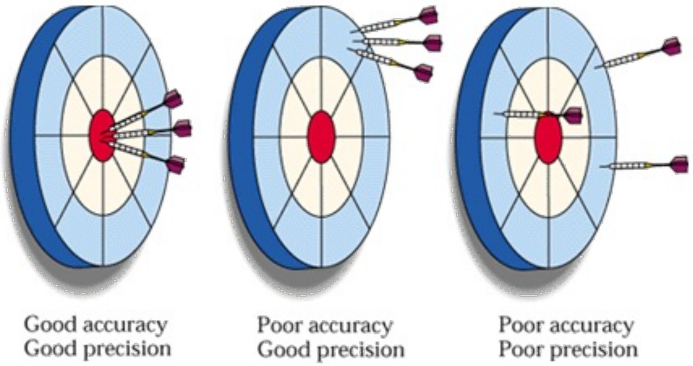

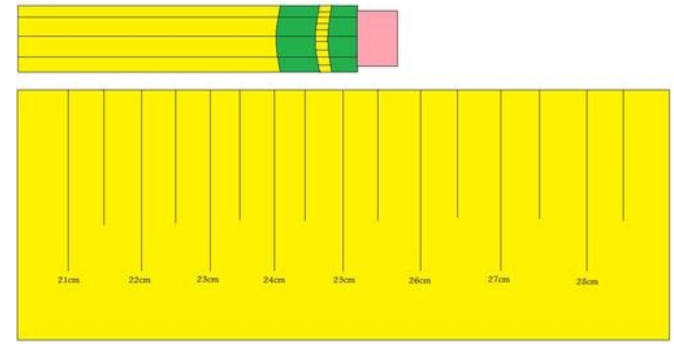

Error Analysis
Dr. Forrest

PHYS 3110

University of Houston

http://wps.prenhall.com/wps/media/objects/165/169061/SW-comart/fig1_2_5.gif
http://chemwiki.ucdavis.edu/@api/deki/files/430/Measuring_pencil.jpg?size=bestfit&width=423&height=241&revision=1

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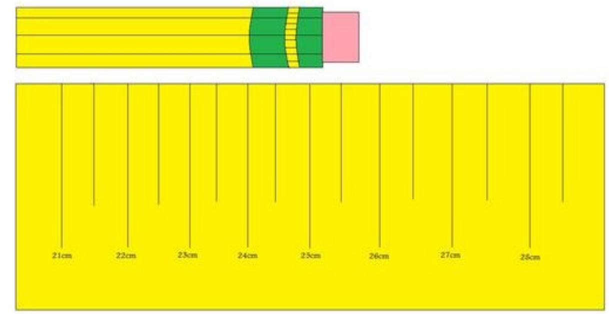
Types of Error

- Instrumental
- Observational
- Environmental
- Theoretical

Types of Error

- Instrumental
  - Accuracy limits of instrument
  - Poorly calibrated instrument
  - Fluctuating signal
  - Broken instrument
- Observational
- Environmental
- Theoretical



http://www.designworldonline.com/wp-content/uploads/2010/02/digital-meter-

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Types of Error

- Instrumental
- Observational
  - Parallax
  - Misused instrument
- Environmental
- Theoretical

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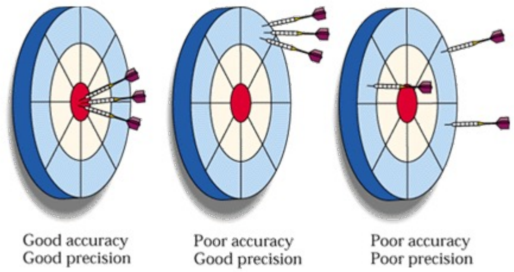
Types of Error

- Instrumental
- Observational
- Environmental
  - Electrical power brown-out, causing low current
  - Local magnetic field not accounted for
  - wind
- Theoretical

