Chapter 10
Rotational Kinematics and Energy
10-1 Angular Position, Velocity, and Acceleration

Definition of Angular Position, $\theta$

$\theta = \text{angle measured from reference line}$

SI unit: radian (rad), which is dimensionless
10-1 Angular Position, Velocity, and Acceleration

Sign Convention for Angular Position
By convention:
\( \theta > 0 \) counterclockwise rotation from reference line
\( \theta < 0 \) clockwise rotation from reference line

Degrees and revolutions:

\[ 1 \text{ rev} = 360^\circ \]
10-1 Angular Position, Velocity, and Acceleration

Arc length $s$, measured in radians:

$$s = r\theta$$

Reference line
10-1 Angular Position, Velocity, and Acceleration

**Definition of Average Angular Velocity, \( \omega_{av} \)**

\[
\omega_{av} = \frac{\Delta \theta}{\Delta t}
\]

SI unit: radian per second (rad/s) = \( s^{-1} \)
Definition of Instantaneous Angular Velocity, $\omega$

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t}$$

SI unit: rad/s = $s^{-1}$
10-1 Angular Position, Velocity, and Acceleration

**Definition of Period, \( T \)**

\[
T = \frac{2\pi}{\omega}
\]

SI unit: second, \( s \)

**Definition of Average Angular Acceleration, \( \alpha_{av} \)**

\[
\alpha_{av} = \frac{\Delta \omega}{\Delta t}
\]

SI unit: radian per second per second (\( \text{rad/s}^2 \)) = \( \text{s}^{-2} \)
10-1 Angular Position, Velocity, and Acceleration

Definition of Instantaneous Angular Acceleration, $\alpha$

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t}$$

SI unit: rad/s$^2 = s^{-2}$

Angular speed increases

$\alpha > 0$
$\omega > 0$

Angular speed decreases

$\alpha < 0$
$\omega > 0$

Angular speed increases

$\alpha < 0$
$\omega < 0$

Angular speed decreases

$\alpha > 0$
$\omega < 0$
If the angular acceleration is constant:

\[ \omega = \omega_0 + \alpha t \]
# 10-2 Rotational Kinematics

## Analogies between linear and rotational kinematics:

<table>
<thead>
<tr>
<th>Linear Quantity</th>
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<td>( x )</td>
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### Linear Equation
\( a = \text{constant} \)

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Example: A high speed dental drill is rotating at $3.14 \times 10^4$ rads/sec. Through how many degrees does the drill rotate in 1.00 sec?

Given: $\omega = 3.14 \times 10^4$ rads/sec; $\Delta t = 1$ sec; $\alpha = 0$

Want $\Delta \theta$.

$$\theta = \theta_0 + \omega_0 \Delta t + \frac{1}{2} \alpha \Delta t^2$$

$$\theta = \theta_0 + \omega_0 \Delta t$$

$$\Delta \theta = \omega_0 \Delta t = \left(3.14 \times 10^4 \text{ rads/sec}\right)(1.0 \text{ sec})$$

$$= 3.14 \times 10^4 \text{ rads} = 1.80 \times 10^6 \text{ degrees}$$
10-3 Connections Between Linear and Rotational Quantities

Tangential Speed of a Rotating Object

\[ v_t = r \omega \]

SI unit: m/s
10-3 Connections Between Linear and Rotational Quantities

Centripetal Acceleration of a Rotating Object

\[ a_{cp} = r\omega^2 \]

SI unit: m/s²
10-3 Connections Between Linear and Rotational Quantities

**Tangential Acceleration of a Rotating Object**

\[ a_t = r\alpha \]

SI unit: m/s^2

**Tangential Versus Centripetal Acceleration**

\[ a_t = r\alpha \]  \text{ due to changing angular speed}

\[ a_{cp} = r\omega^2 \]  \text{ due to changing direction of motion}
This merry-go-round has both tangential and centripetal acceleration.
10-4 Rolling Motion

If a round object rolls without slipping, there is a fixed relationship between the translational and rotational speeds:

\[ v = \frac{2\pi r}{T} \quad \text{and} \quad v = r\omega = v_t \]
10-4 Rolling Motion

We may also consider rolling motion to be a combination of pure rotational and pure translational motion:

\[ v = r \omega \]

\[ v = 2r \omega \]

\[ v = r \omega \]

\[ v = 0 \]
10-5 Rotational Kinetic Energy and the Moment of Inertia

For this mass,

\[ K = \frac{1}{2} m v^2 = \frac{1}{2} m (r \omega)^2 = \frac{1}{2} (mr^2) \omega^2 \]
10-5 Rotational Kinetic Energy and the Moment of Inertia

We can also write the kinetic energy as

\[ K = \frac{1}{2} I \omega^2 \]

Where \( I \), the moment of inertia, is given by

\[ I = \sum m_i r_i^2 \]
Example: (a) Find the moment of inertia of the system below. The masses are $m_1$ and $m_2$ and they are separated by a distance $r$. Assume the rod connecting the masses is massless.

