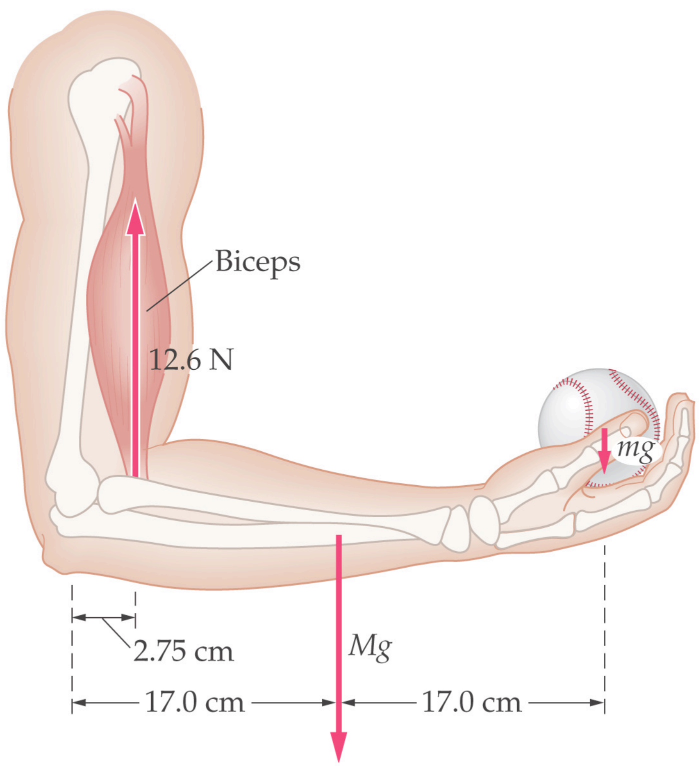
Chapter 11

Rotational Dynamics and Static Equilibrium



Units of Chapter 11

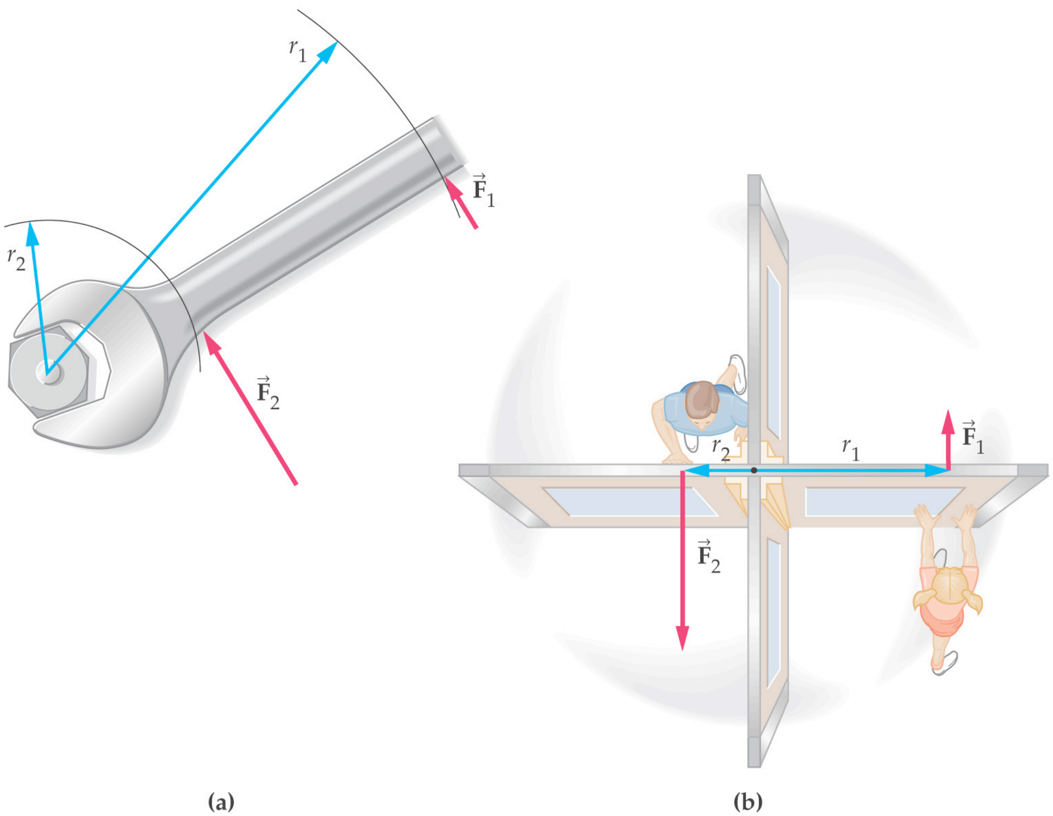
- Torque
- Torque and Angular Acceleration
- Zero Torque and Static Equilibrium
- Center of Mass and Balance
- Dynamic Applications of Torque
- Angular Momentum

Units of Chapter 11

- Conservation of Angular Momentum
- Rotational Work and Power
- The Vector Nature of Rotational Motion

11-1 Torque

From experience, we know that the same force will be much more effective at rotating an object such as a nut or a door if our hand is not too close to the axis.



This is why we have long-handled wrenches, and why doorknobs are not next to hinges.

11-1 Torque

We define a quantity called torque:

Definition of Torque, τ, for a Tangential Force

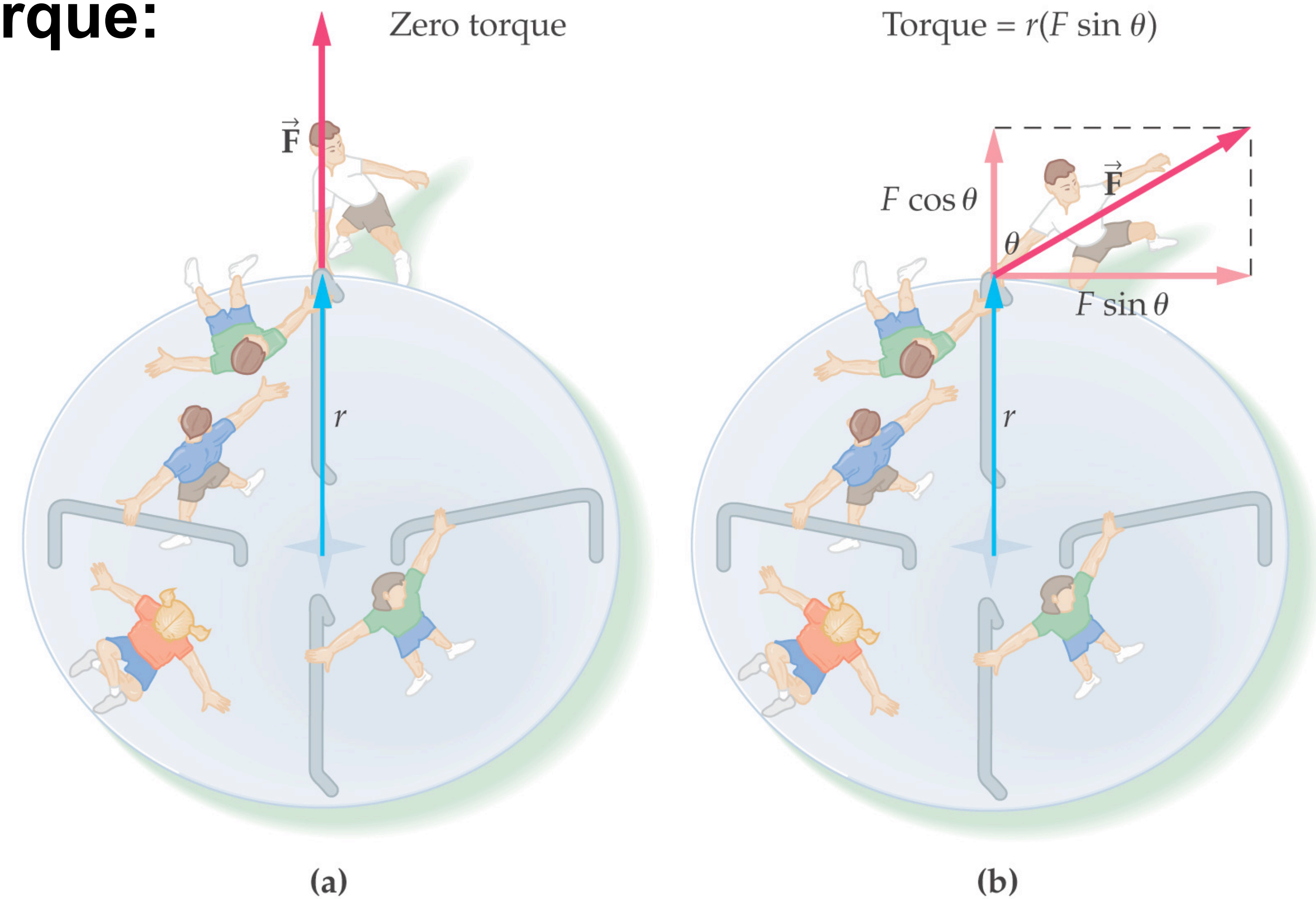
$$\tau = rF$$

SI unit: $\text{N} \cdot \text{m}$

The torque increases as the force increases, and also as the distance increases.