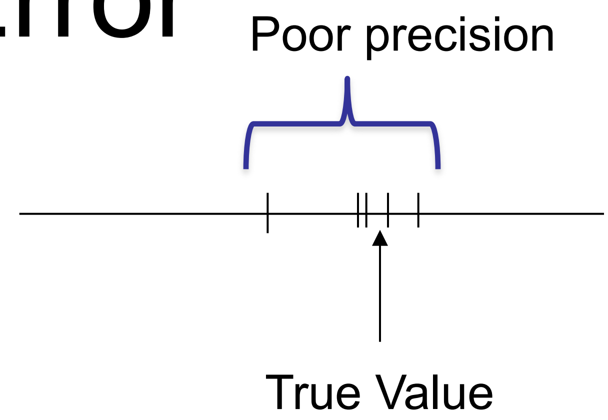
Types of Error

- Instrumental
- Observational
- Environmental
- Theoretical
  - Effects not accounted for or incorrectly ignored
  - Friction
  - Error in equations

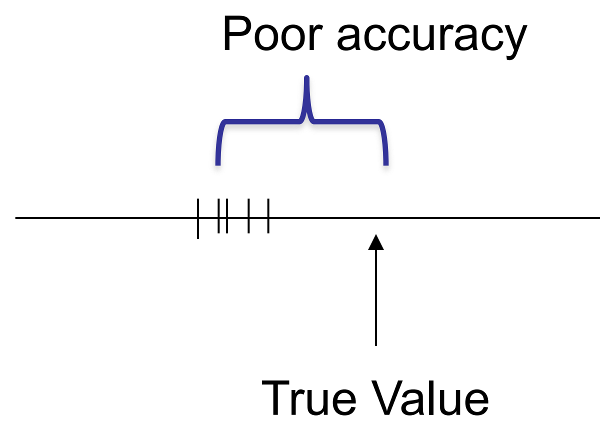


Types of Error

- Random
  - Can be quantified by statistical analysis



- Systematic
  - Try to identify and get rid of
  - Hopefully found during analysis; may need to repeat experiment!



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Error for Different Types of Quantities

- Measured Quantities
  - N Independent measurements of the same physical quantity
  - Measurement of a series of N quantities that are dependent on an independent value, i.e. x(t) or I(V).
- Calculated quantities
  - Propogation of error

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Statistical Analysis of Random Error

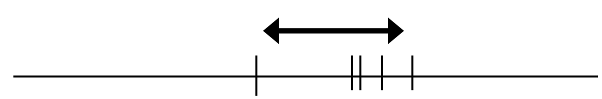
- For n independent measurements, they should group around the true value. For large n, the average should tend to the true value

$$\bar{x} \to x$$

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

- If the measurements are independent, can find the standard deviation, σ. σ proportional to width of the distribution.

$$\sigma = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$



precision

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Standard deviation of the mean

- $\sigma_m = \sigma/n^{1/2}$

$$\sigma_m = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

- For n > 1 measured values, report : $x = \bar{x} \pm \sigma_m$
  - If σ_m = 0 or < roughly 1/10 precision, use accuracy of measurement device for reported error.
- If there is no systematic error, there is a ~2/3 probability that the true value is within $\pm\sigma_m$

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Reporting Error

$$\bar{x} \pm \sigma_m$$

- Significant figures:
  - σ_m: one (sometimes two) sig. figs.
  - x_{ave}: same precision as σ_m

$G = 6.674\ 30 \times 10^{-11}$
$\pm\ 0.000\ 15 \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$

NIST, Fundamental Constants, https://physics.nist.gov/cgi-bin/cuu/Value?bg|search_for=universal_in!
accessed 8/29/23.

Length = 1.53 ± 0.05 m

σ and σ_m

- σ represents the error in one measurement
- σ_m represents the error in the mean of n measurements

