \rho/\epsilon$$

$$\vec{\nabla} \times \vec{B} + ik\mu z\vec{E} = \mu\vec{J} + \mu\vec{\nabla} \times \vec{M}$$

$$\vec{\nabla} \times \vec{E} - (ikz/\mu)\vec{B} = 0$$

In the above $z = \sqrt{\mu/\epsilon}$ and $k = \omega/c = \omega\sqrt{\mu\epsilon}$. Now we need to re-write the equations in terms of divergence less fields so that we may use the multipole radiation development in the previous sections. Note the equation of continuity $i\omega\rho = \vec{\nabla} \cdot \vec{J}$. Then, define a new field $\vec{E}'$

$$\vec{E}' = \vec{E} + \frac{i}{\omega\epsilon}\, \vec{J}$$

Substitute into Maxwell's equations to obtain ;

$$\vec{\nabla} \cdot \vec{B} = 0 \;\; \vec{\nabla} \cdot \vec{E}' = 0$$

$$\vec{\nabla} \times \vec{B} + (ik\mu/z)\vec{E}' = \vec{\nabla} \times \mu\vec{M}$$

$$\vec{\nabla} \times \vec{E}' - (ikz/\mu)\vec{B} = (i/\omega\epsilon)\vec{\nabla} \times \vec{J}$$

Use these equations to find the inhomogeneous Helmholtz equations for the fields (now let $\vec{E}' \to \vec{E}$) ;

$$(\nabla^2 + k^2)\vec{B} = -mu\vec{\nabla} \times (\vec{J} + c\vec{\nabla} \times \vec{M})$$

$$(\nabla^2 + k^2)\vec{E} = -izk\vec{\nabla} \times (\vec{M} + (1/k^2)\vec{\nabla} \times \vec{J})$$

The additional divergence equations of the fields also must equal zero. The multipole components are determined from the solutions with $\hat{r} \cdot \vec{B}$ and $\hat{r} \cdot \vec{E}$.

$$\hat{r} \cdot \vec{B} = (1/k)\sum_{lm} \sqrt{l(l+1)}\, Y_{lm}\, g_l\, a_M$$

$$\hat{r} \cdot \vec{E} = -(1/k)\sum_{lm} \sqrt{l(l+1)}\, Y_{lm}\, f_l\, a_E$$

Now for any vector, $\vec{\alpha}$

$$\vec{r} \cdot (\vec{\nabla} \times \vec{\alpha}) = (\vec{r} \times \vec{\nabla}) \cdot \vec{\alpha} = -i\vec{L} \cdot \vec{\alpha}$$

We let $\vec{\alpha} = \vec{B}$ or $\vec{E}$ so that $\vec{\nabla} \cdot \vec{\alpha} = 0$. Then use Cartesian coordinates to verify that.

$$\nabla^2(\vec{r} \cdot \vec{\alpha}) = \vec{r} \cdot (\nabla^2\vec{\alpha}) + 2\vec{\nabla} \cdot \vec{\alpha}$$

As $\vec{\nabla} \cdot \vec{\alpha} = 0$ we consider the wave equation for ;

$$(\nabla^2 + k^2)(\vec{r} \cdot \vec{B}) = -i\mu\vec{L} \cdot (\vec{J} + \vec{\nabla} \times \vec{M})$$

$$(\nabla^2 + k^2)(\vec{r} \cdot \vec{E}) = -kz\vec{L} \cdot (\vec{M} + (1/k^2)\vec{\nabla} \times \vec{J})$$

In the above we have used, for example;

$$\vec{r} \cdot [(\vec{\nabla} \times (\vec{J} + \vec{\nabla} \times \vec{M})] = -i\vec{L} \cdot (\vec{J} + \vec{\nabla} \times \vec{M})$$

We need solutions for the above inhomogeneous, scalar wave equation. This is obtained using the Green's function technique. The Green's function in spherical coordinates is;

$$g(\vec{x}, \vec{x}') \;=\; \frac{e^{i|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}$$

