Examples of radiation problems

Lecture 13

1 A center fed linear antenna

Because we removed the time dependence of Maxwell's equations by assuming harmonic source terms, the radiation problem can be determined by a static solution to the field equations. Consider radiation from a linear, center-fed antenna as shown in Figure 1. This problem is more complicated than it first appears. The antenna is fed by a harmonic potential which drives charge along the antenna wire. These currents depend on the voltage, the impedance of the wire, and the radiation field as it acts back on the source. In general this results in an integral equation. However, we know that the current must vanish at the ends of the wire, the wire is thin so that radial currents can be neglected, and the actual radiation field is rather insensitive to the actual current distribution. Therefore we can make an approximation for the current using the form;

$$I(z,t) \,=\, I_0\, sin(kd/2 \,-\, k|z|)\, e^{-i\omega t}$$

$$-d/2 \le z \le d/2$$

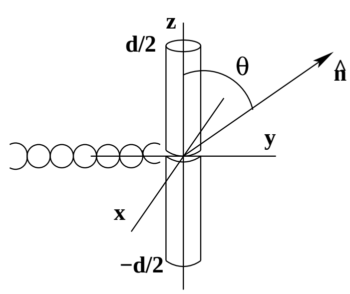

Figure 1: The linear center fed antenna. Neglect the radial width of the antenna wire

Take the long wavelength approximation as in the last lecture. The electric multipoles are given by the coefficient, $a_E(l,m)$. The first term is obtained by intergration the expression above $(-ic/kr)\dfrac{\partial r^2 \rho}{\partial r}$ by parts

$$a_E(l,m) \,=\, \frac{k^2}{i\sqrt{l(l+1)}} \int d^3x\, Y_{lm}^* (\rho \frac{\partial}{\partial r}[r j_l(kr)] \,+\, (ik/c)(\vec{r}\cdot\vec{J})\, j_l(kr) \,-\, (ik)\vec{\nabla}\cdot(\vec{r}\times \vec{M}) j_l(kr))$$

We will find below that the magnetic multipole vanishes, and there is no permanent magnetization so that $\vec{M} = 0$. The current density has the form;

$$\vec{J} = \hat{r}\,\frac{I(r)}{2\pi r^2}[\delta(\theta - 0) - \delta(\theta - \pi)]$$

Also note that $\vec{r} \times \vec{J} = 0$ (thin wire), and therefore $a_M = 0$. The equation of continuity is $\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$. Thus;

$$\rho = (1/i\omega)\frac{dI}{dr}[\delta(\theta - 0) - \delta(\theta - \pi)]/(2\pi r^2)$$

Substituting into the expression for a_E results in;

$$a_E(l, m) = \int_0^{d/2} dr\,[(kr)j_l(kr)\,I(r) - (1/k)(\frac{d}{dr}[r j_l(kr)])] \int d\Omega\, Y^*_{lm}[\delta(\theta - 0) - \delta(\theta - \pi)]$$

Integration over angles to obtain the total power radiated requires ;

$$2\pi[Y_{l0}(0) - Y_{l0}(\pi)] = \sqrt{4\pi(2l + 1)} \quad \text{l odd.}$$

Substitute the harmonic form for the current, $I(z)$, as assumed above and complete the integration over the radial variable. The result is;

$$a_E(l, 0) = \frac{I_0}{\pi d}[\frac{4\pi(2l + 1)}{l(l + 1)}]^{1/2}\,[(\frac{kd}{2})^2\, j_l(kd/2)] \;\; \text{l odd}$$

The radiated power is obtained by substitution into the power equation obtained in the last lecture. Generally one takes only the leading, non-zero term in the multipole expansion, in this case the dipole term, $l = 1$ is the leading term.

