The fields and potentials of a point charge

Lecture 14

1 Advanced and retarded potentials

In order to develop the potential formulation for an arbitrarily moving point charge, begin from the covarient form of Maxwell's equations written in terms of the field tensor. (ignore covarient and contravarient notation)

$$\frac{\partial F_{ik}}{\partial x_k} = (1/c) j_i$$

Remember that this equation reproduces Ampere's law and Gauss's Law. The other 2 equations are automaticaly satisfied by the definition of the form for the field strength tensor.

$$F_{ik} = \frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k}$$

Substitute this expression into Maxwell's equations and apply the Lorentz condition;

$$\frac{\partial A_i}{\partial x_i} = 0$$

The result is;

$$\frac{\partial^2 A_i}{\partial x_k^2} = (1/c) j_i$$

The above can be expanded in terms of the 4-vector components;

$$\nabla^2 \vec{A} - (1/c^2)\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}$$

$$\nabla^2 V - (1/c^2)\frac{\partial^2 V}{\partial t^2} = -\rho/\epsilon$$

We used these are the inhomogeneous wave equations previously. Now recall the Green function which was derived for the scalar wave equation. This required that the solution have a retarded (or advanced) form, and was obtained by expansion of the solution to the wave equation with a delta function source in position and time written in terms of Fourier components. The solution has the form;

$$G(\vec{x}, \vec{x}') = \frac{\delta(\tau - R/c) - \delta(\tau + R/c)}{R}$$

with $R = |\vec{x} - \vec{x}'|$ and $\tau = t - t'$. The delta function requires $|\vec{x} - \vec{x}'|/c = \pm (t - t')$ which is either a retarded or advanced time condition. In most cases the retarded time solution is chosen to satisfy causality. Application of the Green function then provides the solution for an extended source.

$$V = \frac{1}{4\pi\epsilon} \int d^3x' \, \frac{\rho(\vec{x}', \, t' = t - R/c)}{R}$$

$$\vec{A} = \frac{\mu}{4\pi} \int d^3x' \, \frac{\vec{J}(\vec{x}', t' = t - R/c)}{R}$$

2 Lienard-Wiechert potentials

Now apply the above potentials to a point charge moving in an arbitrary direction. We note that at any instant of time the potential is due to one at a previous point in space time, *ie* a retarded point on the world line. The time t' corresponding to this point is determined by;

$$t' = t - R/c$$

To determine the potential we need to assume that the charge has finite dimensions. This is not so obvious, but the spatial dimension causes a shift in the time depending on when the EM energy radiates from the charge. This effect does not vanish in the limiting case when the size of the distribution is reduced to zero and must be included in the source. The integrand for the scalar potential uses the charge density $\rho(\vec{r}', t_r)$ where t_r is the retarded time at the point, $\vec{r}'$. However, the integral over the spatial dimensions of the source, $\int d^3x' \, \rho$ does not return the charge of the particle. To get the charge, one must integrate over the distribution at the same instant of time, but as written, ρ is evaluated over the world line in which the time changes as the particle moves. Thus a point charge must be regarded as a limit of an extended charge. In general the apparent volume is related to the actual volume by the factor;

$$d^3x' = d^3x/[1 - \hat{r} \cdot \vec{\beta}]$$

This includes the Lorentz contraction as observed from the angle θ, and is illustrated in Figure 1. Note that this is obtained by observing that the time for the light to travel the extra distance from the trailing edge is L'/c while the volume moves a distance $L - L'$ with a velocity v. Thus;

$$L'/c = (L - L')/v$$

Observed from an angle;

$$L' \, cos(\theta)/c = (L - L')/v$$

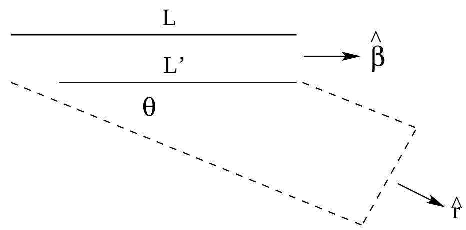


Figure 1: The length element for the volume as viewed at the angle θ

It then follows that the scalar potential is obtained from the expression;

$$V = \frac{1}{4\pi\epsilon} \frac{q}{R - \vec{R} \cdot \vec{\beta}}$$

The vector potential is obtained in a similar way for each component.

$$\vec{A} = \frac{1}{4\pi\epsilon c} \frac{q\vec{\beta}}{[R - \vec{R} \cdot \vec{\beta}]}$$

These are the Lienard-Wiechert potential forms.