The solutions are;

$$\vec{r}\cdot\vec{B} \;=\; \frac{i\mu}{4\pi}\int d^3x'\, g(\vec{x}, \vec{x}')\, \vec{L}'\cdot[\vec{J}(\vec{x}') + \vec{\nabla}'\times\vec{M}(\vec{x}')]$$

$$\vec{r}\cdot\vec{E} \;=\; \frac{-kz}{4\pi}\int d^3x'\, g(\vec{x}, \vec{x}')\, \vec{L}'\cdot[\vec{M}(\vec{x}') + (1/k^2)\vec{\nabla}'\times\vec{J}(\vec{x}')]$$

To satisfy the outgoing boundary condition, we use the spherical Hankel function, $h_l^{(1)}(kr)$ to obtain the correct boundary condition as $r\rightarrow\infty$. The expansion of the Green's function for $\vec{r} > \vec{r}'$ is then;

$$\frac{e^{i|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \;=\; ik\sum_{l=0}^{\infty} j_l(kr)h_l^{(1)}(kr)\sum_m Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)$$

Multiply by $Y^*_{lm}(\theta,\phi)$ and integrate over $d\Omega$

$$\int d\Omega\, Y^*_{lm}\,\frac{e^{i|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \;=\; 4\pi ikj_l(kr')h_l^{(1)}(kr)Y^*_{lm}(\theta',\phi')$$

The solution for $\vec{r}\cdot\vec{B}$ is;

$$\vec{r}\cdot\vec{B} \;=\; -k\sum_{lm}\int d^3x'\, j_l(kr')\, h_l^{(1)}(kr)\, Y^*_{lm}(\hat{r}')\, Y_{lm}(\hat{r})\, (\vec{L}'\cdot[\vec{J}(\vec{x}') \;+\; \vec{\nabla}\cdot'\times\vec{M}(\vec{x}')])$$

Then;

$$\int d\Omega(\vec{r}\cdot\vec{B})\, Y^*_{lm}(\hat{r}) \;=\; (1/k)\sqrt{l(l+1)}\, a_M\, h_l^{(1)}(kr)$$

$$\int d\Omega(\vec{r}\cdot\vec{B})\, Y^*_{lm}(\hat{r}) \;=\; -k\int d^3x'\, j_l(kr')\, h_l^{(1)}(kr)\, Y^*_{lm}(\hat{r}')\vec{L}'\cdot[\vec{J}(\vec{x}') \;+\; \vec{\nabla}\cdot\times\vec{M}(\vec{x}')]$$

Equating the expressions;

$$a_M \;=\; -\frac{k^2}{\sqrt{l(l+1)}}\int d^3x'\, \vec{L}'\cdot[\vec{J}(\vec{x}') \;+\; \vec{\nabla}\cdot\times\vec{M}(\vec{x}')]$$

An expression for a_E can be obtained in the same way;

$$a_E \;=\; -\frac{ik^3}{\sqrt{l(l+1)}}\int d^3x'\, \vec{L}'\cdot[\vec{M}(\vec{x}') \;+\; (1/k^2)\vec{\nabla}\cdot\times\vec{J}(\vec{x}')]$$

The above equations give the values of the multipole fields outside the source dimensions in terms of integrals over the source densities. If the source dimensions are small compared to a wave length then $kr_{max} \ll 1$. We then approximate the spherical Hankel functions by;

$$j_l^{(1)}(x) \rightarrow \frac{x^l}{(2l+1)!!}$$

Use $\vec{\nabla} \cdot \vec{J} = i\omega\rho$ and keep terms to lowest order in $(kr)^l$.

23 EM scattering of an incident plane wave

A plane, monochromatic, linearly polarized wave is incident on a scattering center. The incident wave is;

$$\vec{E}_{in} = \hat{x} E_0 \, e^{ik_{in} r cos(\theta)}$$

$$\vec{B}_{in} = \hat{k_{in}}/c \times \vec{E}_{in}$$

These fields induce dipole moments $(\vec{p}, \vec{m})$ in the scattering center. The dipoles oscillate with the frequency of the incident wave and radiate energy. This radiation can be calculated in the multipole approximation which just uses the static fields obtained from the multipole moments. The dipole multipole fields are;