11-1 Torque

Only the tangential component of force causes a torque:



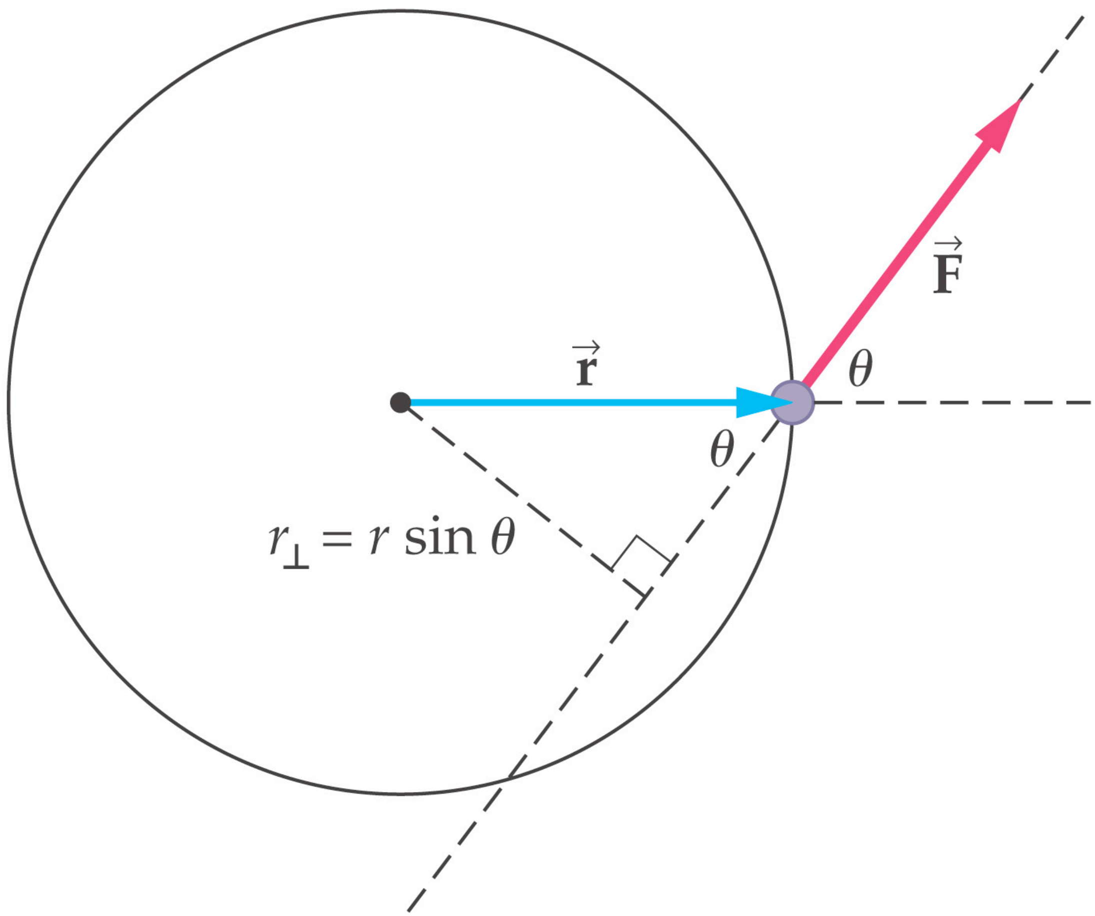
11-1 Torque

This leads to a more general definition of torque:

General Definition of Torque, τ

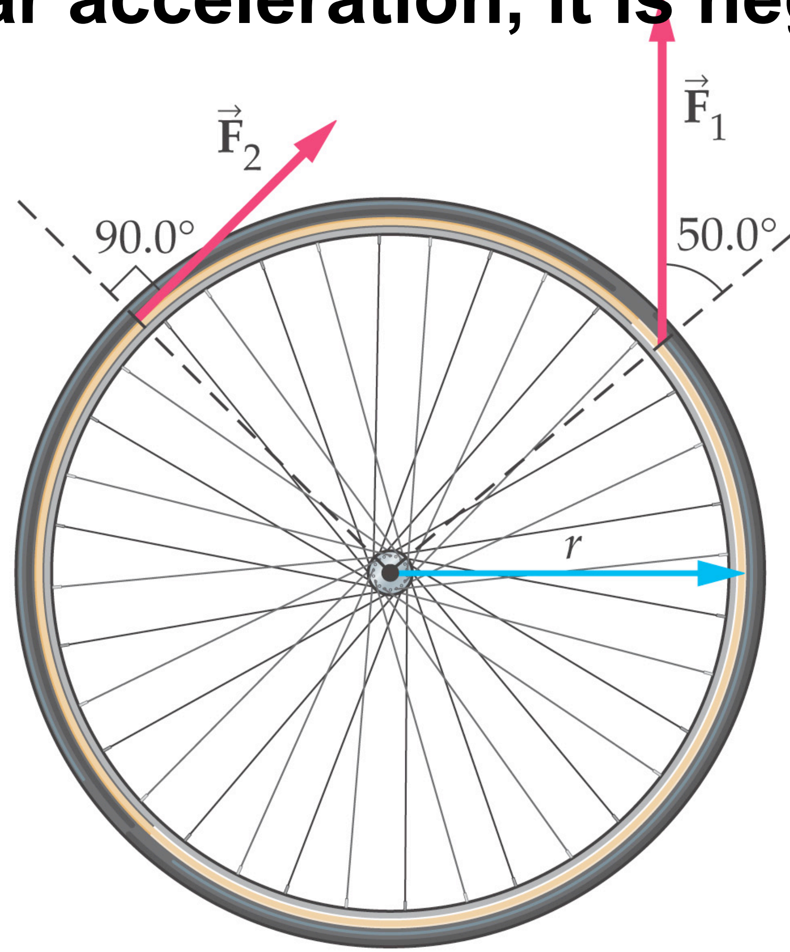
$$\tau = r(F \sin \theta)$$

SI units: N · m

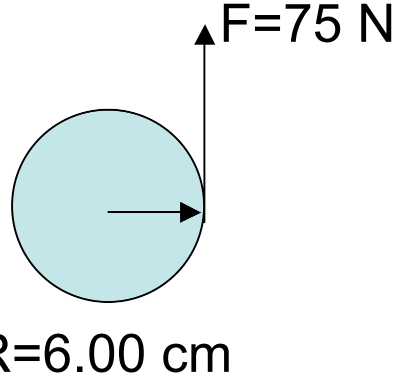


11-1 Torque

If the torque causes a counterclockwise angular acceleration, it is positive; if it causes a clockwise angular acceleration, it is negative.

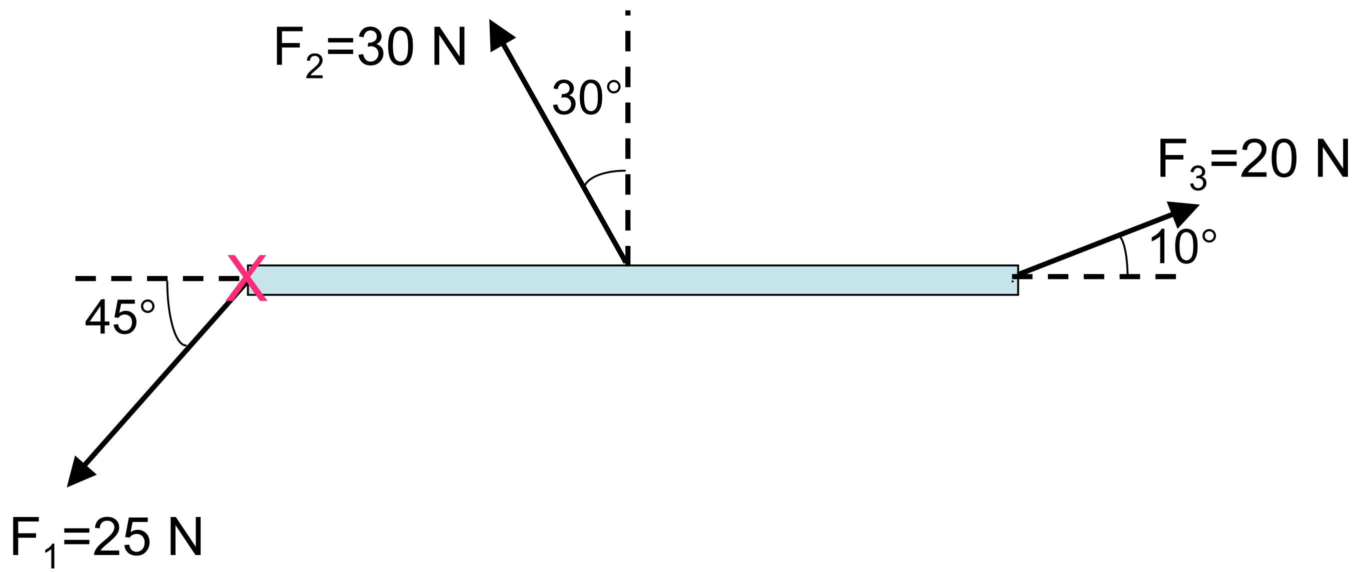


Example (text problem 8.12): The pull cord of a lawnmower engine is wound around a drum of radius 6.00 cm, while the cord is pulled with a force of 75.0 N to start the engine. What magnitude torque does the cord apply to the drum?