3 The covarient Green's function

The text develops the Lienard-Wiechert potentials throught the use of the covarient Green's function, and use this technique here. First attempt to find a solution to the 4-D equation;

$$\Box D = \delta^4(x - x') = \delta(x^0 - x^0{}') \delta(\vec{x} - \vec{x}')$$

$$\Box = \frac{\partial}{\partial x_{nu}} \frac{\partial}{\partial x^\nu} = \nabla^2 - (1/c^2) \frac{\partial^2}{\partial t^2}$$

Here $\Box\, D$ is the D'Alembertian operation on D. The D'Alembertian is the covarient Laplacian operator for the wave equation. Sometimes it is written $\Box^2$. Apply a Fourier transform to write;

$$D(x) = \frac{1}{(4\pi)^2} \int d^4k\, \mathcal{D}\, e^{-ikx}$$

$$\delta^{(4)}(x) = \frac{1}{(2\pi)^4} \int d^4k\, e^{-ikx}$$

The solution for the transform $\mathcal{D}$ is;

$$\mathcal{D} = -\frac{1}{2\pi} \frac{1}{k^\alpha k_\alpha}$$

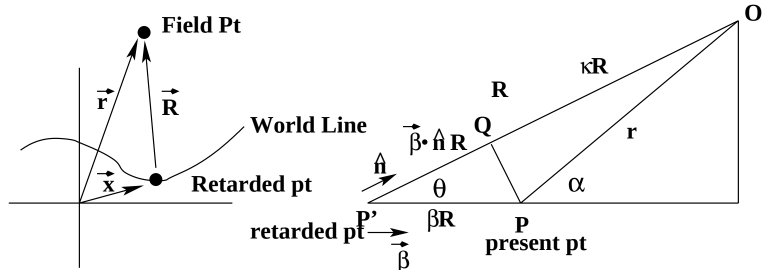


Figure 2: The geometry of the field and source points for a moving point charge

Now $kx = k_0 x_0 - \vec{x} \cdot \vec{k}$ which gives;

$$D = -\frac{1}{2\pi} \int d^4k \, \frac{e^{-ikx}}{k_0^2 - k^2}$$

Integrate the time component in the complex plane. The choice of the contour just above or below the real axis leads to a Θ function which determines whether the solution has a retarded or advanced form. If we select the retarded form then;

$$D = \frac{1}{2\pi} \Theta(x_0 - x_0') \, \delta[(\vec{x} - \vec{x}')^2]$$

Written in this way, the Green's function is a Lorentz scalar. The text demonstrates the development of the Lienard-Wiechert potentials using this function. This technique is more elegent, mathematically, but leaves out physical intuition.

4 Fields from the Lienard-Wiechert potentials

The fields are obtained using;

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

The geometry is shown in Figure 2.

In the figure $R = |\vec{r} - \vec{x}| = c(t - t_r)$, where t_r is the retarded time. The fields are obtained as per the development in the text. After some careful algebra;

$$\vec{E} = \frac{q}{4\pi\epsilon} [\frac{\hat{n} - \vec{\beta}}{\gamma^2 (1 - \vec{\beta} \cdot \hat{n})^3 R^2}]_r + \frac{q}{4\pi\epsilon c} [\frac{\hat{n} \times ([\hat{n} - \vec{\beta}] \times \dot{\vec{\beta}})}{(1 - \vec{\beta} \cdot \hat{n})^3 R}]_r$$