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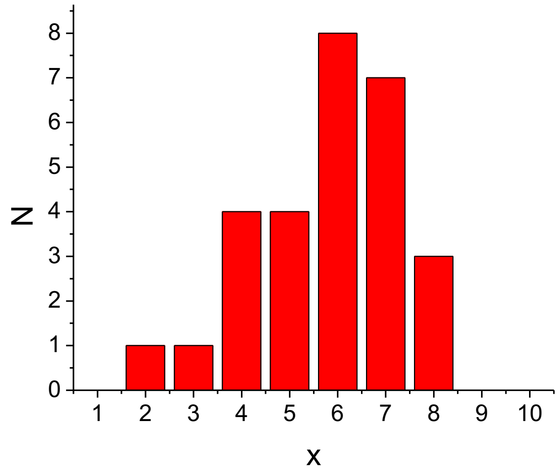
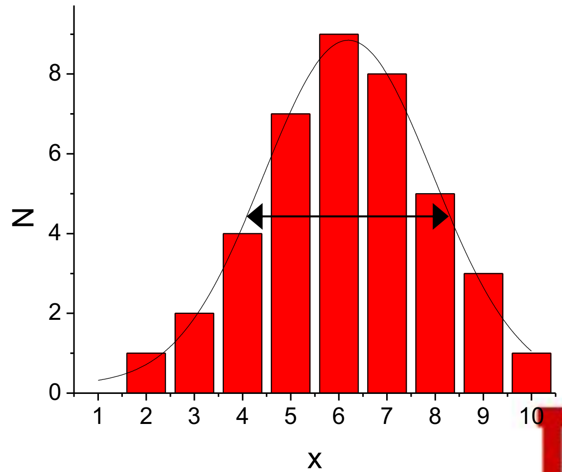
Gaussian Distribution

- Plot of measured value, versus number (N) of times that value was measured.
- IF error is random, for large n, this distribution tends to a Gaussian distribution.

$$N(x) = \frac{n}{\sqrt{2\pi}\ \sigma} e^{\frac{-(x-\bar{x})^2}{2\sigma^2}}$$





Fwhm ≅ 2.4σ

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Gaussian Distribution

- Probability of one measurement being x, P(x) = N(x)/n

$$P(x) = \frac{1}{\sqrt{2\pi}\ \sigma} e^{\frac{-(x-\bar{x})^2}{2\sigma^2}}$$

- The probability of a measurement being within $\pm\sigma$ of x_{ave}:

$$P(\text{within } \sigma) = \int_{\bar{x}-\sigma}^{\bar{x}+\sigma} P(x)\,dx$$

- Probability of being within nσ:

  $\pm 1\sigma$ = 68.3%

  $\pm 2\sigma$ = 95.5%

  $\pm 3\sigma$ = 99.7%

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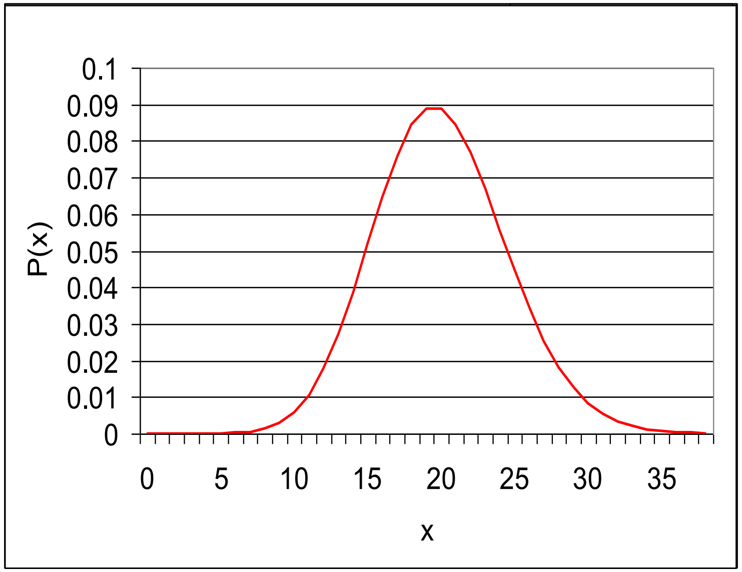
Poisson Distribution

$$P(x) = \frac{(\bar{x})^x e^{-\bar{x}}}{x!}$$



- Applies to processes described by an exponential, such as radioactive decay
- standard deviation $\sigma = \sqrt{x}$
- For large x_{ave}, i.e. for long counting times, the Poisson distribution tends to a Gaussian distribution with the same ave. and σ.