2 Spherical center fed antenna

The geometry of the problem is shown in Figure 2. We need to solve the static problem of a conducting sphere with potential, V_0 on its surface. Proceed by obtaining the solution for the potential in spherical coordinates for $\nabla^2 V = 0$. The solution using separation of variables is;

$$V = \frac{1}{4\pi\epsilon} \sum_l A_l (1/r)^{l+1}\, P_l(cos(\theta))$$

We must match to the boundry condition for the potential at $r = a$, which for the perfectly conducting sphere is a constant value, V_0. Use the orthorgonality of the Legendre polynomials to find the expansion coefficients, A_l . Thus, multiply by P_l and integrate over $dx = d\,cos(\theta)$

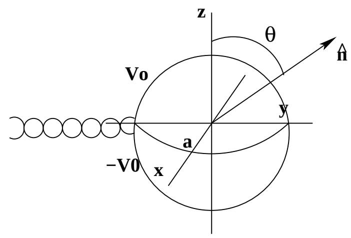

Figure 2: The geometry of a spherical antenna. A potential $\pm V_0$ is applied to the surfaces.

$$A_l = (4\pi\epsilon)\tfrac{(a^{l+1})(2l+1)}{2} \int_{-1}^{1} dx\, V_0\, P_l(x)$$

$$A_l = (4\pi\epsilon)\frac{(a^{l+1})}{2}\,(\frac{-1}{2})^{(l-1)/2}\,\frac{(2l+1)(l-2)!!}{2((l+1)/2)!}\,V_0 \quad \text{l odd}$$

Then the first 2 non-zero terms are;

$$A_1 = (4\pi\epsilon)a^2(3/2)\,V_0$$

$$A_3 = -(4\pi\epsilon)a^4(7/8)\,V_0$$

In general the solution is;

$$V = \sum_{l\,odd} [(a/r)^{l+1}(-1/2)^{(l-1)/2}\frac{(2l+1)(l-2)!!}{2((l+1)/2)!}]\sqrt{\frac{4\pi}{2l+1}}\,V_0\,Y_l^0$$

It is always important to think about any result to dermine if there is some glaring mistake. Clearly the solution must be independent of azmuthal angle by symmetry. It must be reflection symmetric about the (x, y) plane hence dependent only on odd l. We next obtain the surface charge density, σ from;

$$\sigma = -\epsilon\frac{\partial\phi}{\partial r}|_{r=a} = \frac{\sqrt{4\pi}\,\epsilon}{2a}\sum_{l\,odd}(-1/2)^{(l-1)/2}\frac{(l+1)(l-2)!!\sqrt{2l+1}}{((l+1)/2)!}\,V_0\,Y_l^0$$

In the long wavelength approximation evaluate, Q_{lm}.

$$Q_{lm} = \int d\Omega\, a^{l+2}\,\sigma\, Y_l^{*\,m}$$

$$Q_{l0} = \frac{\sqrt{4\pi}\,\epsilon}{2}\sum_{l\,odd} a^{l+1}(-1/2)^{(l-1)/2}\frac{(l+1)(l-2)!!\sqrt{2l+1}}{((l+1)/2)!}\,V_0$$

The electric multipole coefficient is then;

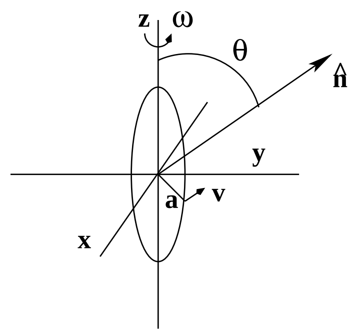

Figure 3: The geometry of spinning loop. The static charge per unit length on the loop is λ

$$a_E \, = \, \frac{ck^{l+2}}{i(2l+1)!!} \, (\frac{l+1}{l})^{1/2} \, \frac{\sqrt{4\pi}\,\epsilon}{2} \, a^{l+1}(-1/2)^{(l-1)/2} \frac{(l+1)(l-2)!!\sqrt{2l+1}}{((l+1)/2)!}$$

There are no currents in the $\hat{r}$ direction. However, there is a surface current $\vec{J} \, = \, J(\theta)\hat{\theta}$ restricted by azmuthal symmetry. Note that since $\vec{\nabla} \cdot (\vec{r} \times \vec{J}) = 0$ there is no m component. Substitution for the lowest multipole gives electric dipole radiation.