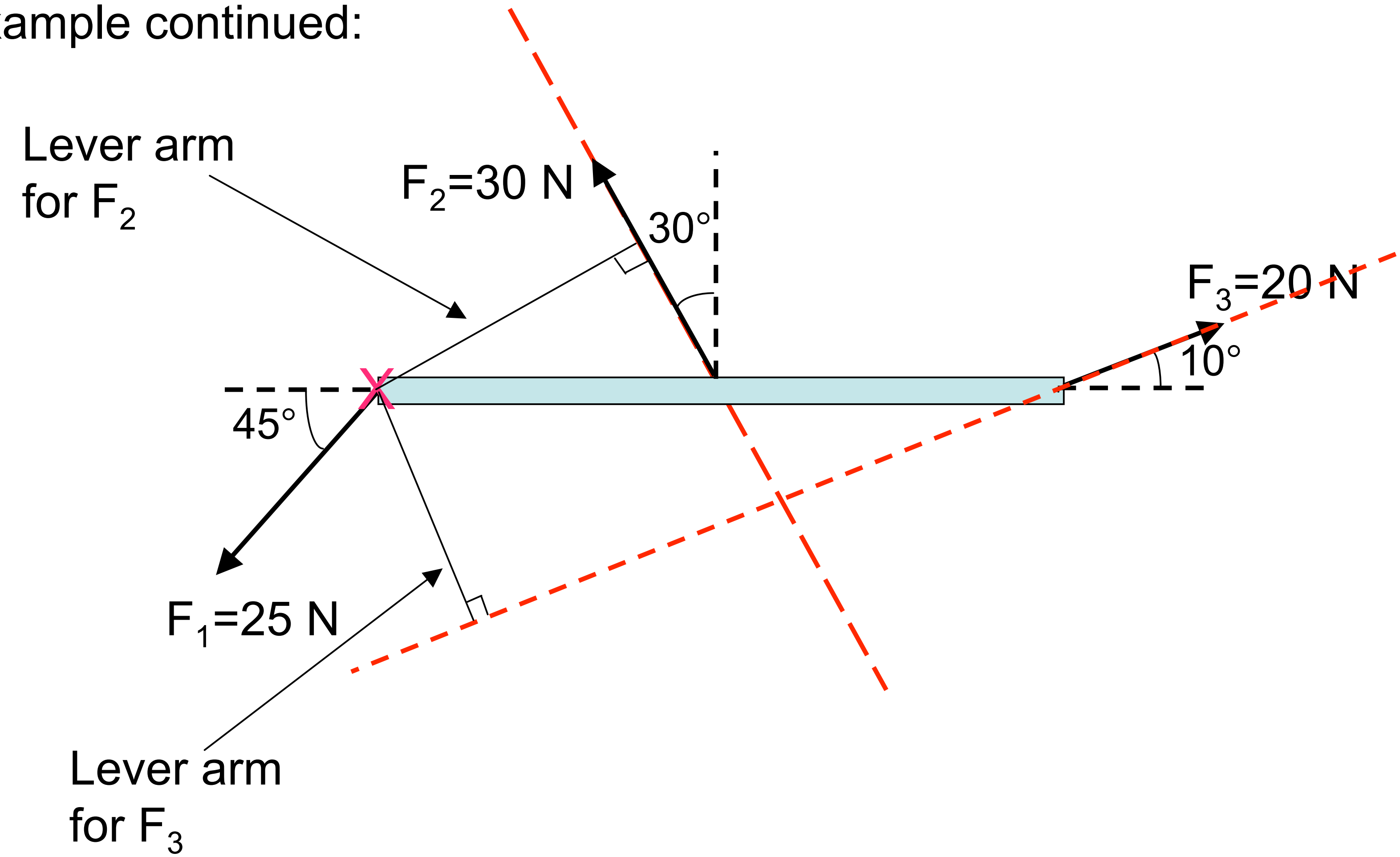


$$\tau = r_{\perp}F$$
$$= rF$$
$$= (0.06\ \text{m})(75.0\ \text{N}) = 4.5\ \text{Nm}$$

Example: Calculate the torque due to the three forces shown about the left end of the bar (the red X). The length of the bar is 4m and F_2 acts in the middle of the bar.



Example continued:



The lever arms are:

$$r_1 = 0$$

$$r_2 = (2\text{m})\sin 60° = 1.73\text{ m}$$

$$r_3 = (4\text{m})\sin 10° = 0.695\text{ m}$$

Example continued:

The torques are:

$$\tau_1 = 0$$

$$\tau_2 = +(1.73\ \text{m})(30\ \text{N}) = +51.9\ \text{Nm}$$

$$\tau_3 = +(0.695\ \text{m})(20\ \text{N}) = +13.9\ \text{Nm}$$

The net torque is +65.8 Nm and is the sum of the above results.

11-2 Torque and Angular Acceleration

Newton's second law: $a = \frac{F}{m}$

If we consider a mass m rotating around an axis a distance r away, we can reformat Newton's second law to read:

$$\alpha = \frac{rF}{mr^2} = \frac{\tau}{I}$$

Or equivalently,

Newton's Second Law for Rotational Motion

$$\tau = I\alpha$$

11-2 Torque and Angular Acceleration

Once again, we have analogies between linear and angular motion:

Linear Quantity	Angular Quantity
m	I
a	α
F	τ

Example (text problem 8.53): A bicycle wheel (a hoop) of radius 0.3 m and mass 2 kg is rotating at 4.00 rev/sec. After 50 sec the wheel comes to a stop because of friction. What is the magnitude of the average torque due to frictional forces?

$$\sum \tau = I\alpha = MR^2\alpha$$

$$\omega_i = 4.00\frac{\text{rev}}{\text{sec}}\left(\frac{2\pi\ \text{rad}}{1\,\text{rev}}\right) = 25.1\,\text{rad/sec}$$

$$\omega_f = 0$$

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_i}{\Delta t} = -0.50\ \text{rad/s}^2$$

$$\left|\tau_{av}\right| = MR^2\left|\alpha\right| = 0.09\ \text{Nm}$$

Example (text problem 8.25): A flywheel of mass 182 kg has a radius of 0.62 m (assume the flywheel is a hoop).

(a) What is the torque required to bring the flywheel from rest to a speed of 120 rpm in an interval of 30 sec?

$$\omega_f = 120\,\frac{\text{rev}}{\text{min}}\left(\frac{2\pi\text{ rad}}{1\text{ rev}}\right)\left(\frac{1\text{ min}}{60\text{ sec}}\right) = 12.6\text{ rad/sec}$$

$$\tau = rF = r(ma) = rm(r\alpha) = mr^2\left(\frac{\Delta\omega}{\Delta t}\right)$$

$$= mr^2\left(\frac{\omega_f - \omega_i}{\Delta t}\right) = mr^2\left(\frac{\omega_f}{\Delta t}\right) = 29.4\text{ Nm}$$

Example continued:

(b) How much work is done in this 30 sec period?

$$W = \tau\Delta\theta = \tau\left(\omega_{av}\Delta t\right)$$

$$= \tau\left(\frac{\omega_i + \omega_f}{2}\right)\Delta t = \tau\left(\frac{\omega_f}{2}\right)\Delta t = 5600\ \mathrm{J}$$

11-3 Zero Torque and Static Equilibrium

Static equilibrium occurs when an object is at rest – neither rotating nor translating.