Reporting Error from a Poisson Process

- When measuring a physical process that you expect to follow a Poisson distribution, the error in one measurement is
$x = x \pm \sigma = x \pm \sqrt{x}$.
- Example:
  - Measuring the intensity of radiation emitted by an α source and scattered by gold nuclei, at an angle θ, over 30 seconds.

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Propagation of Errors

- Determining the error in a quantity calculated from measured data.
- Let x, y, z be measured values
- Let δx, δy, δz be the corresponding estimated errors in the measurements.
  $x \pm \delta x$, etc.
- If one measurement, δx = accuracy of the instrument. If n measurements of x, then use $x \pm \sigma_x$, where σ_x is the standard deviation of the mean of x.

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Propagation of Errors

- Let w(x,y,z) be a function of measured values. We want to find δw, the error in w.
- If the errors are uncorrelated, we assume dx ≈ δx,
- If the errors are correlated, there are cross terms like

differential:

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz$$

$$\delta w = \sqrt{\left(\tfrac{\partial w}{\partial x}\delta x\right)^2 + \left(\tfrac{\partial w}{\partial y}\delta y\right)^2 + \left(\tfrac{\partial w}{\partial z}\delta z\right)^2}$$

$$\delta w^2 = \sum_i \left(\tfrac{\partial w}{\partial x_i}\delta x_i\right)^2$$

$$\tfrac{\partial w}{\partial \mathrm{x}}\tfrac{\partial w}{\partial \mathrm{y}}\operatorname{cov}(\delta x, \delta y)$$

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Propagation of Errors

$$\delta w^2 = \sum_i \left(\frac{\partial w}{\partial x_i} \delta x_i \right)^2$$

Some Examples:

- w = ax + by + cz

$$\delta w = \sqrt{(a\delta x)^2 + (b\delta y)^2 + (c\delta z)^2}$$

- w = k x^a y^b z^c

$$\delta w = \sqrt{\left(\frac{\mathrm{aw}\delta x}{x}\right)^2 + \left(\frac{\mathrm{bw}\delta y}{y}\right)^2 + \left(\frac{\mathrm{cw}\delta z}{z}\right)^2}$$

∂w/∂x = a k x^{a-1} y^b z^c

∂w/∂x = aw/x

$$\frac{\delta w}{w} = \sqrt{\left(\frac{\mathrm{a}\delta x}{x}\right)^2 + \left(\frac{\mathrm{b}\delta y}{y}\right)^2 + \left(\frac{\mathrm{c}\delta z}{z}\right)^2}$$

- During lab, for a quick error estimate, just use the biggest term.

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% Error

- Precision = % error = $(\delta x/x)$ *100%
- Accuracy = % error compared to accepted
  $= |x_{calculated} - x_{accepted}|/x_{accepted}$ * 100%
- % difference = $|x_1 - x_2|/(|x_1 + x_2|/2)$ * 100%

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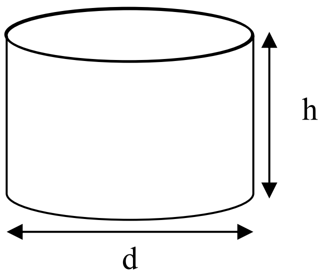
Example: Density of a Cylinder

$\rho = m/V$

$\rho = m/(\pi r^2 h)$

$$\rho = \frac{\overline{m}}{\frac{\pi}{4}\overline{d}^2\overline{h}}$$



$$\delta\rho = \sqrt{\left(\frac{\partial\rho}{\partial m}\delta m\right)^2 + \left(\frac{\partial\rho}{\partial d}\delta d\right)^2 + \left(\frac{\partial\rho}{\partial h}\delta h\right)^2}$$

$$\delta\rho = \sqrt{\left(\frac{\rho}{m}\delta m\right)^2 + \left(\frac{2\rho}{d}\delta d\right)^2 + \left(\frac{\rho}{h}\delta h\right)^2}$$

Check your units!