$$\vec{E}_{sc} = \frac{k^2}{4\pi\epsilon} \frac{e^{ikr}}{r} [\hat{n} \times \vec{p} \times \hat{n}] - \frac{\sqrt{\mu/\epsilon} k^2}{4\pi} \frac{e^{ikr}}{r} [\hat{n} \times \vec{m}]$$

$$\vec{B}_{sc} = (1/c) \hat{n} \times \vec{E}_{sc}$$

In the above, $\vec{p}$ and $\vec{m}$ are the electric and magnetic dipole moments, respectively. Using the Poynting vector we find the differential scattering cross section into the scattering solid angle, $d\Omega$. The cross section is the ratio of the scattered power divided by the incident flux.

$$\frac{d\sigma}{d\Omega} = \frac{r^2 |\hat{a} \cdot \vec{E}_{sc}|^2}{|\hat{z} \cdot \vec{E}_{in}|^2}$$

The unit vector $\hat{a}$ defines directions perpendicular to the observation direction $\hat{n}$. Substitution for the field gives

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon)^2} |\hat{a} \cdot \vec{p} + (1/c)(\hat{n} \times \hat{a}) \cdot \vec{m}|^2$$

For a small dielectric sphere of radius, η, the electric dipole moment for the sphere is;

$$\vec{p} = 4\pi\epsilon_0 \, (\frac{\epsilon - 1}{\epsilon + 2}) \eta^3 \vec{E}_{in}$$

Use the geometry shown in fig. 2. The cross section is obtained for the scattered polarization perpendicular $\hat{a}_\perp$ and parallel, $\hat{a}_\parallel$ to the scattering plane as defined by the plane containing

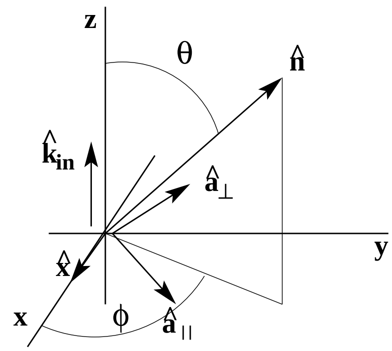

Figure 2: The geometry used to obtain the polarization vectors for scattering from a dielectric sphere

the incident and scattering vector directions. From the figure;

$$\hat{a}_{\|} \cdot \hat{x} \,=\, cos(\theta)\, cos(\phi)$$

$$\hat{a}_{\perp} \cdot \hat{x} \,=\, -sin(\phi)$$

Averaging over all possible initial polarizations *ie* averaging over ϕ, we obtain

$$\frac{d\sigma_{\|}}{d\Omega} \,=\, \frac{k^4 \eta^6}{2} \,|\frac{\epsilon - 1}{\epsilon + 2}|^2 \, cos^2(\theta)$$

$$\frac{d\sigma_{\perp}}{d\Omega} \,=\, \frac{k^4 \eta^6}{2} \,|\frac{\epsilon - 1}{\epsilon + 2}|^2$$

The total differential cross section is;

$$\frac{d\sigma}{d\Omega} \,=\, \frac{d\sigma_{\|}}{d\Omega} \,+\, \frac{d\sigma_{\perp}}{d\Omega}$$

The total cross section is obtained by integration over $d\Omega$.

$$\sigma_T \,=\, \frac{8\pi k^4 \eta^6}{3} \,|\frac{\epsilon - 1}{\epsilon + 2}|^2$$

The polarization of the scattered wave is;

$$P(\theta) \,=\, \frac{d\sigma_{\perp}/d\Omega \,-\, d\sigma_{\|}/d\Omega}{d\sigma_{\perp}/d\Omega \,+\, d\sigma_{\|}/d\Omega} \,=\, \frac{sin^2(\theta)}{1 + cos^2(\theta)}$$