Conditions for Static Equilibrium

For an extended object to be in static equilibrium, the following two conditions must be met:

(i) The net force acting on the object must be zero,

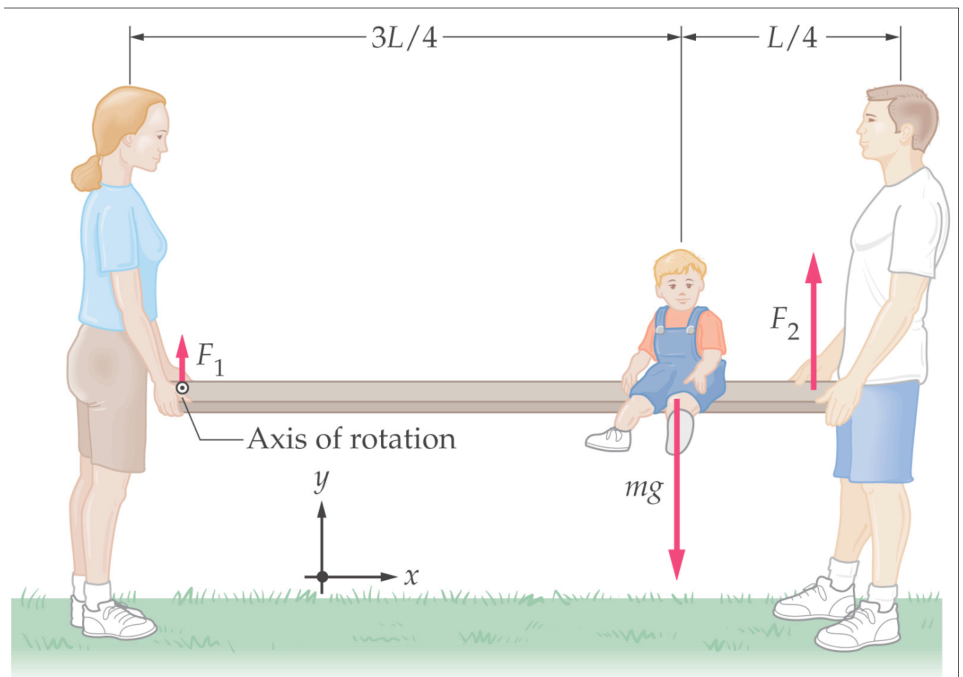
$$\sum F_x = 0, \sum F_y = 0 \qquad \textbf{11–5}$$

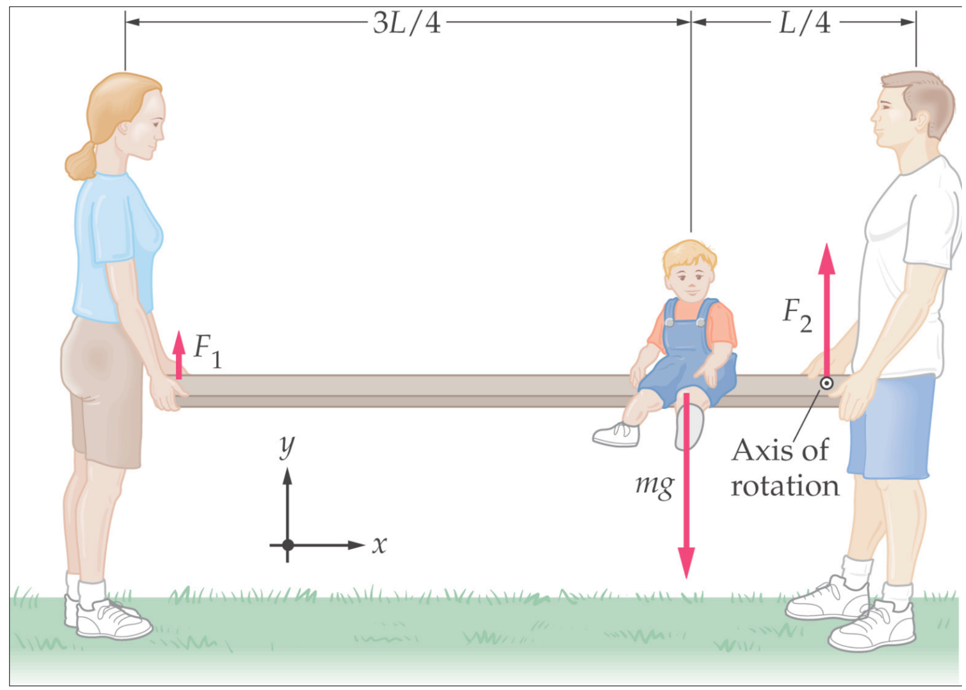
(ii) The net torque acting on the object must be zero,

$$\sum \tau = 0 \qquad \textbf{11–6}$$

11-3 Zero Torque and Static Equilibrium

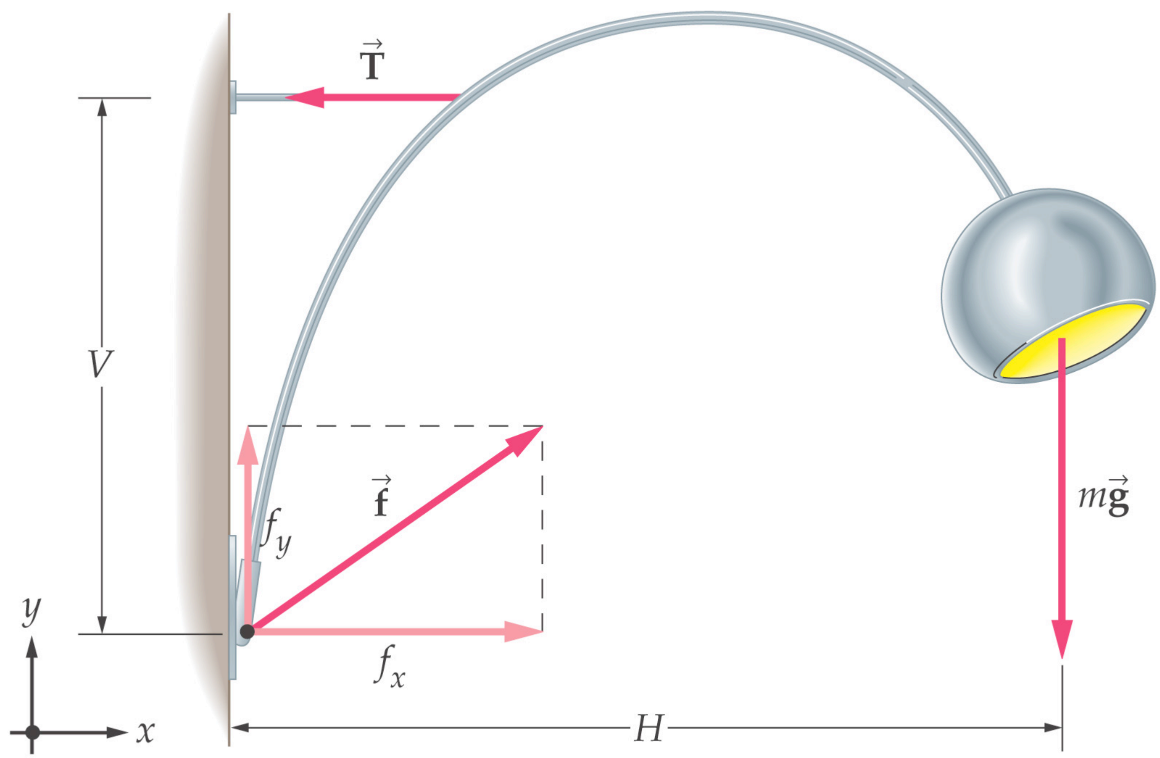
If the net torque is zero, it doesn't matter which axis we consider rotation to be around; we are free to choose the one that makes our calculations easiest.