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Density of a Cylinder

$$\rho = \frac{m}{\frac{\pi}{4} d^2 h} \qquad \delta\rho = \sqrt{\left(\frac{\rho}{m}\delta m\right)^2 + \left(\frac{2\rho}{d}\delta d\right)^2 + \left(\frac{\rho}{h}\delta h\right)^2}$$

Assume, after measuring d, h, and m three times each, you get

m = 492.0 $\pm$ 0.5 g

h = 11.00 $\pm$ 0.01 cm

d = 4.00 $\pm$ 0.02 cm, ρ = 3.559 g/cm^3

But precession of tools are 0.5 g and 0.15 cm.

$$\delta\rho = 3.559 \frac{g}{cm^3} \sqrt{\left(\frac{0.5\ g}{492.0\,g}\right)^2 + \left(\frac{2\times 0.02\ cm}{4.00\,cm}\right)^2 + \left(\frac{0.15\ cm}{11.00\,cm}\right)^2}$$

$$\delta\rho = 3.559\ \frac{g}{cm^3} \sqrt{0.001^2 + 0.010^2 + 0.014^2} = 0.0613\ \frac{g}{cm^3}$$

Sig. Figs.

ρ = 3.56 $\pm$ 0.06 g/cm^3, $\delta\rho/\rho$ = 0.02, 2% error

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Density of a Cylinder

$$\rho = \frac{m}{\frac{\pi}{4} d^2 h} \qquad \delta\rho = \sqrt{\left(\frac{\rho}{m}\delta m\right)^2 + \left(\frac{2\rho}{d}\delta d\right)^2 + \left(\frac{\rho}{h}\delta h\right)^2}$$

$$\delta\rho = 3.559\,\frac{g}{cm^3}\sqrt{\left(\frac{0.5\ g}{492.0\,g}\right)^2 + \left(\frac{2\times 0.02\ cm}{4.00\,cm}\right)^2 + \left(\frac{0.15\ cm}{11.00\,cm}\right)^2}$$

$$\delta\rho = 3.559\ \frac{g}{cm^3}\sqrt{0.001^2 + 0.010^2 + 0.014^2} = 0.0613\ \frac{g}{cm^3}$$

Which term contributes most to error?

How can you reduce the error?

Rule of thumb: If more than 10 measurements of a single variable are needed, use a better instrument or method to reduce σ_m.

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Weighted Average

- n values, each with their own error, $x_i \pm \sigma_i$
- The error in this weighted mean can be found from propagation of error

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i \left(\frac{1}{\sigma_i}\right)^2}{\sum_{i=1}^{n} \left(\frac{1}{\sigma_i}\right)^2}$$

$$\frac{\partial \bar{x}}{\partial x_i} = \frac{\left(\frac{1}{\sigma_i}\right)^2}{\sum_{i=1}^{n} \left(\frac{1}{\sigma_i}\right)^2}$$

$$\delta \bar{x} = \frac{\sqrt{\sum_{i=1}^{n} \left(\frac{1}{\sigma_i}\right)^4 \sigma_i^2}}{\sum_{i=1}^{n} \left(\frac{1}{\sigma_i}\right)^2}$$

$$\delta \bar{x} = \frac{\sqrt{\sum_{i=1}^{n} \left(\frac{1}{\sigma_i}\right)^2}}{\sum_{i=1}^{n} \left(\frac{1}{\sigma_i}\right)^2} = \left(\sum_{i=1}^{n} \left(\frac{1}{\sigma_i}\right)^2\right)^{-1/2}$$

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Weighted Mean Examples

- You measure 5 sets of (E, T) data to try to find σ in the Stephan-Boltzmann Law, $E=\sigma T^4$
- You can't average E & T and find one σ from the average.
- Find 5 σ's, find the weighted mean using the 5 $\delta\sigma$'s calculated from propagation of error, then find the uncertainty of the weighted mean.

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Homework

- Find the density of your Cougar One Card (UH ID card) and find the estimated error in the density.
  - Find the standard deviation of the mean for each measured value. Compare it to the uncertainty in the measurements and decide which to use. Explain your choices.
  - Use propagation of error to find the uncertainty in the density. Show the equation you derive.
  - Your precision should be less than 30%.