The 2 terms represent static (first term) and radiation fields (second term) evaluated at the retarded time. Consider the first term. It has a form which decreases with distance as $\frac{1}{r^2}$

which one would expect for the field of a static charge (including a field of a charge moving with constant velocity). Now use geometric relations obtained from Figure 2. Define $\kappa = (1 - \vec{\beta} \cdot \hat{n})$. Also;

$$(\kappa R)^2 \, r^2 \;=\; (PQ)^2 \;=\; R^2 \beta^2 \, sin^2(\theta) \;=\; r^2 - \beta^2 r^2 \, sin^2(\alpha)$$

and;

$$(\hat{n} - \vec{\beta})R = \vec{r}$$

Substitution in the static field above yields;

$$\vec{E}_s \;=\; \frac{q}{4\pi\epsilon} \frac{\vec{r}}{\gamma^2(1 - \beta^2 \, sin^2(\theta)^{3/2}) \, r^3}$$

This is the static field equation obtained by a Lorentz transformation on the field of a charge at rest. Now look at the radiation field (second term). The field decreases with distance as $\frac{1}{r}$ as expected for a radiation field. In the radiation zone (far field) we expect a wave solution of the form $e^{i(\vec{k} \cdot \vec{x} - \omega t)}$. Placed into Faraday's law the magnetic field has the form; $\vec{B} \;=\; (1/c)\hat{k} \times \vec{E}$. Then write the Poynting vector;

$$\vec{S} \;=\; (1/\mu)\vec{E} \times \vec{B} \;=\; (1/\mu c)[\vec{E} \times \hat{k} \times \vec{E}] \;=\; \frac{\hat{k}E^2}{\mu c}$$

Collect terms so that in the MKS system the magnitude of the Poynting vector is ;

$$S \;=\; \frac{\mu c q^2}{16\pi^2}[\frac{1}{R^2}|\frac{\hat{n} \times (\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}}{(1 - \vec{\beta} \cdot \hat{n})^3}|^2]_r$$

Recall that S is the power flowing through an surface area. Choose the infintesmal area $R^2 d\Omega$ and change the time derivative of the power $\frac{dP}{dt}$ from the present time to the retarded time t'. Thus;

$$\frac{dt}{dt'} = \kappa$$

Collect terms to write the Poynting vector representing the power radiated into a solid angle $d\Omega$ (in Gausian units) in the particle's own time frame.

$$\frac{dP}{d\Omega} \;=\; \frac{q^2}{4\pi c} \frac{1}{R^2}[\frac{|\hat{n} \times (\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}|^2}{(1 - \vec{\beta} \cdot \hat{n})^5}]_r$$

5 Limiting cases

Now look at two limiting cases; 1) The case where the velocity is parallel to the acceleration (linear acceleration), and 2) the case where the velocity is perpendicular to the acceleration (circular acceleration).

5.1 Parallel velocity and acceleration

When $\vec{B}$ and $\dot{\vec{\beta}}$ are parallel;

$$\hat{n} \times (\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}} = \hat{n} \times \hat{n} \times \dot{\vec{\beta}}$$

$$|\hat{n} \times \hat{n} \times \dot{\vec{\beta}}|^2 = \dot{\beta}^2 \, sin^2(\theta)$$

where θ is the angle between the velocity (acceleration) and the direction of observation. In the MKS systen the angular distribution of the power (MKS) is;

$$\frac{dPower}{d\Omega} = \frac{\mu q^2}{16\pi^2 c^3} \frac{\beta^2 \, sin^2(\theta)}{\kappa^5}$$

Note that when $\beta = 0$ $\kappa = 1$ the power maximizes at $\theta = \pi/2$ with respect to the velocity. The maximum of the power as a function of angle approaches $\theta = 0$ as $\beta \rightarrow 1$. A contour plot of the power distribution is shown in Figure 3. Integration over the angles gives the total radiated power (MKS);