3 Spinning loop of charge

The geometry of the problem is shown in Figure 3. The charge per unit length of the loop is λ. While the time distribution is harmonic, it requires a more careful analysis. The static charge density is given by;

$$\rho \, = \, \frac{2\lambda}{a\,sin(\theta)} \delta(\phi - \phi_0)\,\delta(r - a)$$

Check the above equation for the charge density by integration over the spherical volume.

$$Q \, = \, \int \, d^3x \, \rho \, = \, \frac{2\lambda}{a} \, \int \, \delta(r-a)\, r^2 \, dr \, \int \, \frac{sin(\theta)}{sin(\theta)} \, d\theta \, \int \, d\phi \, \delta(\phi - \phi_0)$$

$$Q \, = \, 2\pi a \lambda$$

Now let $\phi = \omega t$. The current density is $\vec{J} \, = \, \rho \vec{v} \, = \, \omega a \, sin(\theta) \, \rho \, \hat{\phi}$

Then assume a Fourier series expansion in time. The period is $T \, = \, \pi/\omega$ with $\phi = \omega t$.

$$\rho(r,t) \, = \, \rho_0 \, + \, Re[\sum_n \, \rho_n \, e^{-in\omega t}]$$

Thus;

$$\rho_n \;=\; (1/T) \int dt \,\frac{2\lambda}{a\, sin(\theta)}\, \delta(\phi-\phi_0)\, \delta(r-a)\, cos(n\omega t)$$
$$\rho_n \;=\; \frac{\lambda\delta(r-a)}{\pi a\, sin(\theta)}\, cos(n\phi)$$

Choose only the time independent term, ρ_n, above and substitute into Q_{lm} to get the electric moment. The time independent form is to be multiplied by $e^{-im\omega t}$

$$Q_{lm} \;=\; \int d^3x\, r^l \rho_n\, Y_l^{*,m}$$
$$Q_{lm} \;=\; (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}(2a^{l+1}\lambda) \int dx\, \frac{P_l^m(x)}{\sqrt{1-x^2}}\, \delta_{mn}$$

Then

$$\int_0^\pi d\theta\, P_l^{-m}(\theta) \;=\; \frac{2^{-m}\pi\Gamma((m+1)/2)\gamma((1-m)/2)}{\Gamma((1+l)/2)\Gamma((1-l)/2)\Gamma((l+m+1)/2)\Gamma((m-l+1)/2)}$$
$$1 \pm m > 0$$

Then for the magnetic multipoles with $\vec{M} = 0$;

$$\vec{r} \times \vec{J} \;=\; -\omega\rho a^2\, sin(\theta)\, \hat{\theta}$$
$$\vec{\nabla}\cdot(\vec{r}\times\vec{J}) \;=\; -\frac{\lambda\omega cos(\theta)}{\pi\, \sin(\theta)}\, \delta(r-a)\, \delta(n\phi)$$
$$M_{lm} \;=\; -\frac{1}{l+1} \int d^3x\, r^l\, \vec{\nabla}\cdot(\vec{r}\times\vec{J})\, Y_l^{*\,m}$$
$$M_{lm} \;=\; (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}(2a^{l+2}\lambda) \int dx\, \frac{P_l^m(x)}{\sqrt{1-x^2}}\, [\delta_{m,n} + \delta_{m,-n}]$$

4 Atomic transition probability

Here we are to find the angular pattern of the radiation and rate of radiated energy of a atomic transition in atomic hydrogen. The hydrogen wave function has the form;

$$\psi_{lmn} \;=\; \sqrt{(\alpha)^3([2l+1]/(8\pi\hbar))([l-m]!/[l+m]!)([n-l-1]!/([n+1]!)^3)} \times (\alpha r)^l\, e^{im\phi}\, P_l^m\, e^{-\alpha r}\, L_{n-l+1}^{2l+1}(\alpha r)$$

In the above;