24 A center fed linear antenna

Because we have assumed harmonic source terms, the radiation problem can be determined by a solution to the field obtained from a static source. We first look at the radiation from a linear center-fed antenna as shown in Fig 3. This problem is more complicated than it first appears. The antenna is fed by a harmonic potential which drives charge along the antenna wire. These currents depend on the voltage, the impedance of the wire, and the radiation field, and in general are part of the solution. The radiation field thus couples back into the source. However, we know that the current must vanish at the ends of the wire, the wire is thin so that radial currents can be neglected, and the actual radiation field is rather insensitive to the actual current distribution. Therefore we assume that we can approximate the current by the form;

$$I(z,t) \,=\, I_0\, sin(kd/2 \,-\, k|z|)\, e^{-i\omega t}$$

$$-d/2 \leq z \leq d/2$$

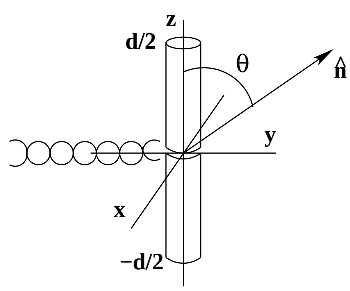

Figure 3: The linear center fed antenna. We neglect the radial width of the antenna wire

If we make the long wavelength approximation;

$$a_E(l,m) \,=\, \frac{k^2}{i\sqrt{l(l+1)}} \int d^3x\, Y^*_{lm}(\rho \frac{\partial}{\partial r}[r j_l(kr)] \,+\, (ik/c)(\vec{r}\cdot\vec{J})\, j_l(kr) \,-\, (ik)\vec{\nabla}\cdot(\vec{r}\times \vec{M}) j_l(kr))$$

We see below that the magnetic multipole vanishes. There is no permanent magnetization so that $\vec{M} = 0$. The current density has the form;

$$\vec{J} \,=\, \hat{r}\, \frac{I(r)}{2\pi r^2}[\delta(\theta - 0) \,-\, \delta(\theta - \pi)]$$

We also note that $\vec{r}\times\vec{J} = 0$ (thin wire). Therefore $a_M = 0$. The equation of continuity is $\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} \,=\, 0$. Therefore;

$$\rho \,=\, (1/i\omega)\frac{dI}{dr}[\delta(\theta - 0) \,-\, \delta(\theta - \pi)]/(2\pi r^2)$$

Substitution into the expression for a_E ;

$$a_E(l,m) \,=\, \int_0^{d/2} dr\,(kr)j_l(kr)\,I(r) -$$
$$(1/k)(\frac{d}{dr}[rj_l(kr)]) \int\, d\Omega\, Y^*_{lm}[\delta(\theta - 0) - \delta(\theta - \pi)]$$

Integration over angles gives ;

$$2\pi[Y_{l0}(0) - Y_{l0}(\pi)] \,=\, \sqrt{4\pi(2l+1)} \quad \text{l odd.}$$

Then substitute the harmonic form for the current as given above for $I(z)$, and complete the integration over the radial variable. The result is;

$$a_E(l,0) \,=\, \frac{I_0}{\pi d}[\frac{4\pi(2l+1)}{l(l+1)}]^{1/2}\,[(\frac{kd}{2})^2\,j_l(kd/2)] \;\; \text{l odd}$$

The radiated power is then obtained by substitution into the power equation obtained previously. Generally we take only the leading, non-zero term in the multipole expansion, in this case the dipole term, $l = 1$ is the leading term.

25 EM scattering from a long, perfectly conducting wire

Consider a plane wave incident perpendicular to a long perfectly conducting wire with the $\vec{E}$ field polarized parallel to the length of the wire. We wish to find the scattered wave. The geometry is illustrated in Figure 4. The radius of the wire is R, which is assumed small compared to the wave length.

Begin by solving the homogeneous wave equation in cylindrical coordinates. The wave equation is;

$$\nabla^2\vec{E} \,+\, k^2\vec{E} \,=\, 0$$

An incident wave polarized in the $\hat{x}$ direction has the form, $\vec{E} \,=\, E(\rho,\theta)\hat{x}$. The solution is independent of x as the wire is essentially infinite in length, and because of cylindrical symmetry around the $\hat{x}$ axis, we expect a solution in θ of the form $cos(\nu\theta)$. Therefore we have a solution;