$r_1$ and $r_2$ are the distances between mass 1 and the rotation axis and mass 2 and the rotation axis (the dashed, vertical line) respectively.
Example continued:

Take $m_1 = 2.00 \, \text{kg}$, $m_2 = 1.00 \, \text{kg}$, $r_1 = 0.33 \, \text{m}$, and $r_2 = 0.67 \, \text{m}$.

$$ I = \sum_{i=1}^{2} m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 $$

$$ = (2.00 \, \text{kg})(0.33 \, \text{m})^2 + (1.00 \, \text{kg})(0.67 \, \text{m})^2 $$

$$ = 0.67 \, \text{kg m}^2 $$

(b) What is the moment of inertia if the axis is moved so that it passes through $m_1$?

$$ I = \sum_{i=1}^{2} m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 $$

$$ = (2.00 \, \text{kg})(0.00 \, \text{m})^2 + (1.00 \, \text{kg})(1.00 \, \text{m})^2 $$

$$ = 1.00 \, \text{kg m}^2 $$
Moments of inertia of various regular objects can be calculated:

- **Hoop or cylindrical shell**
  \[ I = MR^2 \]

- **Disk or solid cylinder** (axis at rim)
  \[ I = \frac{1}{2} MR^2 \]

- **Disk or solid cylinder** (axis at rim)
  \[ I = \frac{3}{2} MR^2 \]

- **Hollow sphere**
  \[ I = \frac{2}{3} MR^3 \]

- **Solid sphere**
  \[ I = \frac{2}{5} MR^2 \]

- **Solid sphere** (axis at rim)
  \[ I = \frac{7}{5} MR^2 \]
Example: What is the rotational inertia of a solid iron disk of mass 49.0 kg with a thickness of 5.00 cm and a radius of 20.0 cm, about an axis through its center and perpendicular to it?

From the previous slide:

\[ I = \frac{1}{2} MR^2 = \frac{1}{2} (49.0 \text{ kg})(0.2 \text{ m})^2 = 0.98 \text{ kg m}^2 \]
10-6 Conservation of Energy

The total kinetic energy of a rolling object is the sum of its linear and rotational kinetic energies:

\[ K = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2 \]

\[ = \frac{1}{2} m v^2 + \frac{1}{2} I \left( \frac{v}{r} \right)^2 = \frac{1}{2} m v^2 \left( 1 + \frac{I}{m r^2} \right) \]

The second equation makes it clear that the kinetic energy of a rolling object is a multiple of the kinetic energy of translation.
10-6 Conservation of Energy

If these two objects, of the same mass and radius, are released simultaneously, the disk will reach the bottom first – more of its gravitational potential energy becomes translational kinetic energy, and less rotational.

$h$
Example: Two objects (a solid disk and a solid sphere) are rolling down a ramp. Both objects start from rest and from the same height. Which object reaches the bottom of the ramp first?

The object with the largest linear velocity (v) at the bottom of the ramp will win the race.
Example continued:

Apply conservation of mechanical energy:

\[ E_i = E_f \]

\[ U_i + K_i = U_f + K_f \]

\[ mgh + 0 = 0 + \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2 = \frac{1}{2} mv^2 + \frac{1}{2} I\left(\frac{v}{R}\right)^2 \]

\[ mgh = \frac{1}{2} \left( m + \frac{I}{R^2} \right) v^2 \]

Solving for \( v \):

\[ v = \sqrt{\frac{2mgh}{\left( m + \frac{I}{R^2} \right)}} \]
Example continued:

The moments of inertia are:

\[ I_{\text{disk}} = \frac{1}{2} mR^2 \]
\[ I_{\text{sphere}} = \frac{2}{5} mR^2 \]

For the disk: \[ v_{\text{disk}} = \sqrt{\frac{4}{3} gh} \]

For the sphere: \[ v_{\text{sphere}} = \sqrt{\frac{10}{7} gh} \]

Since \( V_{\text{sphere}} > V_{\text{disk}} \) the sphere wins the race.

Compare these to a box sliding down the ramp. \[ v_{\text{box}} = \sqrt{2gh} \]
Summary of Chapter 10

• Period: \[ T = \frac{2\pi}{\omega} \]

• Counterclockwise rotations are positive, clockwise negative

• Linear and angular quantities:

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## Summary of Chapter 10

- **Linear and angular equations of motion:**

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### Tangential speed:

\[ v_t = r\omega \]

### Centripetal acceleration:

\[ a_{cp} = r\omega^2 \]

### Tangential acceleration:

\[ a_t = r\alpha \]
Summary of Chapter 10

• Rolling motion: \( \omega = \frac{v}{r} \)

• Kinetic energy of rotation: \( K = \frac{1}{2} I \omega^2 \)

• Moment of inertia: \( I = \sum m_i r_i^2 \)

• Kinetic energy of an object rolling without slipping:

\[
K = \frac{1}{2} m \omega^2 + \frac{1}{2} I \omega^2
\]

• When solving problems involving conservation of energy, both the rotational and linear kinetic energy must be taken into account.