$\alpha = 2M\eta^2/(\hbar^2 n)$

$L_k^l(br)$ is the Laguerre polynomial

M is the reduced mass of the hydrogen atom

$\eta^2 = e/(4\pi\epsilon)$

$E = -M\eta^4/(2n^2\hbar^2) \;\; n = l+1, l+2, \cdots$

We need the transition densities, so as an example choose the transition, $012 \to 001$

$$\rho \;=\; \psi_{001}\psi_{012}\, e^{-i(E_2-E_1)t/\hbar}$$

$$\vec{J} \;=\; \hbar/M\, Im[\psi_{001}\vec{\nabla}\psi_{012}]\, e^{-i(E_2-E_1)t/\hbar}$$

The wave function is written as;

$$\psi \;=\; A_{mln}\,(\alpha r)^l e^{-\alpha r} L_{n-l+1}^{2l+1}(\alpha r) Y_l^m$$

In the long wavelength limit;

$$Q_{lm} \;=\; \int\, d^3x\, r^l \rho\, Y_l^{*\,m}$$

$$Q_{lm} \;=\; A_{001}A_{012} \int\, d\Omega\, Y_0^0\, Y_1^0\, Y_1^{*\,m} \;\int\, dr\, r^{l+2}(\alpha r) e^{-2\alpha r} L_2^1(\alpha r) L_2^3(\alpha r)$$

$$Q_{10} \;=\; A_{001}A_{012}\alpha/(4\pi) \int\, dr\, r^4\, e^{-2\alpha r}\, L_2^1 L_2^3$$

For the magnetic multipole, use $\vec{J}$ from the above and substitute into;

$$M_{lm} \;=\; -(1/(l+1)) \int\, d^3x\, r^l\, \vec{\nabla}\cdot(\vec{r}\times\vec{J}/c)\, Y_l^{*\,m}$$

Manipulation of the operations and integration shows that the magnetic multipole vanishes.

5 Angular momentum

EM radiation not only contains power (energy), but contains angular momentum. The asymptotic behavior of the magnetic field for the electric multipole has the form;

$$\vec{H} \;\propto\; -(k/l)\, \vec{L}\, Y_{lm}/r^{l+1}$$

The electric field obtained from this field is;

$$\vec{E} \propto -(i/l)\, z\, \vec{\nabla} \times \vec{L}(Y_{lm}/r^{l+1})$$

Apply the identity, $i(\vec{\nabla} \times \vec{L}) = \vec{r}\nabla^2 - \vec{\nabla}(1+r\frac{\partial}{\partial r})$ and extract the largest value of $\vec{E}$ as $r \to 0$;

$$\vec{E} \propto -z\, \vec{\nabla}(Y_{lm}/r^{l+1})$$

The above is just the static electric multipole field. In the near-field the magnetic field is smaller than the electric field by a factor of kr. However in the far-field the electric to magnetic field is proportional to z.
The time averaged energy density in the EM wave has the form;

$$\mathcal{E} = (\epsilon/4)[\vec{E} \cdot \vec{E}^* + z^2 \vec{H} \cdot \vec{H}^*]$$

The angular momentum is obtained from the angular momentum density;

$$\vec{\mathcal{L}} = (1/(2c^2))\, Re[\vec{r} \times (\vec{E} \times \vec{H}^*)]$$

The above reduces to;

$$\vec{\mathcal{L}} = (\mu/2\omega)\, Re[\vec{H}^*(\vec{L} \cdot \vec{H})]$$

The angular momentum in a spherical shell between r and $r + dr$ is

$$\frac{d\mathcal{L}_z}{dr} = \frac{ds\mu}{2\omega k^2}\, Re \sum_{mm'} a_E^*(l, m')a_E^*(l, m)|^2 \int d\Omega\, Y_{lm}^* Y_{lm}$$

The explicit values are somewhat detailed and are reproduced in the text for the various angular momentum components. Only the $\hat{z}$ component has a relatively simple form. For a multipole of a single m value the angular momentum components along $\hat{x}$ and $\hat{y}$ vanish. This obviously is expected from our prior knowledge of the quantization of a photon with eigenvalues of (l, m) where only the angular momentum component along $\hat{z}$ can be precisely known.