$$P = \frac{\mu g^2}{6\pi c^3} \beta^2 \gamma^6$$

5.2 Perpendicular velocity and acceleration

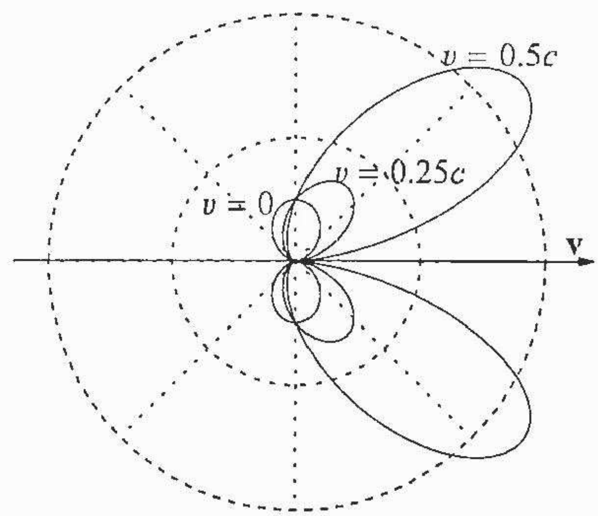
If the motion is circular, choose the coordinate system shown in Figure 4. The motion of the charge is in the (x, z) plane. Evaluate;

$$|\hat{n} \times (\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}|^2 = \dot{\beta}^2 [\kappa^2 + \frac{sin^2(\theta) \cos^2(\phi)}{\gamma^2}]$$

This then gives for the angular distribution of the power (MKS);

$$\frac{dP}{d\Omega} = \frac{\mu g^2}{16\pi^2 c^3} \frac{\dot{\beta}^2}{\kappa^3} [1 - \frac{sin^2(\theta) \cos^2(\phi)}{\gamma^2 \kappa^2}]$$

Integrated over angle the total power is (MKS)



Polar diagram of the energy loss angular distribution factor $\sin^2\theta/(1 - v\cos\theta/c)^5$ during bremsstrahlung for particle speeds $v = 0$, $v = 0.25c$, and $v = 0.5c$.

Figure 3: A contour plot of the power distribution as a function of the velocity of a linearly accelerated charge

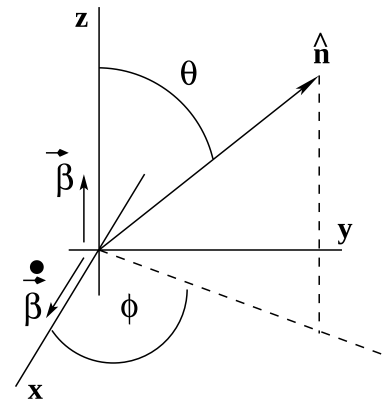


Figure 4: The geometry for an acceleration perpendicular to the velocity

$$P = \frac{\mu g^2}{6\pi c^3}\dot{\beta}^2\gamma^4$$

5.3 Other examples

Other examples of the electric field generated by an accelerating charge are shown in Figure 5. Note the concentration of the transverse components of the field at varions positions. Remember the transverse field is a radiation component while the longitudinal field is a static one. The transversity correlates with the value of acceleration. Now suppose we have a charge that moves harmonically along the z axis of a cartesian reference frame.

$$z(t') = a\,cos(\omega t')$$

$$\dot{z}(t') = -a\omega\,sin(\omega t')$$

$$\ddot{z}(t') = -a\omega^2\,cos(\omega t')$$

Evaluate the radiative power as follows;

$$|\hat{n}\times(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}|^2 = (\hat{n}\cdot\dot{\vec{\beta}})^2(\hat{n}-\vec{\beta})^2 + [\hat{n}\cdot(\hat{n}-\vec{\beta})]^2\dot{\beta}^2 - 2(\hat{n}-\vec{\beta})\cdot\dot{\vec{\beta}}(\hat{n}\cdot\dot{\vec{\beta}})[\hat{n}\cdot(\hat{n}-\vec{\beta})]$$

$$(\hat{n}-\vec{\beta})^2 = 1+\beta^2-2\hat{n}\cdot\vec{\beta}$$

Collect terms;

$$|\hat{n}\times(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}|^2 = \dot{\beta}^2\,sin^2(\theta)$$