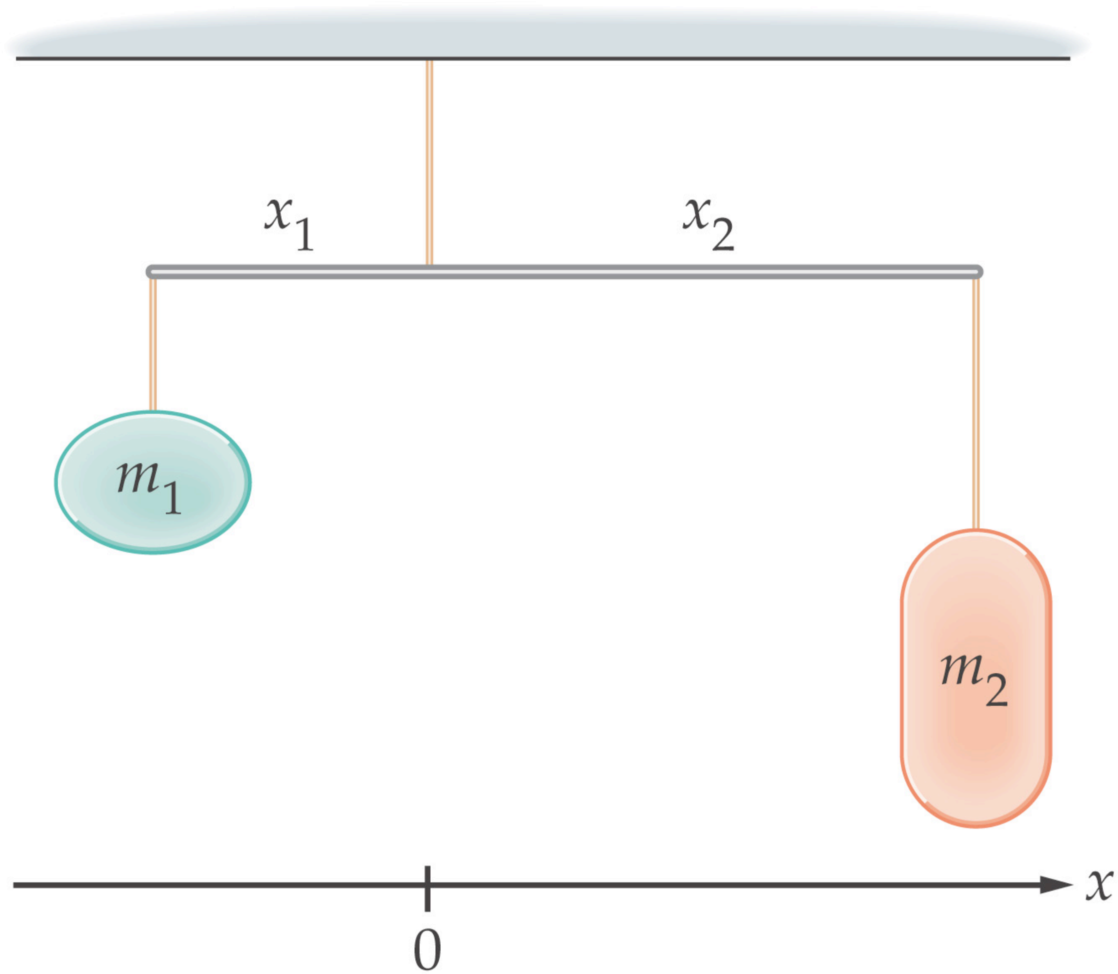
11-3 Zero Torque and Static Equilibrium

When forces have both vertical and horizontal components, in order to be in equilibrium an object must have no net torque, and no net force in either the *x*- or *y*-direction.

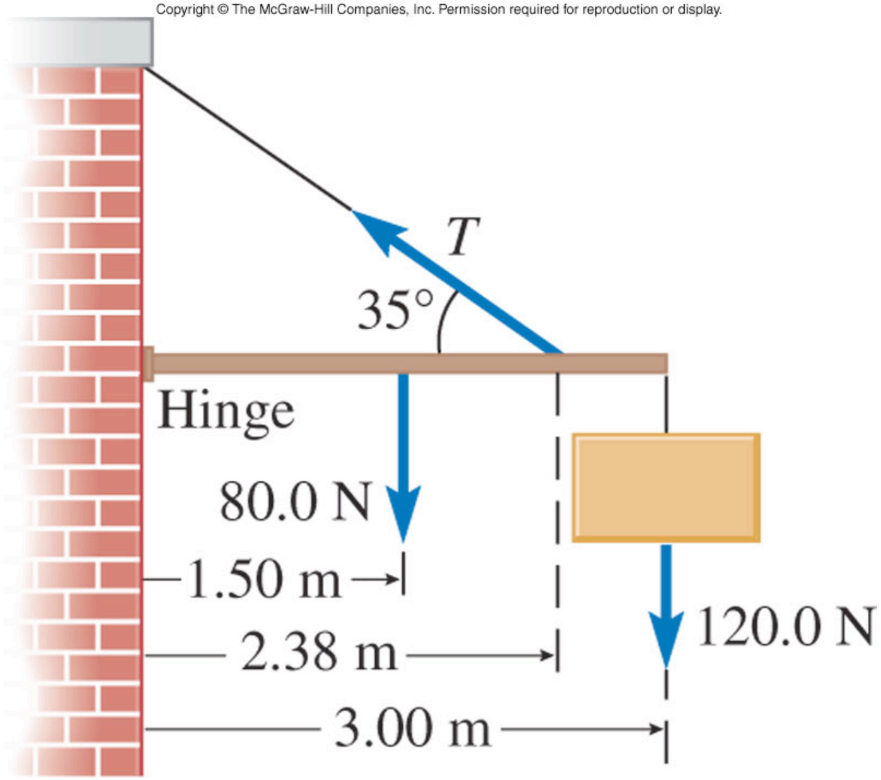


11-4 Center of Mass and Balance

If an extended object is to be balanced, it must be supported through its center of mass.

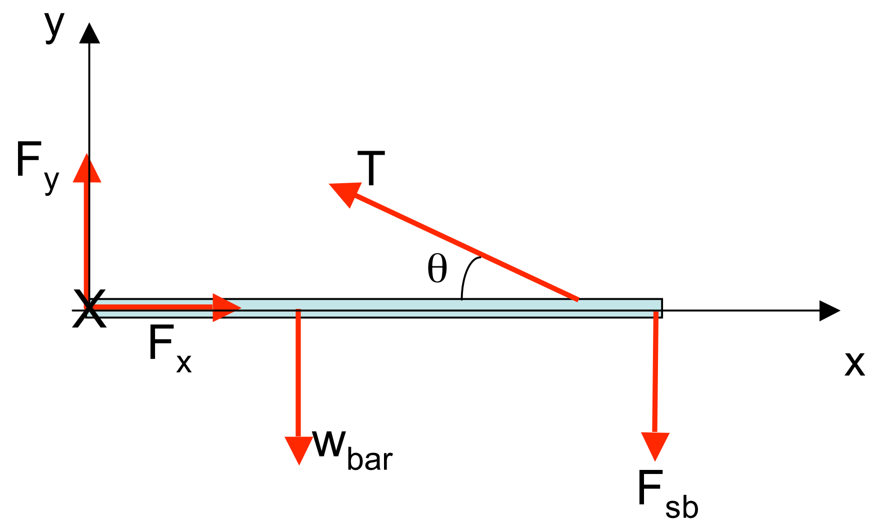


Example (text problem 8.35): A sign is supported by a uniform horizontal boom of length 3.00 m and weight 80.0 N. A cable, inclined at a 35° angle with the boom, is attached at a distance of 2.38 m from the hinge at the wall. The weight of the sign is 120.0 N. What is the tension in the cable and what are the horizontal and vertical forces exerted on the boom by the hinge?



Example continued:

FBD for the bar:



Apply the conditions for equilibrium to the bar:

$$(1)\ \sum F_x = F_x - T\cos\theta = 0$$

$$(2)\ \sum F_y = F_y - w_{\text{bar}} - F_{sb} + T\sin\theta = 0$$

$$(3)\ \sum \tau = -w_{\text{bar}}\left(\frac{L}{2}\right) - F_{sb}(L) + (T\sin\theta)x = 0$$

Example continued:

Equation (3) can be solved for T:

$$T = \frac{w_{\text{bar}}\left(\frac{L}{2}\right) + F_{sb}(L)}{x\sin\theta}$$

$$= 352\ \text{N}$$

Equation (1) can be solved for F_x:

$$F_x = T\cos\theta = 288\ \text{N}$$

Equation (2) can be solved for F_y:

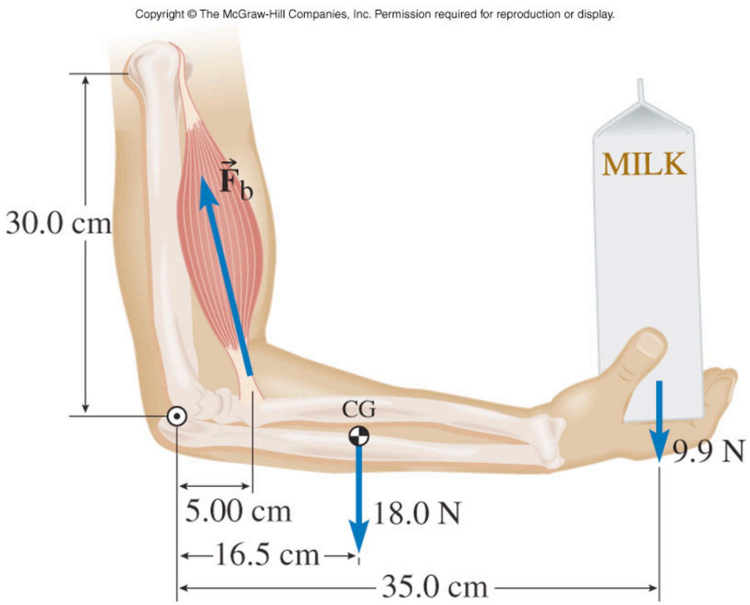
$$F_y = w_{\text{bar}} + F_{sb} - T\sin\theta$$