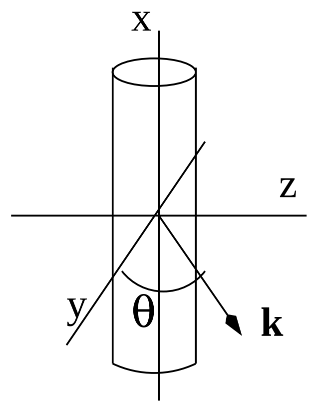

Figure 4: The geometry of the wire, showing the incident wave polarization and the direction of the scattered wave.

$$E \,=\, \sum_{\nu} [A_{\nu}\, J_{\nu}(k\rho) \,+\, B_{\nu}\, N_{\nu}(k\rho)]\, cos(\nu\theta)$$

The cylindrical Bessel functions J_{ν} and N_{ν} have been used. Now E must vanish for $\rho = R$ since the wire is perfectly conducting. This means that we use a set of functions defined by;

$$S_{\nu}(k\rho) \,=\, [\frac{J_{\nu}(k\rho)}{J_{\nu}(kR)} \,-\, \frac{N_{\nu}(k\rho)}{N_{\nu}(kR)}]$$

$$E \,=\, \sum_{\nu} C_{\nu}\, S_{\nu}(k\rho)\, cos(\nu\theta)$$

We now must satisfy the asymptotic condition as $\rho \rightarrow \infty$. Thus begin by expanding the incident plane wave in the $cos(\nu\theta)$ eigenfunctions.

$$E_{in} \,\rightarrow\, E_{in}\, e^{ik\rho\, cos(\theta)} \,=\, \sum_{\nu} f_{\nu}(\rho)\, cos(\nu\theta)$$

$$f_{\nu}(\rho) \,=\, (1/\pi) \int_{0}^{\pi} d\theta\, e^{ik\rho\, cos(\theta)}\, cos(\nu\theta)$$

Use the integral relation for the Bessel function.

$$J_{\nu}(\alpha) \,=\, \frac{i^{-\nu}}{\pi} \int_{0}^{\pi} d\theta\, e^{i\alpha\, cos(\theta)}\, cos(\nu\theta)$$

to write;

$$f_{\nu}(\rho) \,=\, i^{\nu}\, J_{\nu}(k\rho)$$

$$e^{ik\rho\, cos(\theta)} \,=\, \sum_{\nu} i^{\nu}\, J_{\nu}(k\rho)\, cos(\nu\theta)$$

Then use the asymptotic values of the Bessel functions.

$$\lim_{\rho \to \infty} J_\nu(k\rho) = \sqrt{\frac{2}{\pi k\rho}}\, cos(k\rho - \nu\pi/2 - \pi/2)$$
$$\lim_{\rho \to \infty} N_\nu(k\rho) = \sqrt{\frac{2}{\pi k\rho}}\, sin(k\rho - \nu\pi/2 - \pi/2)$$

The asymptotic form of the wave equation for outgoing waves is;

$$\psi = \psi_{in} + \psi_{out}$$

The form for ψ_{in} has been evaluated above. The outgoing wave is represented by the cylindrical Hankel function, $H^1_\nu(kr)$

$$\lim_{\rho \to \infty} H^1_\nu(k\rho) = \sqrt{\frac{2}{\pi k\rho}}\, e^{ik\rho - \nu(\pi/2) - \pi/4}$$

Therefore the wave satisfying the boundary conditions has the form;

$$\sum_\nu i^\nu J_\nu(k\rho)\, cos(\nu\theta) + \sum_\nu C_\nu\, H^1_\nu(k\rho)\, cos(\theta) =$$
$$\sum_\nu A_\nu\, S_\nu(k\rho)\, cos(\nu\rho)$$

Then when $\rho = R$ the right side vanishes so that;

$$C_\nu = -\frac{i^\nu J_\nu(kR)}{J_\nu(kR) + iN_\nu(kR)}$$

In the limit $\rho \to \infty$ use the asymptotic forms of the Bessel functions to write;

$$i^\nu[\frac{e^{i(k\rho-\phi)} + e^{-i(k\rho-\phi)}}{2} + C_\nu\, e^{ik\rho} = A_\nu[\frac{e^{i(k\rho-\phi)}}{2J_\nu(kR)} - \frac{e^{i(k\rho-\phi)}}{2N_\nu(kR)}]$$

