Chapter 4
Two-Dimensional Kinematics
Units of Chapter 4

• Motion in Two Dimensions
• Projectile Motion: Basic Equations
• Zero Launch Angle
• General Launch Angle
• Projectile Motion: Key Characteristics
4-1 Motion in Two Dimensions

If velocity is constant, motion is along a straight line:

\[
\begin{align*}
\vec{v}_y &= v_0 \sin \theta \\
\vec{v}_0 &= v_0 \\
\theta &= 25^\circ
\end{align*}
\]

\[
\begin{align*}
x &= v_{0x} t \\
y &= v_{0y} t
\end{align*}
\]
4-1 Motion in Two Dimensions

Motion in the $x$- and $y$-directions should be solved separately:

<table>
<thead>
<tr>
<th>TABLE 4–1 Constant-Acceleration Equations of Motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Position as a function of time</td>
</tr>
<tr>
<td>$x = x_0 + v_{0x}t + \frac{1}{2}a_xt^2$</td>
</tr>
<tr>
<td>$y = y_0 + v_{0y}t + \frac{1}{2}a_yt^2$</td>
</tr>
</tbody>
</table>
4-2 Projectile Motion: Basic Equations

Assumptions:

• ignore air resistance

• $g = 9.81 \text{ m/s}^2$, downward

• ignore Earth’s rotation

If $y$-axis points upward, acceleration in $x$-direction is zero and acceleration in $y$-direction is $-9.81 \text{ m/s}^2$
What is the motion of a struck baseball? Once it leaves the bat (if air resistance is negligible) only the force of gravity acts on the baseball.
The baseball has $a_x = 0$ and $a_y = -g$, it moves with constant velocity along the $x$-axis and with nonzero, constant acceleration along the $y$-axis.
Example: An object is projected from the origin. The initial velocity components are $v_{ix} = 7.07$ m/s, and $v_{iy} = 7.07$ m/s. Determine the x and y position of the object at 0.2 second intervals for 1.4 seconds. Also plot the results.

$$\Delta y = v_{iy}\Delta t + \frac{1}{2}a_y\Delta t^2$$
$$\Delta x = v_{ix}\Delta t$$

Since the object starts from the origin, $\Delta y$ and $\Delta x$ will represent the location of the object at time $\Delta t$. 
Example continued:

<table>
<thead>
<tr>
<th>t (sec)</th>
<th>x (meters)</th>
<th>y (meters)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.2</td>
<td>1.41</td>
<td>1.22</td>
</tr>
<tr>
<td>0.4</td>
<td>2.83</td>
<td>2.04</td>
</tr>
<tr>
<td>0.6</td>
<td>4.24</td>
<td>2.48</td>
</tr>
<tr>
<td>0.8</td>
<td>5.66</td>
<td>2.52</td>
</tr>
<tr>
<td>1.0</td>
<td>7.07</td>
<td>2.17</td>
</tr>
<tr>
<td>1.2</td>
<td>8.48</td>
<td>1.43</td>
</tr>
<tr>
<td>1.4</td>
<td>9.89</td>
<td>0.29</td>
</tr>
</tbody>
</table>
4-2 Projectile Motion: Basic Equations

The acceleration is independent of the direction of the velocity:

\[ a_x = 0 \]
\[ a_y = -g \]
These, then, are the basic equations of projectile motion:

\[ x = x_0 + v_{0x}t \]
\[ y = y_0 + v_{0y}t - \frac{1}{2}gt^2 \]

\[ v_x = v_{0x} \]
\[ v_y = v_{0y} - gt \]

\[ v_x^2 = v_{0x}^2 \]
\[ v_y^2 = v_{0y}^2 - 2g\Delta y \]
4-3 Zero Launch Angle

Launch angle: direction of initial velocity with respect to horizontal
4-3 Zero Launch Angle

In this case, the initial velocity in the \( y \)-direction is zero. Here are the equations of motion, with \( x_0 = 0 \) and \( y_0 = h \):

\[
x = v_0 t
\]

\[
y = h - \frac{1}{2} gt^2
\]

\[
v_x = v_0 = \text{constant}
\]

\[
v_y = -gt
\]

\[
v_x^2 = v_0^2 = \text{constant}
\]

\[
v_y^2 = -2g \Delta y
\]
4-3 Zero Launch Angle

This is the trajectory of a projectile launched horizontally:
4-3 Zero Launch Angle

Eliminating $t$ and solving for $y$ as a function of $x$:

$$y = h - \frac{1}{2}g \left( \frac{x}{v_0} \right)^2 = h - \left( \frac{g}{2v_0^2} \right)x^2$$

This has the form $y = a + bx^2$, which is the equation of a parabola.