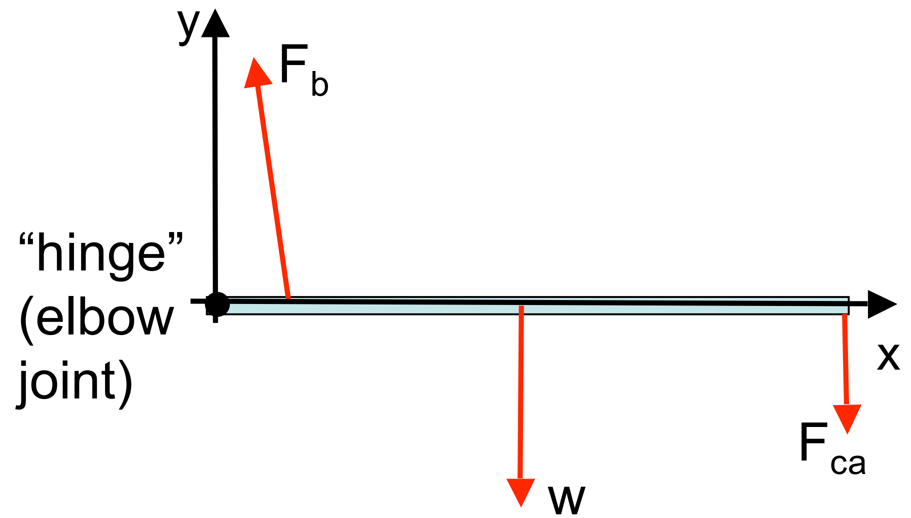
$$= -2.00\ \text{N}$$

Equilibrium in the Human Body

Example: Find the force exerted by the biceps muscle in holding a one liter milk carton with the forearm parallel to the floor. Assume that the hand is 35.0 cm from the elbow and that the upper arm is 30.0 cm long. The elbow is bent at a right angle and one tendon of the biceps is attached at a position 5.00 cm from the elbow and the other is attached 30.0 cm from the elbow. The weight of the forearm and empty hand is 18.0 N and the center of gravity is at a distance of 16.5 cm from the elbow.

Example continued:



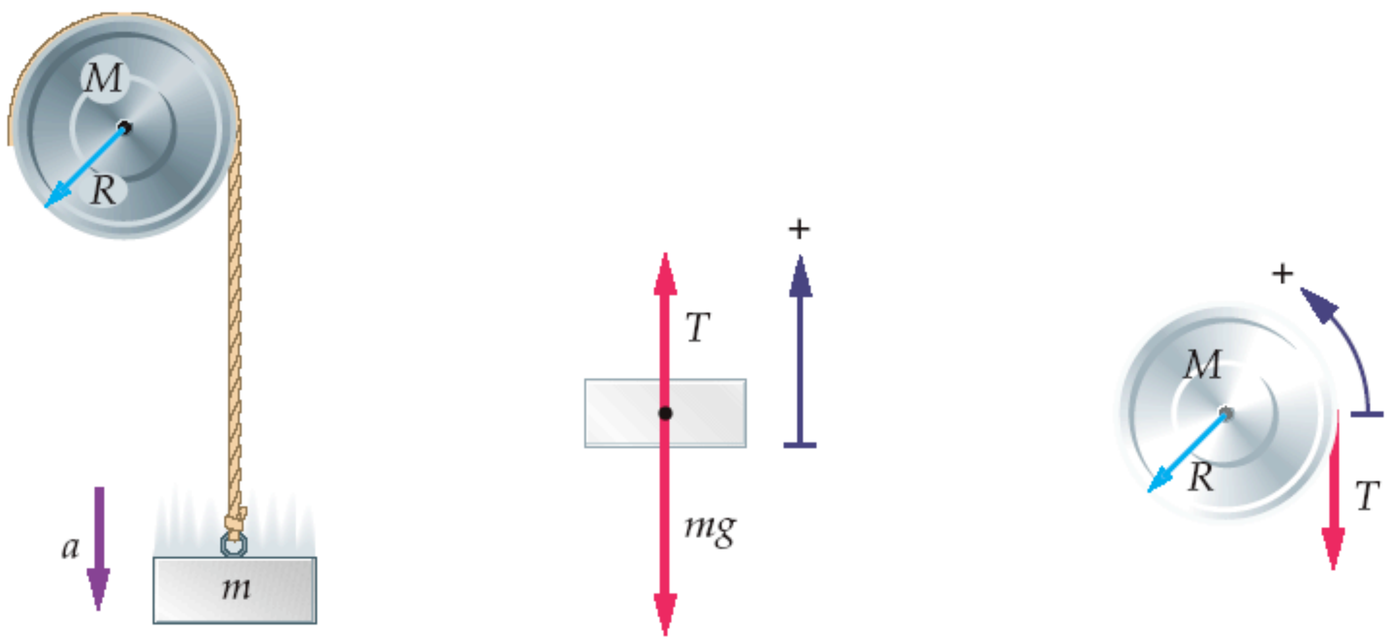


$$\sum \tau = F_b x_1 - w x_2 - F_{ca} x_3 = 0$$

$$F_b = \frac{w x_2 + F_{ca} x_3}{x_1} = 130\ \mathrm{N}$$

11-5 Dynamic Applications of Torque

When dealing with systems that have both rotating parts and translating parts, we must be careful to account for all forces and torques correctly.



(a) Physical picture

(b) Free-body diagram for mass

(c) Free-body diagram for pulley

11-6 Angular Momentum

Definition of the Angular Momentum, *L*

$$L = I\omega$$

SI unit: $\text{kg} \cdot \text{m}^2/\text{s}$

Using a bit of algebra, we find for a particle moving in a circle of radius *r*,

$$L = rmv = rp$$

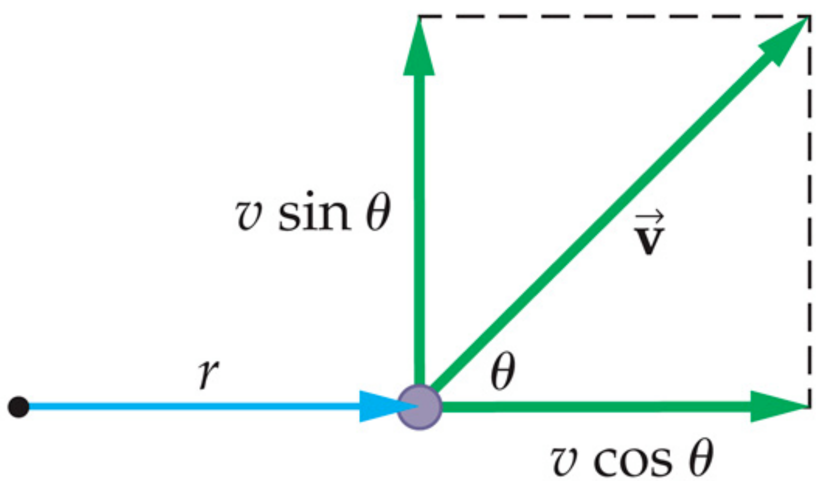
11-6 Angular Momentum

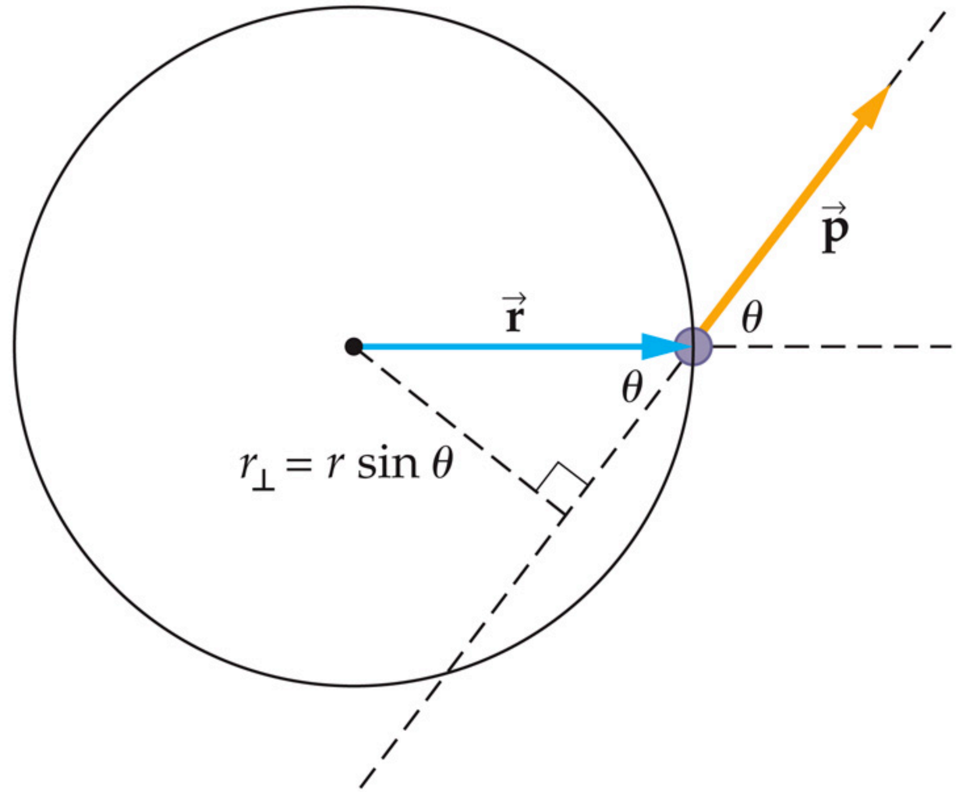
For more general motion,

Angular *Momentum*, *L*, for a Point Particle

$$L = rp\sin\theta = rmv\sin\theta$$

SI unit: $kg \cdot m^2/s$





11-6 Angular Momentum

Looking at the rate at which angular momentum changes,

$$\frac{\Delta L}{\Delta t} = I\frac{\Delta\omega}{\Delta t}$$

$$= I\alpha$$

Newton's Second Law for Rotational Motion

$$\tau = I\alpha = \frac{\Delta L}{\Delta t}$$

Example (text problem 8.69): A turntable of mass 5.00 kg has a radius of 0.100 m and spins with a frequency of 0.500 rev/sec. What is the angular momentum? Assume a uniform disk.