Solving for A_ν

$$A_{nu} = i^\nu \frac{J_\nu(kR) N_\nu(kR)}{J_\nu(kR) + iN_\nu(kR)}$$
$$C_{nu} = i^\nu\, e^{i\phi} \frac{J_\nu(kR)}{J_\nu(kR) + iN_\nu(kR)}$$

The scattered wave is then,

$$\vec{E}_{scat} = \sqrt{\frac{2}{\pi k\rho}} \sum_\nu C_\nu\, e^{ik\rho}\, cos(\nu\theta)$$

This can be used to obtain the averaged Poynting vector for a polarized wave in the asymptotic region, $P = \frac{\epsilon c}{2}|E|^2$. The Poynting vector times the differential area, $r^2 d\Omega$ gives the scattered power into $d\Omega$ and this divided by the incident flux results in the differential scattering cross section.

26 1-D wave transmission with a potential

Consider a 1-D Schrodinger equation with a potential function, $U(x)$. This equation is identical to a wave equation with a source equal to $U\psi$ (similar to the radiation problem discussed earlier).

$$\frac{d^2\psi}{dx^2} + [k^2 - \lambda U(x)]\psi = 0$$

The boundary conditions are;

$$\psi_{\lim x \to \infty} = A e^{ikx}$$

$$\psi_{\lim x \to -\infty} = e^{ikz} + B e^{-ikx}$$

The Green's function is $G(x, x') = -\frac{1}{2\pi i} e^{ik|x-x'|}$ so that;

$$\frac{d^2 G}{dx^2} + k^2 G = -\delta(x - x')$$

The integral equation solution is then;

$$\psi(x) = e^{ikx} - \lambda \int_{-\infty}^{\infty} dx'\, G(x, x')\, U(x')\, \psi(x')$$

One can show that the boundary conditions are satisfied by the above form when letting $x \to \pm\infty$. Apply a Fourier transform to obtain the solution for ψ or use the Born approximation and iterate the solution.

27 Flow of a perfect fluid

27.1 The equation of motion

A perfect fluid is incompressible and can support no shear. Thus begin by considering the equation of continuity.

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{V}) = 0$$

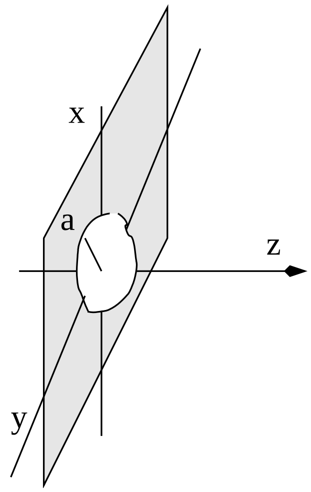

Figure 5: The geometry for fluid flow through an aperture of radius, a in the $\hat{z}$ direction

In the above, ρ is the mass density, and $\vec{V}$ is the fluid velocity. For the fluid to be irrotational $\vec{\nabla} \times \vec{V} = 0$. Since the fluid is irrotational $\vec{V} = -\vec{\nabla}\phi$ where ϕ is a scalar velocity potential. Now expand the equation of continuity.

$$\frac{\partial \rho}{\partial t} + \vec{V} \cdot \vec{\nabla}\rho + \rho\vec{\nabla} \cdot \vec{V} = 0$$

$$\frac{d\rho}{dt} + \rho\vec{\nabla} \cdot \vec{V} = 0$$

In the case when the fluid is incompressible $\frac{d\rho}{dt} = 0$. Therefore;

$$\vec{\nabla} \cdot \vec{V} = 0 = \nabla^2\phi$$

27.2 Flow of a perfect fluid through an aperture

Assume a perfect fluid flow through a circular aperture of radius a in an infinite wall. The wall has negligible thickness. Choose a geometry with the origin at the center of the aperture and the z axis perpendicular to the wall, pointing in the direction of the flow. We use cylindrical coordinates as shown in Figure 5.