The landing point can be found by setting $y = 0$ and solving for $x$:

$$x = v_0 \sqrt{\frac{2h}{g}}$$
4-4 General Launch Angle

In general, \( v_{0x} = v_0 \cos \theta \) and \( v_{0y} = v_0 \sin \theta \)

This gives the equations of motion:

\[
\begin{align*}
x &= (v_0 \cos \theta)t \\
y &= (v_0 \sin \theta)t - \frac{1}{2}gt^2
\end{align*}
\]

\[
\begin{align*}
v_x &= v_0 \cos \theta \\
v_y &= v_0 \sin \theta - gt \\
v_x^2 &= v_0^2 \cos^2 \theta \\
v_y^2 &= v_0^2 \sin^2 \theta - 2g\Delta y
\end{align*}
\]
4-4 General Launch Angle

Snapshots of a trajectory; red dots are at \( t = 1 \) s, \( t = 2 \) s, and \( t = 3 \) s
4-5 Projectile Motion: Key Characteristics

Range: the horizontal distance a projectile travels

If the initial and final elevation are the same:

\[ R = \left( \frac{v_0^2}{g} \right) \sin 2\theta \]
4-5 Projectile Motion: Key Characteristics

The range is a maximum when $\theta = 45^\circ$:

$$R_{\text{max}} = \frac{v_0^2}{g}$$
4-5 Projectile Motion: Key Characteristics

Symmetry in projectile motion:
Example (text problem 3.50): An arrow is shot into the air with $\theta = 60^\circ$ and $v_i = 20.0 \text{ m/s}$.

(a) What are $v_x$ and $v_y$ of the arrow when $t = 3$ sec?

The components of the initial velocity are:

$\begin{align*}
  v_{ix} &= v_i \cos \theta = 10.0 \text{ m/s} \\
  v_{iy} &= v_i \sin \theta = 17.3 \text{ m/s}
\end{align*}$

At $t = 3$ sec:

$\begin{align*}
  v_{fx} &= v_{ix} + a_x \Delta t = v_{ix} = 10.0 \text{ m/s} \\
  v_{fy} &= v_{iy} + a_y \Delta t = v_{iy} - g \Delta t = -12.1 \text{ m/s}
\end{align*}$
Example continued:

(b) What are the x and y components of the displacement of the arrow during the 3.0 sec interval?

\[ \Delta r_x = \Delta x = v_{ix} \Delta t + \frac{1}{2} a_x \Delta t^2 = v_{ix} \Delta t + 0 = 30.0 \text{ m} \]

\[ \Delta r_y = \Delta y = v_{iy} \Delta t + \frac{1}{2} a_y \Delta t^2 = v_{iy} \Delta t - \frac{1}{2} g \Delta t^2 = 7.80 \text{ m} \]
Example: How far does the arrow in the previous example land from where it is released?

The arrow lands when $\Delta y = 0$.  

\[ \Delta y = v_{iy} \Delta t - \frac{1}{2} g \Delta t^2 = 0 \]

Solving for $\Delta t$:  

\[ \Delta t = \frac{2v_{iy}}{g} = 3.53 \text{ sec} \]

The distance traveled is:  

\[ \Delta x = v_{ix} \Delta t + \frac{1}{2} a_x \Delta t^2 \]

\[ = v_{ix} \Delta t + 0 = 35.3 \text{ m} \]
Summary of Chapter 4

• Components of motion in the $x$- and $y$-directions can be treated independently

• In projectile motion, the acceleration is $-g$

• If the launch angle is zero, the initial velocity has only an $x$-component

• The path followed by a projectile is a parabola

• The range is the horizontal distance the projectile travels