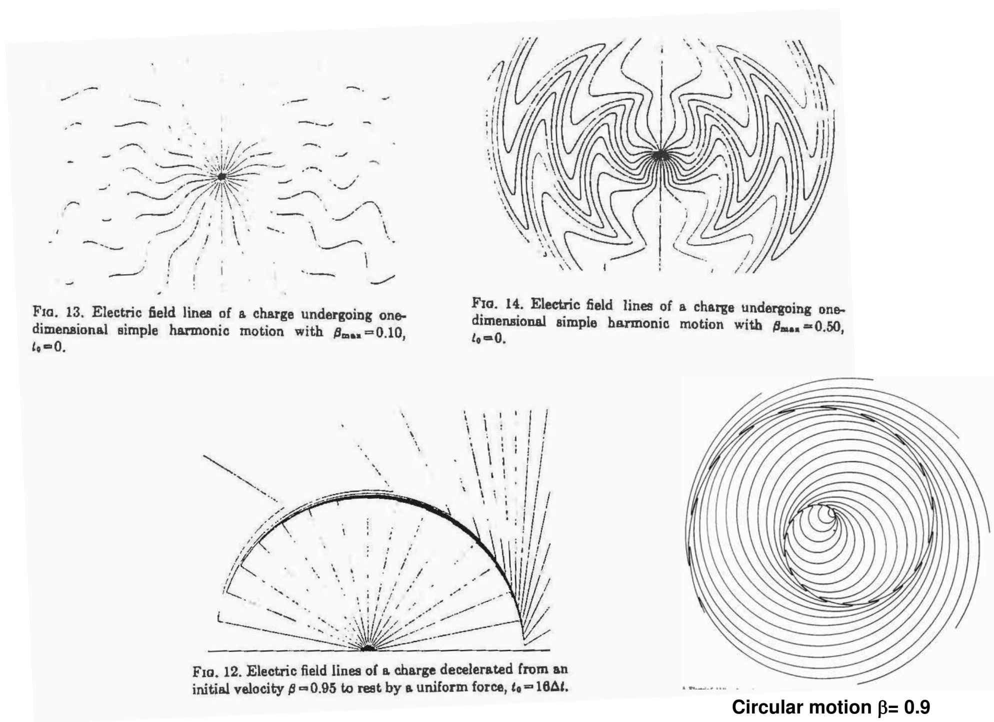


Figure 5: Examples of the electric field for an accelerated charge

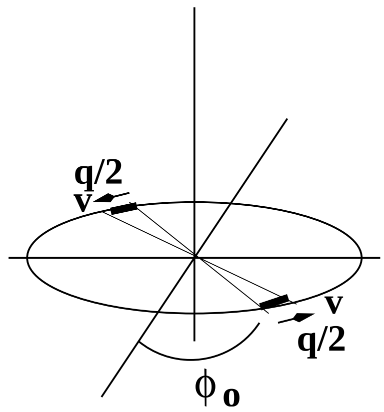


Figure 6: Radiation from sets of charge rotating in a circle

$$\frac{dP}{d\Omega} = \frac{\mu q^2}{16\pi^2 c^3} \frac{\dot{\beta}^2 \, sin^2(\theta)}{\kappa^5}$$

Substitute for $\dot{\beta}$

$$\frac{dP}{d\Omega} = \frac{\mu q^2}{16\pi^2 c^3} \frac{a^2 \omega^4 \, sin^2(\theta)}{\kappa^5}$$

Note that $qa = p$ the electric dipole moment. You will find electric dipole radiation written as;

$$\frac{dP}{d\Omega} = \frac{\mu}{16\pi^2 c^3} \, p^2 \omega^4 \, sin^2(\theta)$$

The factor, κ^5, is a relativistic effect which corrects the classical value for high frequencies.