$$\omega = 0.500\frac{\text{rev}}{\text{sec}}\left(\frac{2\pi\ \text{rad}}{1\,\text{rev}}\right) = 3.14\ \text{rad/sec}$$

$$L = I\omega = \left(\frac{1}{2}MR^2\right)\omega = 0.079\ \text{kg m}^2/\text{s}$$

Example (text problem 8.75): A skater is initially spinning at a rate of 10.0 rad/sec with I=2.50 kg m^2 when her arms are extended. What is her angular velocity after she pulls her arms in and reduces I to 1.60 kg m^2?

The skater is on ice, so we can ignore external torques.

$$L_i = L_f$$

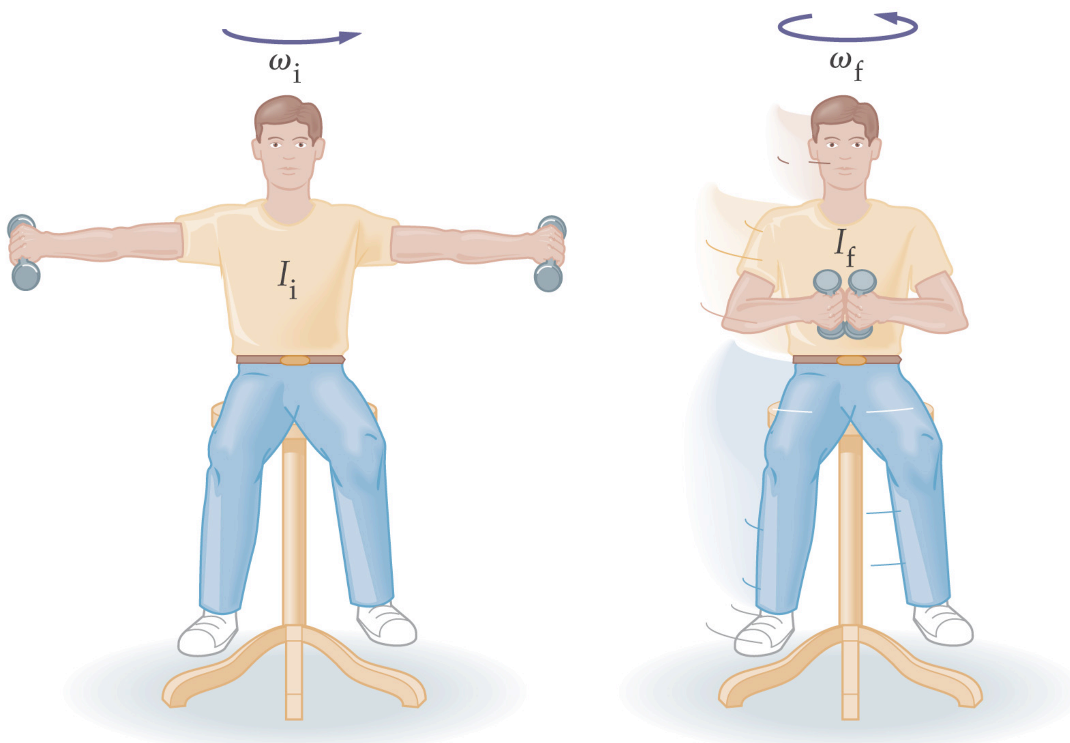
$$I_i\omega_i = I_f\omega_f$$

$$\omega_f = \left(\frac{I_i}{I_f}\right)\omega_i = \left(\frac{2.50\ \text{kg m}^2}{1.60\ \text{kg m}^2}\right)(10.0\ \text{rad/sec}) = 15.6\ \text{rad/sec}$$

11-7 Conservation of Angular Momentum

If the net external torque on a system is zero, the angular momentum is conserved.

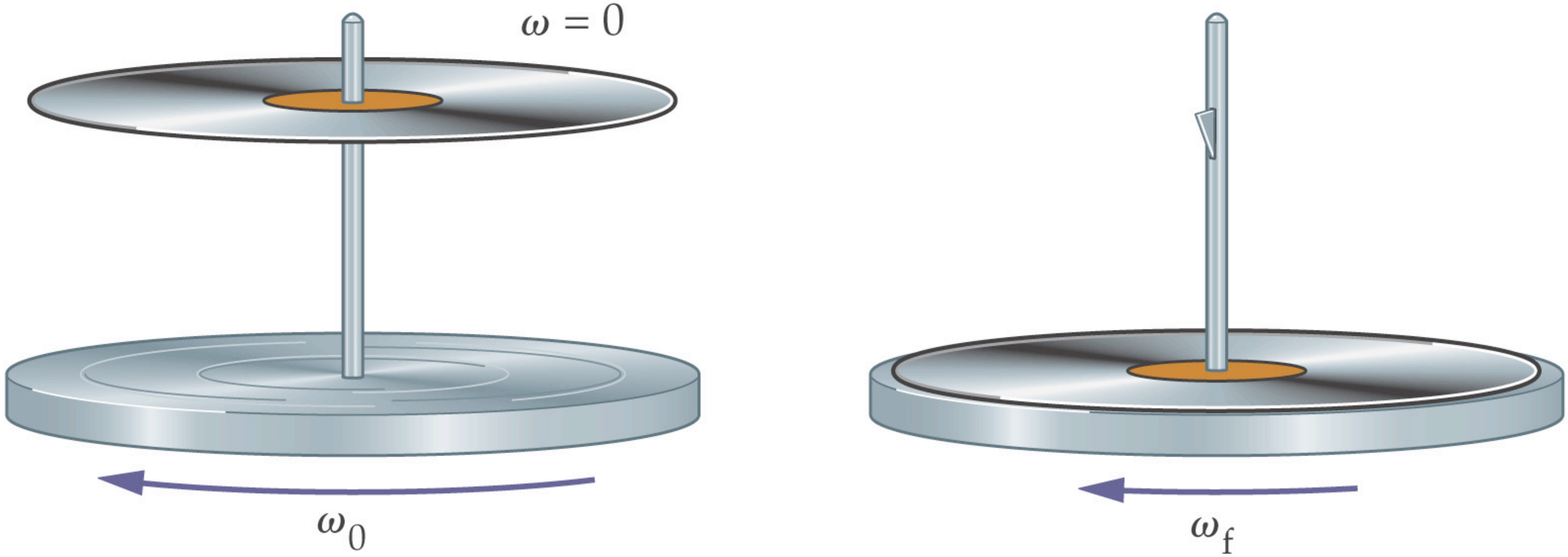
The most interesting consequences occur in systems that are able to change shape:



11-7 Conservation of Angular Momentum

As the moment of inertia decreases, the angular speed increases, so the angular momentum does not change.

Angular momentum is also conserved in rotational collisions:



11-8 Rotational Work and Power

A torque acting through an angular displacement does work, just as a force acting through a distance does.

Work Done by Torque

$$W = \tau \Delta\theta$$

The work-energy theorem applies as usual.

11-8 Rotational Work and Power

Power is the rate at which work is done, for rotational motion as well as for translational motion.