The velocity potential has the functional form, $\phi = \phi(\rho, z)$, and because of symmetry, the solution it is independent of θ. Laplace's equation takes the form;

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + (1/r)\frac{\partial \phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} = 0$$

The boundary conditions are;

$$\phi(r,0) \;=\; g(r) \quad r < a$$

$$\frac{\partial\phi(r,0)}{\partial z} \;=\; 0 \quad r > a$$

The function for the fluid velocity in the aperture, $g(r)$, is known.

27.3 Hankel Transform

Clearly we expect the solution has a radial form described by Bessel functions. Apply a Hankel transform. Multiply the equation by $r\,J_0(\alpha r)$ and integrate over r.

$$\int_0^\infty dr\,[\frac{\partial^2\phi}{\partial r^2} \,+\, (1/r)\frac{\partial\phi}{\partial r}]\,r\,J_0(\alpha r) \,+\, \int_0^\infty dr\,\frac{\partial^2\phi}{\partial z^2}\,r\,J_0(\alpha r) \;=\; 0$$

The Hankel transform is;

$$\int_0^\infty dr\,[\frac{\partial^2\phi}{\partial r^2} \,+\, (1/r)\frac{\partial\phi}{\partial r}]\,r\,J_0(\alpha\,r) \;=\; -\alpha^2 F_0(\alpha)$$

In the above $F_0(\alpha)$ is the Hankel transform of $f(r)$. Therefore, the transformed equation becomes;

$$\frac{d^2\Phi}{dz^2} \,-\, \alpha^2\Phi \;=\; 0$$

where $\Phi(\alpha, z)$ is the Hankel transform of ϕ.

$$\Phi(\alpha, z) \;=\; \int_0^\infty dr\,\phi(r,z)\,r\,J_0(\alpha\,r)$$

The solution to the transformed equation is

$$\Phi \;=\; A(\alpha)\,e^{-\alpha\,z}$$

We also need the transform of the derivative of ϕ in order to apply the boundary conditions.

$$\int_0^\infty dr\,\frac{\partial\phi(r,z)}{\partial z}\,r\,J_0(\alpha\,r) \;=\; -\alpha\,A(\alpha)\,e^{-\alpha z}$$

Then invert the Hankel transform of the two boundary conditions.

$$\phi \;=\; \int_0^\infty d\alpha\,\alpha\,J_0(\alpha\,r)A(\alpha)\,e^{-\alpha z}$$

$$\frac{\partial \phi}{\partial z} \;=\; -\int_0^\infty d\alpha\, \alpha^2\, J_0(\alpha\, r) A(\alpha)\, e^{-\alpha z}$$

Thus a set of dual integral equations is developed from the Hankel transform of the fluid flow. These coupled integral equations are due to the dual boundary conditions.

27.4 Solution of the dual integral equations

The dual integral equations are set equal to the boundary conditions at $z = 0$. Let $\rho = r/a$, $F(u) = A(u/a)$, and $G(\rho) = a^2 g(r)$. The integral equations become;

$$\int_0^\infty du\, J_0(\rho u) F(u) \;=\; G(\rho) \quad 0 < \rho < 1$$

$$\int_0^\infty du\, u\, J_0(\rho u) F(u) \;=\; 0 \quad \rho > 1$$

These equations may be solved by Mellin transforms. The Mellin transformation of the functional form $y^\alpha J_\nu(xy)$ is;

$$J_\alpha(s) \;=\; \int_0^\infty y^{\alpha+s-1}\, J_\nu(xy)\, dy \;=\; \frac{2^{\alpha+s-1}}{x^{\alpha+s}}\, \frac{\Gamma([\alpha+\nu+s]/2)}{\Gamma([2-\alpha-s+\nu]/2)}$$

Application of the Mellin transform and its inverse is mathematically complicated, eventually requiring integration in the complex plane. The solution of the above dual equations can be shown to be;

$$F(u) \;=\; (2/\pi)\, cos(u) \int_0^1 dy\, \frac{yG(y)}{(1-y^2)^{1/2}} + (2/\pi) \int_0^1 dy\, \frac{y}{(1-y^2)^{1/2}}\, [\int_0^1 du\, G(yu)\, xu\, sin(xu)]$$