6 Radiation and coherence

Now an accelerated charge radiates energy. Suppose we set 2 charges in symmetric positions on a loop and let them spin around the z axis as shown in Figure 6. This results in radiation because the charges are accelerated. Now suppose we consider the loop filled with a continuium charge distribution which rotates. Contrary, perhaps to intuition, there is no radiation. This occurs because of coherence of the radiation from each of the elemental charge distributions. We see this as follows. The charge distribution of the two elements as shown in the figure can be written;

$$\rho = (q/2) \frac{\delta(r-a)}{a^2} \, \delta(\theta - \pi/2)[\delta(\phi + \phi_0) + \delta(\phi - \phi_0)]$$

Rewrite the charge per unit length so that we can add additional symmetric units of charge;

$$\lambda \;=\; \frac{q}{(n+1)\,\pi a}\delta(\phi - \pi k/N - \phi_0) \quad k = 0, 1, 2, \cdots, N-1$$

All of the charge elements rotate with the frequence ω. The charge density is then written;

$$\rho_k \;=\; \frac{q}{(n+1)\pi}\frac{\delta(r-a)}{a^2}\,\delta(\theta - \pi/2)\,\delta(\phi - \pi k/N - \omega t)$$

Now work out the Fourier components of the distribution.

$$\rho_k \;=\; \sum A_n\, cos(n[\phi - \omega t])$$

The coefficients A_n are found using othorgonality of the Fourier functions, so that;

$$\rho_k \;=\; \frac{q}{(n+1)2\pi}\frac{\delta(r-a)}{a^2}\,\delta(\theta - \pi/2)\,\sum_n cos(2\pi kn/N)\,cos(n[\phi - \omega])$$

Then;

$$\rho \;=\; \sum_{k=0}^{N-1} \rho_k$$

The sum over k provides the following;

$$\sum_{k=0}^{N-1} cos(2\pi nk/N) \;=\; cos(n\pi[1 - 1/N])\,\frac{sin(n\pi)}{\sin(n\pi/N)}$$

This is zero unless $n = N$ or 0. Thus the radiation has multipolarity N. But if $N \to \infty$ there will be no radiation. In effect the radiation from each charge element contributes coherently, cancelling the radiated power.

7 Covarient formulation

We begin by writing the total radiated power in Gaussian units for the linear and circular acceleration. The power radiated for linear acceleration is;

$$P_l \;=\; \frac{2g^2}{3c}\dot{\beta}^2\,\gamma^6$$

The power radiated for circular acceleration is;

$$P_c \;=\; \frac{2g^2}{3c}\dot{\beta}^2\,\gamma^4$$

Look at the expression for linear motion using $p = momentum$; and with energy amd momentum expressed in energy units

$$\frac{dp}{dt} = \frac{d\gamma\beta m}{dt} = m \cdot \beta\gamma^3$$

For circular motion, γ is constant so;

$$\frac{dp}{dt} = m \cdot \beta\gamma$$

Substitution gives;

$$P_l = \frac{2g^2}{3c^3}(\frac{dp}{dt})^2$$

$$P_c = \frac{2g^2}{3c^3}\gamma^2(\frac{dp}{dt})^2$$

Note that these equations show that the power loss due to radiation per force required, $\frac{dp}{dt}$, is larger for circular acceleration than linear acceleration by a factor of γ^2. Thus linear particle accelerators which reach high values of γ, in particular electron machines, are designed to be linear rather than circular.

Now to make the above equations covarient we need to look at the form;

$$\frac{dp_u}{d\tau}\frac{dp^u}{d\tau} = -(\frac{d\vec{p}}{d\tau})^2 + (1/c^2)(\frac{dE}{d\tau})^2$$

Then since ;

$$\frac{dE}{d\tau} = \frac{pc}{E}\frac{dp}{\tau}$$

$$\frac{dp_u}{d\tau}\frac{dp^u}{d\tau} = -(\frac{d\vec{p}}{d\tau})^2 + (\beta^2)(\frac{dp}{d\tau})^2 = -(\frac{dp}{dt})^2$$

The final result is

$$P = \frac{2q^2}{3c}\gamma^6[\dot{\beta}^2 - (\vec{\beta} \times \dot{\vec{\beta}})^2]$$

This checks when $\dot{\vec{\beta}}$ is parallel and perpendicualr to the velocity.