Power Produced by a Torque

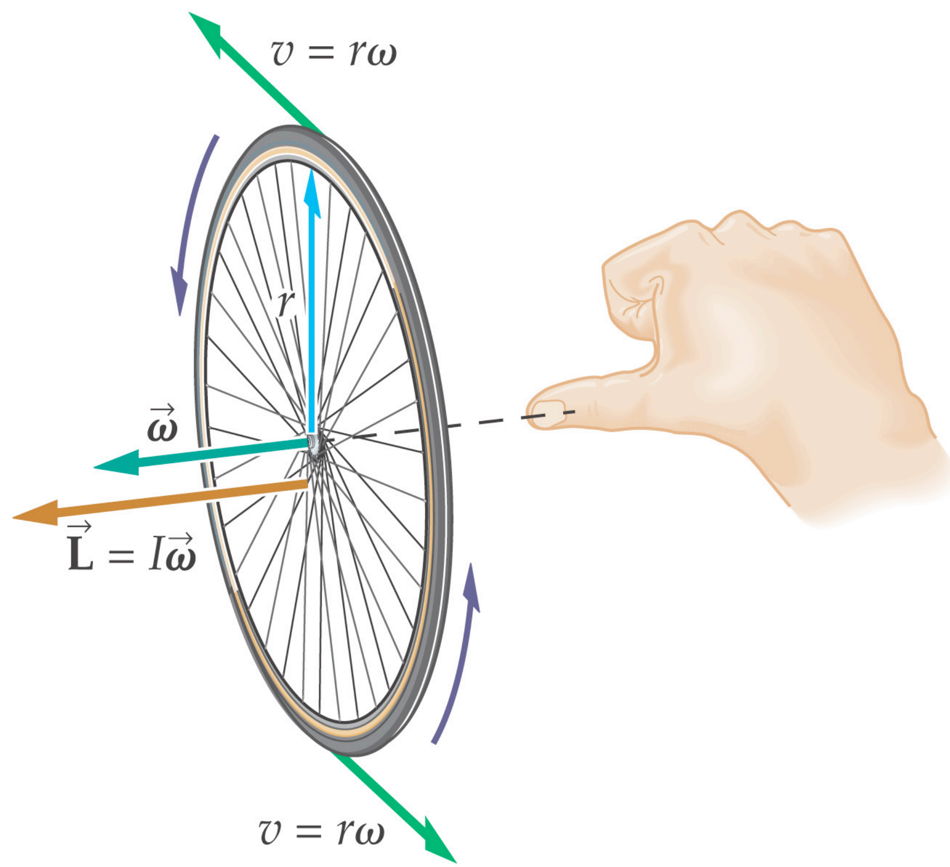
$$P = \frac{W}{\Delta t} = \tau \frac{\Delta\theta}{\Delta t} = \tau\omega$$

Again, note the analogy to the linear form:

$$P = Fv$$

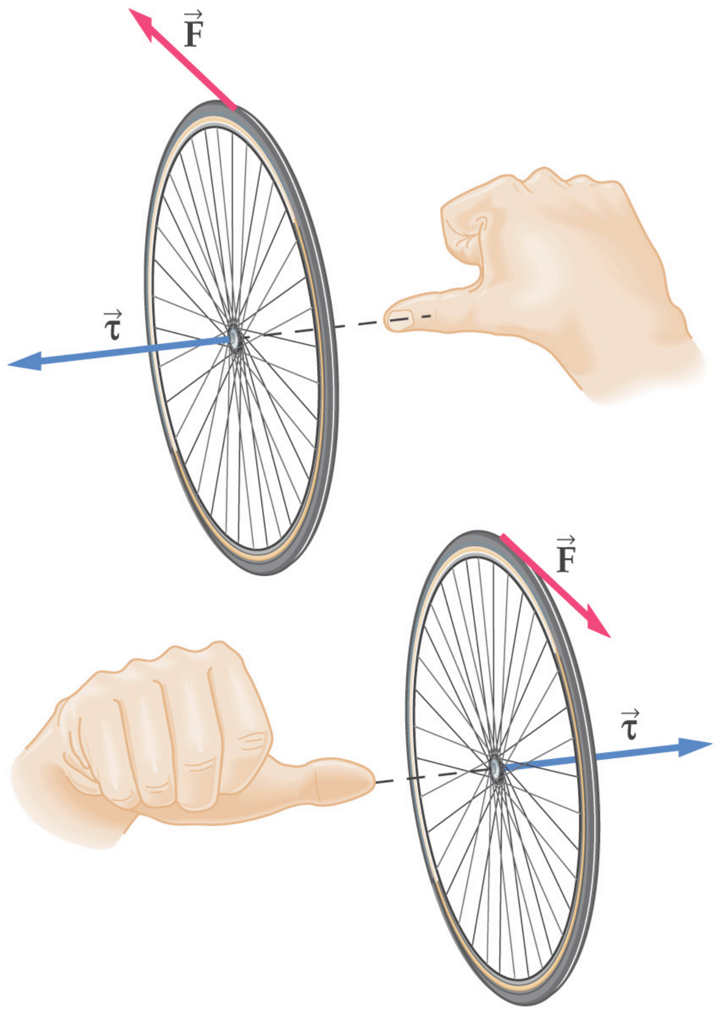
11-9 The Vector Nature of Rotational Motion

The direction of the angular velocity vector is along the axis of rotation. A right-hand rule gives the sign.



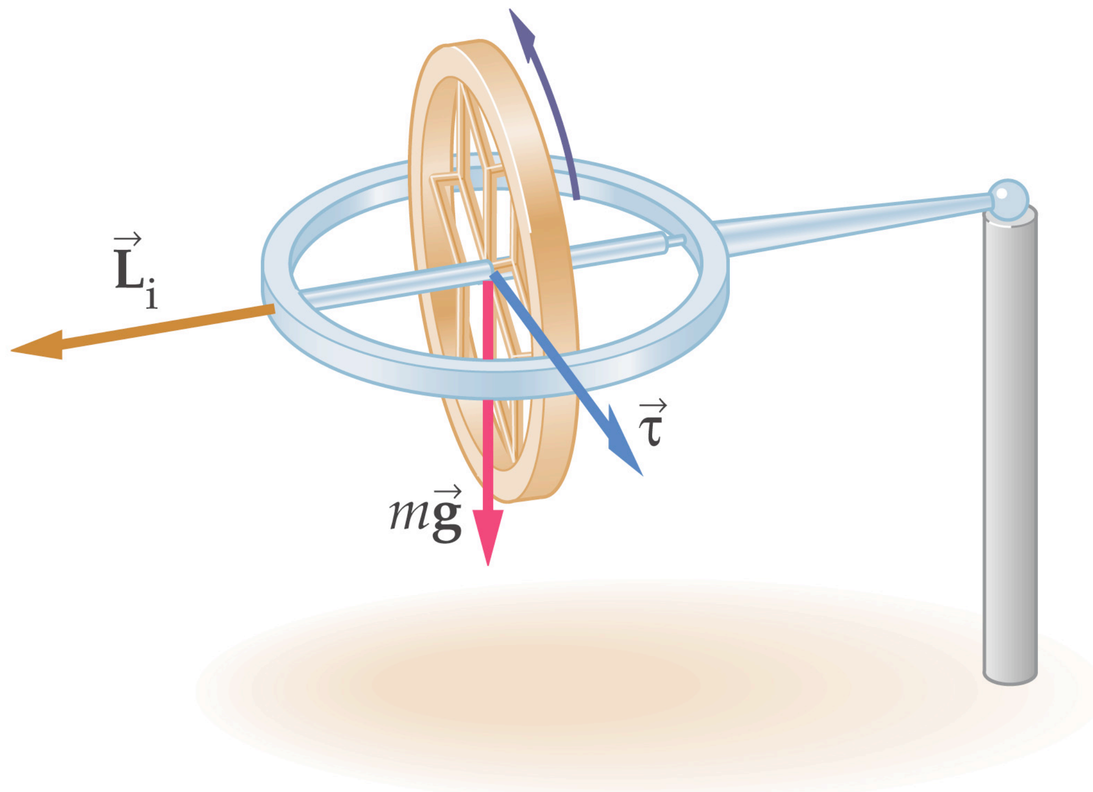
11-9 The Vector Nature of Rotational Motion

A similar right-hand rule gives the direction of the torque.



11-9 The Vector Nature of Rotational Motion

Conservation of angular momentum means that the total angular momentum around any axis must be constant. This is why gyroscopes are so stable.



Summary of Chapter 11

- A force applied so as to cause an angular acceleration is said to exert a torque.
- Torque due to a tangential force: $\tau = rF$
- Torque in general: $\tau = rF \sin\theta$
- Newton's second law for rotation: $\tau = I\alpha$
- In order for an object to be in static equilibrium, the total force and the total torque acting on the object must be zero.
- An object balances when it is supported at its center of mass.

Summary of Chapter 11

- **In systems with both rotational and linear motion, Newton's second law must be applied separately to each.**
- **Angular momentum:** $L = I\omega$
- **For tangential motion,** $L = rmv$
- **In general,** $L = rmv\sin\theta$
- **Newton's second law:** $\tau = \Delta L/\Delta t$
- **In systems with no external torque, angular momentum is conserved.**

Summary of Chapter 11

- **Work done by a torque:** $W = \tau \, \Delta\theta$
- **Power:** $P = Fv$
- **Rotational quantities are vectors that point along the axis of rotation, with the direction given by the right-hand rule.**