The integration is over the triangular area passing through the points \((x,y)\), \((y,x)\), \((x,z)\) as shown in the figure. The integral is:

\[
V = \frac{1}{60}
\]

Carry out the integration to obtain;

\[
V = 1/60
\]

We first note; \(\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}\). Then \(\hat{r} = \vec{r}/|r| = \cos(\phi) \sin(\theta)\hat{x} + \sin(\phi) \sin(\theta) \hat{y} + \cos(\theta) \hat{z}\)

Then \(\vec{\phi}\) moves in the \((x,y)\) plane. Projection of \(\vec{\phi}\) onto the \((x,y)\) axes gives;
\[ \dot{\phi} = -\sin(\phi) \hat{x} + \cos(\phi) \hat{y} \]

Finally the projection of \( \vec{\theta} \) onto the \((z,\rho)\) plane gives;

\[ \hat{\theta} = \cos(\phi) \cos(\theta) \hat{x} + \sin(\phi) \cos(\theta) \hat{y} - \sin(\theta) \hat{z} \]

This can be written as a matrix transformation;

\[
\begin{pmatrix}
\hat{r} \\
\hat{\theta} \\
\hat{\phi}
\end{pmatrix} = \begin{pmatrix}
\sin(\theta) \cos(\phi) & \sin(\theta) \sin(\phi) & \cos(\theta) \\
\cos(\theta) \cos(\phi) & \cos(\theta) \sin(\phi) & -\sin(\theta) \\
-\sin(\phi) & \cos(\phi) & 0
\end{pmatrix} \begin{pmatrix}
\hat{x} \\
\hat{y} \\
\hat{z}
\end{pmatrix}
\]

The transformation matrix, \( A \) has an inverse, \( A^T \), such that \( AA^T = I \) where \( I \) is the identity matrix. This matrix is easily found and gives the inverse transformation;

\[
A^T = \begin{pmatrix}
\sin(\theta) \cos(\phi) & \cos(\theta) \cos(\phi) & -\sin(\phi) \\
\sin(\theta) \sin(\phi) & \cos(\theta) \sin(\phi) & \cos(\phi) \\
\cos(\theta) & -\sin(\theta) & 0
\end{pmatrix}
\]

Thus;

\[
\begin{pmatrix}
\hat{x} \\
\hat{y} \\
\hat{z}
\end{pmatrix} = A^T \begin{pmatrix}
\hat{r} \\
\hat{\theta} \\
\hat{\phi}
\end{pmatrix}
\]

1.44)

In each of these cases the argument of the \( \delta \) function has the form \( \delta(\alpha x + \beta) \) Therefore let \( \eta = \alpha x + \beta \) so that \( x = (\eta - \beta)/\alpha \) and \( dx = d\eta/\alpha \). Then substitute for \( x \) in the integrand and integrate over \( d\eta \). In general

\[
\int_a^b dx f(x) = \int_{\alpha a + \beta}^{\alpha b + \beta} d\eta/\alpha \ f((\eta - \beta)/\alpha) \ \delta(\eta) = f(-\beta/\alpha)/\alpha
\]

1.45)

a) Consider the integral \( \int_a^b dx x f(x) \frac{d(\delta(x))}{dx} dx \). Integrate this by parts.
\[
\begin{align*}
\int_a^b dx \left[ x f(x) \right] \frac{d(\delta(x))}{dx} &= f(x) \bigg|_a^b - \int_a^b dx \left[ \frac{dxf(x)}{dx} \right] \delta(x) \\
\frac{dxf(x)}{dx} &= f(x) + xf'(x)
\end{align*}
\]

The first term vanishes for \( x = a \) and \( x = b \). The second term is;

\[
\int_a^b dx \delta(x) + \int_a^b dx x f'(x)
\]

If \( f'(x) \) does not have a singularity at \( x = 0 \) this term also vanishes. Thus;

\[
\int_a^b dx x f(x) \frac{d(\delta(x))}{dx} = -\int_a^b dx \delta(x) f(x)
\]

b) Use the definition of the step function \( \Theta(x) = 0 \) for \( x < 0 \) and \( \Theta(x) = 1 \) for \( x > 1 \). Integrate by parts in the following;

\[
\int_a^b dx f(x) \frac{d\Theta}{dx} = f(x) \left. \Theta(x) \right|_a^b - \int_a^b dx f'(x) \Theta(x)
\]

The first term is only non-zero for \( x = b \). Thus;

\[
f(x) \left. \Theta(x) \right|_a^b = f(b)
\]

The second term is evaluated only for \( x > 0 \).

\[
\int_0^b dx f'(x) = f(b) - f(a)
\]

Therefore;

\[
\int_a^b dx f(x) \frac{d\Theta}{dx} = f(b) - [f(b) - f(0)] = f(0)
\]